Running Head: MODELING VISUAL PROBLEM-SOLVING

Modeling Visual Problem-Solving as Analogical Reasoning

Andrew Lovett and Kenneth Forbus

Northwestern University

Abstract

We present a computational model of visual problem-solving, designed to solve problems from the Raven's Progressive Matrices intelligence test. The model builds on the claim that analogical reasoning lies at the heart of visual problem-solving, and intelligence more broadly. Images are compared via structure-mapping, aligning the common relational structure in two images to identify commonalities and differences. These commonalities or differences can themselves be reified and used as the input for future comparisons. When images fail to align, the model dynamically re-represents them to facilitate the comparison. In our analysis, we find that the model matches adult human performance on the Standard Progressive Matrices test, and that problems which are difficult for the model are also difficult for people. Furthermore, we show that model operations involving abstraction and re-representation are particularly difficult for people, suggesting that these operations may be critical for performing visual problem-solving, and reasoning more generally, at the highest level.

Keywords: Visual comparison, analogy, problem-solving, cognitive modeling.

Modeling Visual Problem-Solving as Analogical Reasoning

Analogy is perhaps the cornerstone of human intelligence (Gentner, 2003, 2010; Penn, Holyoak, & Povinelli, 2008; Hofstadter & Sander, 2013). By comparing two domains and identifying commonalities in their structure, we can derive useful inferences and develop novel abstractions. Analogy can drive scientific discovery, as when Rutherford famously suggested that electrons orbiting a nucleus were like planets orbiting a sun. But it also plays a role in our everyday lives, allowing us to apply what we've learned in past experiences to the present, as when a person solves a physics problem, chooses a movie to watch, or considers buying a new car.

Analogy's power lies in its abstract nature. We can compare two wildly different scenarios, applying what we've learned in one scenario to the other, based on commonalities in their relational structure. Given this highly abstract mode of thought, and its importance in human reasoning, it may be surprising that when researchers want to test an individual's reasoning ability, they often rely on concrete, visual tasks.

Figure 1 depicts an example problem from Raven's Progressive Matrices (RPM), an intelligence test (Raven, Raven, & Court, 1998). This test requires that participants compare images in a (usually) 3x3 matrix, identify a pattern across the matrix, and solve for the missing image. RPM was designed to measure a subject's *eductive ability* (the ability to discover patterns in confusing stimuli), a term that has now been mostly replaced by *fluid intelligence* (Cattell, 1963). It has remained popular for decades because it is highly successful at predicting a subject's performance on other ability tests—not just visual tests, but verbal and mathematical as well (Burke & Bingham, 1969; Zagar, Arbit, & Friedland, 1980; Snow, Kyllonen, & Marshalek,

1984). A classic scaling analysis which positioned ability tests based on their intercorrelations placed RPM in the center, indicating it was the single most predictive test (Figure 2).

How is a visual test so effective at measuring general problem-solving ability? We believe that despite its concrete nature, RPM tests individuals' abilities to make effective analogies. The connection between RPM and analogy is well-supported by the analysis in Figure 2. In that analysis, visual (or *geometric*), verbal, and mathematical analogy problems were clustered around RPM, suggesting that they correlate highly with it and that they are also strong general measures. Indeed, RPM can be seen as a complex geometric analogy problem, where subjects must determine the relation between the first two images and the last image in the top row and then compute an image that produces an analogous relation in the bottom row. Consistent with this claim, Holyoak and colleagues showed that high RPM performers required less assistance when performing analogical mappings (Vendetti, Wu, & Holyoak, 2014) and retrievals (Kubricht et al., 2015). Furthermore, a meta-analysis of brain imaging studies found that verbal analogies, geometric analogies, and matrix problems engage a common brain region, the left rostrolateral prefrontal cortex, which may be associated with relational reasoning (Hobeika et al., 2016)[1].

Here we argue that the mechanisms and strategies that support effective analogizing are also those that support visual problem-solving. To test this claim, we model human performance on RPM using a well-established computational model of analogy, the Structure-Mapping Engine (SME) (Falkenhainer, Forbus, & Gentner, 1989). While SME was originally developed to model abstract analogies, there is increasing evidence that its underlying principles also apply

[1] Matrix problems, specifically, engage several additional areas, perhaps due to their greater complexity and the requirement that test-takers select from a set of possible answers, both of which may increase working memory demands.

to concrete visual comparisons (Markman & Gentner, 1996; Sagi, Gentner, & Lovett, 2012). RPM provides the opportunity to test the role of analogy in visual thinking on a large scale, and to determine what components are needed to perform this task outside of the analogical mapping that SME provides. In particular, we consider the dual challenges of perception and re-representation: How do you represent concrete visual information in a manner that supports abstract analogical thought, and how do you change your representation when images fail to align?

This approach also allows us to gain new insights about RPM and what it evaluates in humans. By ablating the model's ability to perform certain operations and comparing the resulting errors to human performance, we can identify factors that make a problem easier or more difficult for people. As we show below, problems tend to be more difficult when they a) must be represented more abstractly, or b) require complex re-representation operations. We close by considering whether abstract thinking and re-representation in RPM might generalize to other analogical tasks and thus be central to human intelligence.

We next describe RPM in greater detail, including a well-established previous computational model. Afterwards, we present our theoretical framework, showing how analogical reasoning maps onto RPM and visual problem-solving more broadly. We then describe our computational model, which builds on this framework and follows previous models of other visual problem-solving tasks. We present a simulation of the Standard Progressive Matrices, a 60-item intelligence test. The model's overall performance matches average American adults. We finish with our ablation analysis.

Raven's Progressive Matrices

On a typical RPM problem, test-takers are shown a 3x3 matrix of images, with the lower right image missing. By comparing the images and identifying patterns across each row and column, they determine the answer that best completes the matrix, choosing from eight possible answers. Figures 3-5 show several example problems, which will be used as references throughout the paper and referred to by their respective letters. Note that no actual test problems are shown, but these example problems are analogous to real test problems.

RPM has been a popular intelligence test for decades. It is successful because it does not rely on domain-specific knowledge or verbal ability. Thus, it can be used across cultures and ages to assess *fluid intelligence*, the ability to reason flexibly while solving problems (Cattell, 1963). RPM is one of the best single-test predictors of problem-solving ability: participants who do well on RPM do well on other intelligence tests (e.g., Burke & Bingham, 1969; Zagar, Arbit, & Friedland, 1980; see Raven, Raven, & Court, 2000b, for a review) and do well on other verbal, mathematical, and visual ability tests (Snow, Kyllonen, & Marshalek, 1984; Snow & Lohman, 1989). Thus, RPM appears to tap into core, general-purpose cognitive abilities. However, it remains unclear what exactly those abilities are.

Carpenter, Just, and Shell (1990) conducted an influential study of the Advanced Progressive Matrices (APM), the hardest version of the test. They ran test-takers with an eye tracker, analyzed the problems, and built two computational models.

Carpenter et al. found that participants generally solved problems by looking across a row and determining how each object varied between images. Their analysis produced five rules to explain how objects could vary: 1) *constant in a row*: the object stays the same; 2) *quantitative pairwise progression*: the object changes in some way (e.g., rotating or increasing in size)

between images (problem A, Figure 3); 3) *distribution of three*: there are three different objects in the three images; those objects will be in every row, but their order will vary (problem B); 4) *figure subtraction* or *addition*: add or subtract the objects in the first two images to produce the objects in the last (problem E); and 5) *distribution of two*: each object is in only two of the three images (problems F, H). While the final rule sounds simple, it is actually quite complex. It requires recognizing that there is no corresponding object in one of the three images.

Carpenter et al.'s FAIRAVEN model implements these rules. Given hand-coded, symbolic image representations as input, it analyzes each row via the following steps:

1. Identify corresponding objects. The model uses simple heuristics, such as matching same-shaped objects, or matching leftover objects.

2. For each set of corresponding objects, determine which of the five rules (see above) it instantiates. FAIRAVEN can recognize every rule type except distribution of two.

The model performs these steps on each of the first two rows and then compares the two rows' rules, generalizing over them. Finally, it applies the rules to the bottom row to compute the answer.

The BETTERAVEN model improves on FAIRAVEN in a few ways. Firstly, during correspondence-finding, it can recognize cases where an object is only in two of the three images. Secondly, it checks for the distribution of two rule. Thirdly, it has a more developed goal management system. It compares the rules identified in the top two rows, and if they are dissimilar, it backtracks and looks for alternate rules.

Carpenter et al. argued that this last improvement, better goal management, was what truly set BETTERAVEN apart. They believed that skilled RPM test-takers are adept at goal management, primarily due to their superior work memory capacity. This could explain RPM's

strong predictive power: it may accurately measure working memory capacity, a key asset in other ability tests. In support of this hypothesis, other researchers (Vodegel-Matzen, van der Molen, & Dudink, 1994; Embretson, 1998) have found that as the number and complexity of Carpenter rules in a problem increases, the problem becomes harder. Presumably, a greater number of rules places more load on working memory.

We believe Carpenter et al.'s models are limited in that they fail to capture *perception*, *analogical mapping*, and *re-representation*. As we have suggested above and argue below, these processes are critical in both analogical thought and visual problem-solving. Because they do not analyze these or other general processes, Carpenter et al. can derive only limited connections between RPM and other problem-solving tasks. The primary connection they derive is that RPM requires a high working memory capacity. Below, we briefly describe each of the model's limitations.

1. Perception. The Carpenter models take symbolic representations as input. These representations are hand-coded, based upon descriptions given by participants. While we agree on the use of symbolic representations, this approach is limited in two respects: a) it ignores the challenge of generating symbolic representations from visual input, and b) it makes no theoretical claims about what information should or should not be captured in the symbolic representations. In fact, problem-solving depends *critically* on what information is captured in the initial representations, and the ability to identify and represent the correct information might well be a skill underlying effective problem-solving.

2. Analogical mapping. When the BETTERAVEN model compares images, it identifies corresponding objects using three simple heuristics: a) match up identical shapes, b) match up leftover shapes, c) allow a shape to match with nothing. In contrast, we believe visual

comparison can use the same rich relational alignment process used in abstract analogies (Markman & Gentner, 1996; Sagi, Gentner, & Lovett, 2012).

In addition, we believe one challenge in RPM may be determining which images to compare. For example, in RPM it is typically enough to compare the adjacent images in each row, but for some problems one must also compare the first and last images in the row (e.g., problem E). In our analysis below, we test whether problems requiring this additional image comparison are more difficult to solve.

3. Re-representation. Carpenter et al. mention that in some cases their model is given a second representation to try if the first one fails to produce a satisfying answer. This is an example of re-representation: changing an image representation to facilitate a comparison. However, it appears that re-representation is not performed or analyzed in any systematic way. In fact it is likely not needed in most cases because the initial representations are hand-coded—if the initial representations are constructed to facilitate the comparison, then no re-representation will be required.

We believe re-representation can play a critical role in visual problem-solving, as well as in analogy more broadly. In this paper we show how re-representation is used to solve problems such as G, and we analyze the difficulty of these problems for human test-takers.

Theoretical Framework

We argue that visual thinking often involves analogical processing of the same form that is used elsewhere in cognition (e.g. Gentner & Smith, 2013; Kokinov & French 2003). That is, a problem is encoded (*perception*) and analyzed to ascertain what comparison(s) are needed to be done. The comparisons are carried out via structure-mapping (Gentner, 1983). The results are analyzed and evaluated by task-specific processes, which include methods for re-representation

(Yan et al. 2003), often leading to further comparisons. This is an example of a *map/analyze* cycle (Falkenhainer, 1990; Forbus, 2001; Gentner et al., 1997), a higher-level pattern of analogical processing which has been used in modeling learning from observation and conceptual change. The same overall structure, albeit with vision-specific encoding and analysis processes, appears to be operating in solving some kinds of visual problems, and specifically RPM problems. This provides a straightforward explanation as to why RPM is so predictive of performance on so many non-visual problems: The same processes are being used.

One key claim which should be emphasized is: *Re-representation is driven by comparison*. Rather than a top-down search process that explores different possible representations for each image, we are suggesting that initial, bottom-up representations are changed only when necessary to facilitate a comparison.

Below we provide background on analogical mapping. We then describe five steps for visual problem-solving:

1. *Perception*: Generate symbolic representations from images.
2. *Visual Comparison*: Align the relational structure in two images, identify commonalities and differences.
3. *Perceptual Reorganization*: Re-represent the images, if necessary, to facilitate a comparison.
4. *Difference Identification*: Symbolically represent the differences between images.
5. *Visual Inference*: Apply a set of differences to one image to infer a new image.

We have previously modeled two other visual problem-solving tasks using analogy: a visual oddity task (Lovett & Forbus, 2011a) and geometric analogy (Lovett et al., 2009b) (Figure

6). In what follows, we speak generally of visual problem-solving when possible, and focus on RPM specifics only when necessary.

Analogical Mapping

Cases are represented symbolically, as entities, attributes, and relations. Two representations, a *base* and a *target*, are aligned based on their common relational structure (Gentner, 1983; Hummel & Holyoak, 1997; Larkey & Love, 2003; Doumas & Hummel, 2013). Based on the corresponding attributes and relations, corresponding entities can be identified. The result of a mapping is a set of correspondences between the base and target, and a similarity score based on the depth and breadth of aligned structure (Falkenhainer, Forbus, & Gentner, 1989). According to structure-mapping theory (Gentner, 1983, 2010), mappings are constrained to allow each base item to map to just one target item. Mappings also include *candidate inferences*, where structure in one description is projected to the other. The results of a mapping can be represented symbolically, so that they themselves can play a role in future reasoning (even in future analogies). We use two types of reification that have been used elsewhere in the literature for non-visual comparisons:

1. *Generalization*. Here one constructs an abstraction, sometimes called a *schema*, describing the commonalities in the base and target (Glick & Holyoak, 1983; Kuehne et al., 2000).
2. *Difference Identification*. Here one explicitly represents the differences between the base and target. The most interesting differences are *alignable differences*, where there is some expression in the base and some corresponding but different expression in the target (Gentner & Markman, 1994).

Visual Perception

To perform analogy between two stimuli, one must first generate symbolic representations to describe them (Gentner, 2003, 2010; Penn, Holyoak, & Povinelli, 2008). Here we are interested not in low-level visual processing, but rather in the resulting representations and the ways they can support problem-solving. Visual representations can be characterized as hierarchical hybrid representations (HHRs). They are hierarchical (Palmer, 1977; Marr & Nishihara, 1978; Hummel & Stankiewicz, 1996) in that a given image can be represented at multiple levels of abstraction in a spatial hierarchy; for example, a rectangle could be seen as a single object or as a set of four edges. They are hybrid in that there are separate qualitative and quantitative components at each level in the hierarchy (Kosslyn et al., 1989). The qualitative, or categorical component symbolically describes relations between elements; for example, one object **contains** another, or two edges are **parallel** (Biederman, 1987; Hummel & Biederman, 2002; Forbus, Nielsen, & Faltings, 1991). The quantitative component describes concrete quantitative values for each element, e.g., its location, size, and orientation (Forbus, 1983; Kosslyn, 1996).

Qualitative representations are critical for helping us remember, reproduce, and compare spatial information (e.g., quadrants of a circle: Huttenlocher, Hedes, & Duncan: 1991; angles between object parts: Rosielle & Cooper, 2001; locations: Maki, 1982). We believe that these representations capture structural information about a visual scene, and that they can be compared via the same alignment processes used in analogy (Markman & Gentner, 1996; Sagi, Gentner, & Lovett, 2012). However, as in any analogy, the outcome of the comparison depends heavily on the representations used. In particular, one must select the appropriate level in the spatial hierarchy.

How many hierarchical levels are used in the human visual system is still an open question. Here we assume three levels for two-dimensional perception: groups, objects, and edges. Groups are sets of objects grouped together based on similarity. Objects are individual objects. Edges are the edges that make up each object. We further propose that when an individual views a scene, the highest level is available first—for example, Figure 7A contains no obvious groups, so one would initially perceive this as a row of three objects, including qualitative relations between these objects. This follows reverse hierarchy theory (Hochstein & Ahissar, 2002; see also: Love, Rouder, & Wisniewski, 1999), which claims that visual perception is a bottom-up process, beginning with low-level features, but that large-scale, high-level features are the ones initially available for conscious access; deliberate effort is required to move down the hierarchy and think about the smaller-scale details (e.g., the relations among the edges of each shape in Figure 7A).

Two clarifying points must be made about the spatial hierarchy. Firstly, it is distinct from a *relational* hierarchy, i.e., describing attributes, lower-order relations, and higher-order relations. The level in the spatial hierarchy determines the entities—edges, objects, or groups—but it does not determine whether lower-order or higher-order relations may be applied to these entities. For example, Kroger, Holyoak, and Hummel (2004) found that when comparing images of four colored squares, it was easier to compare the squares' colors directly, and more difficult to compare higher-order relations between the squares (e.g., "The top two squares are the **same** color, and the bottom two squares are a **different** color, so the relations describing the top two vs. the bottom two squares are **different**"). Here, both the attributes and the higher-order relations described the squares' colors, and thus they existed at the same level in the spatial hierarchy.

Secondly, the high-level advantage in the spatial hierarchy is not absolute. For example, Navon (1977) showed that when large letters were made up of arrangements of smaller letters, it was easier to perceive the large letter than to perceive the smaller letters. But follow-up studies showed there were many ways to disrupt this advantage (e.g., varying the absolute size of the letters: Kinchla & Wolfe, 1979; the density of the letters: LaGrasse, 1993; or the spatial frequency components of the letters: Hughes, Norzawa, & Kitterlie, 1996).

Qualitative Vocabulary. We propose that there are qualitative relations and attributes at the level of groups, objects, and edges. One key question is: What are those relations and attributes? That is, what are the visual properties that are important enough to be captured qualitatively? Some qualitative relations are obvious (e.g., one object is **right of** another, or one object **contains** another), while others are strongly supported by psychological evidence (e.g., **concave** angles between edges are highly salient: Hulleman, te Winkel, & Bosiele, 2000; Ferguson, Aminoff, & Gentner, 1996). However, there is no straightforward way to produce a complete qualitative vocabulary. Thus, our approach has been to consider the constraints of the tasks we are modeling. We have developed a qualitative vocabulary that can be used across three different tasks: geometric analogy (Lovett et al., 2009b), the visual oddity task (Lovett et al., 2011a), and RPM. It can be viewed in its entirety at (Lovett et al., 2011b). We see this vocabulary as one important outcome of the modeling work, as it provides a set of predictions about human visual cognition. At the paper's conclusion, we consider how these predictions can be further tested.

Visual Comparison

If visual perception produces a range of representations—qualitative and quantitative components at different hierarchical levels—then visual problem solving consists of a strategic

search through these representations, using comparison in order to find the key similarities and differences between two images. We view this search as proceeding top-down and from qualitative to quantitative, though again we do not claim that high-level representations are universally accessed first. At each step, the search is guided by analogical mapping, which identifies corresponding elements in the two images. Next, we summarize top-down comparison and qualitative/quantitative comparison.

Top-down comparison. Oftentimes, comparisons at a high level in the spatial hierarchy can guide comparisons at a lower level. Consider Figure 7. Each image contains three objects, and each object contains four edges. Thus, an object-level representation would consist of three entities, while an edge-level representation would consist of 12 entities. It is much simpler to compare the objects than to compare the individual edges. However, once the corresponding objects are known, the individual edges may be compared more easily.

A comparison between object-level representations, using analogical mapping, can identify the corresponding objects in the two images. In Figure 7, the leftmost trapezoid in image A goes with the leftmost trapezoid in image B because they occupy the same spot in the relational structure. Once the corresponding objects are identified, one can compare the edge-level representations for each object pair. Thus, instead of comparing two images with 12 edges each, one is comparing two objects with four edges each. These objects may be compared using the shape comparison strategy below.

Qualitative/Quantitative comparison of shapes. This strategy identifies transformations between shapes, e.g., the rotation between trapezoids in Figure 7. It is inspired by research on *mental rotation*, in which participants are shown two shapes and asked whether a rotation of one would produce the other (Shepard & Metzler, 1971; Shepard & Cooper, 1982). A popular

hypothesis is that people perform an analog rotation in their minds, transforming one object's representation to align it with the other. Our approach assumes the existence of two representations: a qualitative, orientation-invariant representation that describes the edges' locations and orientations relative to each other; and a quantitative, orientation-specific representation that describes the absolute location, orientation, size, and curvature of each edge. It works as follows (Lovett et al., 2009b):

1. Using structure-mapping, compare qualitative, orientation-invariant representations for the two shapes. This will identify corresponding parts in the shapes. For example, comparing the two leftmost shapes, the lower edge in Figure 7A goes with the leftmost edge in 7B.
2. Take one pair of corresponding parts. Compute a quantitative transformation between them. Here, there is a 90° clockwise rotation between the two edges.
3. Apply the quantitative transformation to the first shape. Here, we rotate the shape 90°. After the rotation is complete, compare the aligned quantitative representations to see if the locations, sizes, orientations, and curvatures of the corresponding edges match.

Perceptual Reorganization

One limitation of top-down comparison is that it is constrained by the initial bottom-up perception. If one perceives two edges *A* and *B* as being part of one object and some other edge *C* as being part of another, one will never consider how edge *C* relates to edges *A* and *B*. This is a problem because for some comparisons, those extra relations may be critical. For example, consider Figures 8A and 8B. Seeing the similarity between the objects inside the rectangles requires leaving the object level, decomposing them into edge-level representations, and re-grouping the edges differently. We term this process *perceptual reorganization*, as it is a

reevaluation of the *perceptual organization* that initially groups elements of a scene together (Palmer & Rock, 1994). It comes in two forms.

Basic perceptual reorganization. Consider again Figures 8A and 8B. At the object level, each image contains two elements: a rectangle and an inner object. An object-level comparison would indicate that the rectangles correspond and the inner objects correspond. Now, suppose the inner objects' shapes were compared. One might recognize that the left object is a *subshape* of the right object. That is, all the edges in the left object are also found in the right object.

Based on this finding, one could update Figure 8B's representation, segmenting the inner object into two objects (Figure 8C). Now, when the two images are compared, one can better see how the images relate to each other: the right image contains the left image, plus an additional edge.

Complex perceptual reorganization. Sometimes, the strategy above is not enough. Consider Figure 9. At the object level, the left image might contain two objects: a square and an 'X;' while the right image might contain a single object: an hourglass. Suppose the initial comparison aligned the square with the hourglass. These are completely different shapes, so at this stage, the relation between the images is poorly understood. Furthermore, because neither shape is a subshape of the other, basic perceptual reorganization is not an option.

However, one *might* recognize that the corresponding objects share some common parts: horizontal edges at the top and bottom. With this strategy, one explores these commonalities by breaking each object down into its parts. The objects are segmented into separate entities for each edge, and the comparison is repeated. Now, suppose the 'X' shape in the left image corresponds to one of the diagonals in the right image. Again, they share a common part, so the 'X' shape is broken down into its edges.

When the comparison is repeated, one gets a set of corresponding edges in the two images: the two diagonals correspond, and the top and bottom horizontal edges correspond. Because these edges are common across the two images, one can group them back together into a single object. This produces three objects in the left image: an hourglass, and the two vertical edges. It produces one image in the right object: an hourglass. Finally, the comparison is repeated, and a new understanding emerges: the left image contains the right image, plus vertical edges on its right and left sides.

Complex reorganization requires more steps than basic reorganization. It places greater demands on an individual's abstraction ability; in this example, one must move fluidly between the object and edge levels. It also places greater demands on working memory; to solve Figure 9, one must consider all the edges at once, rather than merely the edges in each object. Thus, we would expect comparisons involving complex perceptual reorganization to require significantly more effort than comparisons involving basic perceptual reorganization.

Difference Identification

Both geometric analogy and RPM problems require an individual to compare images and identify the key differences between them—in other words, how are the objects changing between images A and B in a geometric analogy problem (Figure 6B), or across a row of images in an RPM problem (Carpenter, Just, & Shell, 1990). We use the term *pattern of variance* for a symbolic, qualitative representation of the differences across two or more images. Differences may include:

a) Spatial relations being added, removed, or reversed. In Figure 6B, a **contains** relation is removed and a **rightOf** relation is added between A and B

b) Objects being added, removed, or transformed. In Figure 6B, the dot shape is removed.

Top-down comparison provides an effective means for computing patterns of variance. Analogical mapping identifies changes in the relational structure, or objects in one image that have no corresponding object in another image. Shape comparison identifies transformations, such as when an object rotates.

Patterns of variance may be reminiscent of *transformational distance* models of similarity, where two stimuli are compared by computing the transformations between them (e.g., Hahn, Chatter, & Richardson, 2003). However, they are distinct in that patterns of variance are the result of the comparison, and not the actual mechanism of comparison.

Patterns of variance in RPM

RPM is unique among the tasks we've considered in that it requires computing patterns of variance across rows of three objects. These patterns can be complex, and identifying the correct pattern type is critical to solving the problems (Carpenter, Just, & Shell, 1990). We propose that the patterns may be characterized via two binary parameters, resulting in four pattern types. Consider the example problems in Figure 3.

Contrastive/descriptive patterns. Many RPM problems involve the objects changing in some way as you look across each row of the matrix. For example, in problem A, one object moves from the left to the right side of the other object, while it rotates clockwise. Each row of problem A could be represented with a *contrastive* pattern of variance which captures these changes. On the other hand, other problems involve some set of objects appearing in each row. In problem B, each row contains a square, a circle, and a diamond, although the order in which these appears varies across rows. These rows each could be represented with a *descriptive* pattern of variance, which simply describes each image in the row.

While a descriptive pattern doesn't explicitly describes differences, it is sensitive to them. In the top row of problem B, there is an inner circle in each image. Because this circle does not change across the images of the row, it is not included in the pattern of variance. Thus, the pattern only describes the square, circle, and diamond shapes, which are the key information needed to solve the problem.

Holistic/component patterns. Patterns may also be classified as holistic or component. The examples given above are *holistic* patterns, where differences between entire images are represented. In *component* patterns, images must be broken down into their component elements, which are represented independently. For example, problem D requires a *component descriptive* pattern which essentially says: "Each row contains a circle, a square, and a diamond. *Independently*, each row also contains a group of squares, a horizontal line, and an ellipse." Which objects are paired together in the images will vary across rows, and so it can't be a part of the pattern.

A given matrix row can be represented using any of the four pattern types. Thus, to determine if one has chosen the correct type, one must compare the patterns for the top and middle rows. This can be done using the same analogical mapping process. If the patterns align, this confirms that the rows are being represented correctly. If not, it will be necessary to backtrack and represent the rows differently.

One open question is whether some pattern types are more difficult to represent than others. In particular, as the patterns become more abstract and farther from the initial concrete images, will they be processed less fluently (Kroger et al.)? Comparing contrastive and descriptive patterns, contrastive patterns appear more abstract, as they describe the differences between images, rather than the contents of each image. Comparing holistic and component

patterns, component patterns appear more abstract, as they isolate each object from the image in which it appeared. The simulation below presents an opportunities to compare these pattern types and determine their relative difficulty.

Visual Inference

Finally, a set of differences can be applied to one image to infer a new image. Consider our geometric analogy example (Figure 6B). First, images A and B are compared to compute a pattern of variance between them, as described above. Next, images A and C are compared to identify the corresponding objects. Finally, the A/B differences are applied to image C to infer D', a representation of the answer image. In this case, the dot is removed and the 'Z' shape is moved to the left of the pie shape. Note that because this inference is performed on qualitative, symbolic representations, the resulting D' is another representation, not a concrete image. Thus, the problem-solver can infer that the 'Z' shape should be on the left, but they may not know the exact, quantitative locations of the objects.

Visual inference can also be performed on the patterns of variance in RPM. Consider problem A (Figure 3). The pattern of variance for the top row indicates that one object moves to the right while rotating clockwise. By comparing the images in the top and bottom rows, one can determine that the arrow maps to the rectangle, while the circle maps to the trapezoid. Thus, one infers that in the answer image, the rectangle should move to the right of the trapezoid and rotate clockwise.

Note that the above strategy is not the only way to solve problem A. Alternatively, one might use a *second-order comparison* strategy: 1) Compare images in the top row, compute a pattern of variance. 2) For each possible answer, insert it into the bottom row and compute a pattern of variance. 3) Compare the top row pattern to each bottom row pattern, and pick the

answer that produces the closest fit. Later, we will use the model to explain why people might use a visual inference strategy or a second-order comparison strategy.

Model

Our computational model builds on the above claims, in particular: 1) analogy drives visual problem-solving, 2) hierarchical hybrid representations (HHRs) capture visual information, and 3) problem-solving is a strategic search through the representation space. If interested, the reader may download the computational model and run it on example problems A-K[2]. The model possesses two key strengths:

1) The model is not strongly tied to Raven's Matrices. It uses the Structure-Mapping Engine, a general computational model of analogy, and it incorporates operations that have been used to model other visual problem-solving tasks, including geometric analogy (Lovett & Forbus, 2012) and the visual oddity task (Lovett & Forbus, 2011). Furthermore, the visual representations used here are identical to those used in the geometric analogy model, and nearly identical to those used in the oddity task model.[3] Thus, this model allows us to test the generality of our claims.

2) The model possesses multiple strategies for solving a problem. One strategy solves the simpler, more visual problems found in the first section of the Standard Progressive Matrices (SPM) test (e.g., problem I). The other two strategies, described in the next section, capture alternative approaches for solving typical 3x3 problems. While including both strategies is not *necessary* for solving the problems, it allows us to more fully model the range of human problem-solving behavior.

---

[2] http://www.qrg.northwestern.edu/software/cogsketch/CogSketch-v2023-ravens-64bit.zip (Windows only)

[3] The only difference from the oddity task model is the inclusion of an attribute describing the relative size of an object. This attribute has been included in later versions of the oddity task model.

This section is divided up as follows: 1) We provide an overview of how the model solves problems, walking through several examples and covering the special cases where a 3x3 matrix is not used (example problems I, J, and K). 2) We summarize the existing systems on which the model is built. 3) We describe the basic operations used to model RPM and other visual problem-solving tasks. 4) We cover the strategic decisions the model must make during problem-solving. As we shall see, the difficulty of a problem depends greatly on the outcome of the strategic decisions, and thus they allow us to explore what makes a problem difficult and what makes an individual an effective problem-solver.

Figure 10 illustrates the steps the model takes to solve 3x3 matrix problems. Each problem is solved through a series of comparisons, first comparing the images in a row to generate a pattern of variance that describes how the images are changing, then comparing the patterns in the top two rows. Each of these comparisons requires a strategic search for the representation that best facilitates the comparison. Finally, the model solves a problem using one of two strategies: *visual inference* (step 5) or *second-order comparison* (step 6). Below, we walk through the steps using problems A, D, and G as examples.

Step 1. The model automatically generates a representation for each image in the problem (i.e., each cell of the matrix). The model generates the highest-level representation possible, focusing on the big picture (e.g., groupings of objects), rather than the small details (edges within each object). Each representation includes a list of objects and set of qualitative relations between the objects, as well as qualitative features for each object. The set of qualitative relations and features is known as a *structural* representation, and it allows two images to be compared via structure-mapping.

In problem A, the upper leftmost image representation would indicate that they are two objects, one **open** and one **closed**. The **closed** object is **rightOf** the open object. The **closed** object is **curved** and **symmetrical** (this is just a sampling of the representation). In problem D, the three squares in the upper leftmost image would be grouped together based on similarity. The representation would describe a **group** which is located **above** an object. In the problem G, the representation would include two objects: an ellipse and an 'X' shape inside the ellipse.

Step 2. The adjacent images in the top row are compared via top-down comparison: first the images are compared to identify corresponding objects, and then the corresponding objects are compared to identify shape changes and transformations. The full set of differences is encoded as a pattern of variance: a qualitative, symbolic representation of what is changing between the images.

In problem A, the pattern indicates that an object moves from **left of** to **within** to **right of** another object while rotating. In problem D, the pattern indicates that the objects are entirely changing their shapes between each image. In problem G, the pattern, which matches the ellipse shape to an hourglass shape to a simpler, squared-off hourglass shape, is largely unsatisfying. However, the comparison indicates that corresponding objects have some parts in common (the vertical and diagonal edges). Therefore, the model initiates complex perceptual reorganization, breaking the objects down into their component edges. Grouping the corresponding edges back up, it determines that each image contains the squared-off hourglass shape found in the rightmost image. Thus, the change is that there are first two horizontal curves, then two vertical curves, then neither, while the squared-off hourglass remains the same.

Step 3. The same steps are performed for the middle row.

Step 4. The patterns of variance for the top two rows are compared via structure-mapping. A generalized pattern consisting of the common elements to both rows is generated.

In problem A, the rows are a perfect match: they both contain an object moving to the right and rotating. Similarly, in problem G they both contain two horizontal curves, then two vertical curves, then neither. In problem D, however, there is a bad match: in the top row, there is a change from circle to square, whereas in the middle row there is a change from diamond to circle (as described below, the model does not know terms like "square," but it is able to distinguish different shapes). Thus, the model must search through the range of possible representations for a pattern of variance.

The typical representation is a *holistic contrastive* representation, which represents the changes between each image. Here, the model gets better results with a *component descriptive* representation. It is *descriptive* in that it describes what it sees, rather than describing changes. For example, in the top row, it sees a circle, so it expects to find a circle somewhere in the middle row. It is *component* in that it does not group objects together in images. The circle and the group of square are in the same image in the top row, but it does not expect to find them in the same image in the next row. Essentially, the representation says that in each row there is a circle, a square, and a diamond; and also a group, a horizontal edge, and an ellipse. Using this representation, the model finds a perfect fit between the rows.

Step 5. When possible, the model attempts to solve problems via *visual inference*. It projects the differences to the bottom row and infers the answer image. This first requires finding correspondences between the bottom row objects and objects in a row above, again using structure-mapping.

In problem A, the trapezoid and rectangle in the bottom row correspond to the circle and arrow in the top row[4] (based on corresponding relational structure). The model infers that in the missing image, the rectangle should be to the right of the trapezoid, and it should be rotated 90° from its orientation in the middle image. In problem B, the objects in the bottom row correspond to the objects in the top row. The missing objects are a circle and a horizontal edge, so the model infers that the answer should contain these objects.

In problem G, the inference fails. The model required all three images to perform complex perceptual reorganization on the above rows, suggesting that all three images will be needed for a similar perceptual reorganization on the bottom row. Without knowing the third image in the bottom row, the model abandons the visual inference strategy. Another approach must be used to solve for the answer.

Step 6. When visual inference fails, the model falls back on a simpler strategy: *second-order comparison*. It iterates over the list of answers and plugs each answer into the bottom row, computing the bottom row's pattern of variance. It selects the answer whose associated pattern best matches the generalized pattern for the above rows.

In problem G, the model selects answer 4 because when the 'X' is plugged into the bottom row, it produces a similar pattern of two horizontal curves, then two vertical curves, then neither.

The visual inference and second-order comparison approaches map onto classic strategies for analogical problem-solving. Psychologists and modelers studying geometric analogy ("A is to B as C is to…?") have argued over whether people solve directly for the answer (Sternberg, 1977; Schwering et al., 2009) or evaluate each possible answer (Evans, 1968; Mulholland,

[4] Either the top or middle row can be compared to the incomplete bottom row. The model actually uses whichever is closest to the bottom row, in terms of number of elements per image.

Pellegrino, & Glaser, 1980), with some (Bethell-Fox, Lohman, & Snow, 1984) suggesting that we adjust our strategy depending on the problem. We previously showed how a geometric analogy model incorporating both strategies could better explain human response times (Lovett & Forbus, 2012).

Here, we model both strategies not because both are required—the Carpenter model used visual inference exclusively, while our own model could use second-order comparison exclusively without a drop in performance. Rather, we hope to better explain human performance by incorporating a greater range of behavior. Following our geometric analogy model, this model attempts to solve for answers directly via visual inference. When this fails, it reverts to second-order comparison. In our analysis, we consider whether people have greater difficulty on the problems where this happens.

Solving 2x2 Matrices

The Standard Progressive Matrices (SPM) includes simpler 2x2 matrices (e.g., problem K) which lack a middle row. Because there is no way to evaluate the top row representation, the model simply assumes that a contrastive representation is appropriate, skipping steps 2 and 3 above. Problem-solving is otherwise the same.

Solving Non-Matrix Problems

In addition to visual inference and second-order comparison, the model utilizes a *texture completion* strategy for solving more basic problems which are not framed in a matrix (e.g., problem I). Because texture detection is a low-level perceptual operation that does not distinguish objects and their relations (Julesz, 1984; Nothdurft, 1991), this strategy does not rely on qualitative structure. Instead, it operates directly on the concrete image, in the form of a bitmap. It scans across the top half of the image, looking for a repeating pattern (Figure 11A). It

then scans across the bottom half, inserting each possible answer into the missing portion of the image. If one answer produces a pattern that repeats at a similar frequency, that answer is selected.

Some section A problems are more complex (problem J). If texture completion fails, the model turns the image into a 2x2 matrix. It does this by carving out three other pieces of the image (Figure 11B), such that the missing piece will be the bottom right cell in the matrix. Now, the problem can be solved in the way other 2x2 and 3x3 matrices are solved.

Existing Systems

The model builds on two pre-existing systems: the CogSketch sketch understanding system (Forbus et al., 2011), and the Structure-Mapping Engine (SME) (Falkenhainer, Forbus, & Gentner, 1989). CogSketch is used to process the input stimuli, generating visual representations for problem-solving. SME plays a ubiquitous role in the model, performing comparisons between shapes, images, and patterns of variance. The systems are briefly described below.

CogSketch

CogSketch is an open-domain sketch understanding system. It automatically encodes the qualitative spatial relations between objects in a 2-D sketch. To produce a sketch, a user can either a) draw the objects by hand; or b) import 2-D shapes from PowerPoint. Importing from PowerPoint is useful for cognitive modeling because psychological stimuli can be recreated as PowerPoint slides, if they are not in that form already.

CogSketch does not fully model visual perception. It depends on the user to manually segment a sketch into separate objects, essentially telling the system where one object stops and the next begins (as described below, the model can automatically make changes to the user's segmentation when necessary). Given this information, CogSketch computes several qualitative

spatial relations, including topology (e.g., one object **contains** another, or two objects **intersect**) and relative position. The relations are a basic model of the features a person might perceive in the sketch. In this work, we use CogSketch to simplify perception and problem-solving in three ways:

1. Each RPM problem is initially manually segmented into separate objects. Problems are recreated in PowerPoint, drawing one PowerPoint shape for each object[5], and then imported into CogSketch. The experimenters attempt to segment each image into objects consistently, based on the Gestalt grouping rules of closure and good continuation (Wertheimer, 1938), i.e., preferring closed shapes and longer straight lines. We note that the system may revise this segmentation based on its automatic grouping processes, and based upon perceptual reorganization.

2. For non-matrix problems (e.g., problems I-J), the large, upper rectangle is given the label "Problem" within CogSketch. The smaller shape within this rectangle that surrounds the missing piece is given the label "Answer." The model uses this information to locate these objects during problem-solving.

3. Each RPM problem is segmented into separate images (i.e., the cells of the matrix and the list of possible answers) using *sketch-lattices*. A sketch-lattice is an *N*x*N* grid that can be overlaid on a sketch. Each cell of the grid is treated as a separate image, for the purposes of computing image representations. Sketch-lattices can be used to locate particular images (e.g., the upper leftmost image in the top sketch-lattice). RPM problems require two sketch-lattices: one for the problem matrix, and one for the list of possible answers.

CogSketch's objects are the starting point for our model's perceptual processing. The model automatically forms higher-level representations by grouping objects together and lower-

[5] Some objects cannot be drawn as a single shape, due to PowerPoint's limitations. These are drawn as multiple shapes, imported into CogSketch, and then joined together using CogSketch's *merge* function.

level representations by segmenting objects into their edges. It supplements CogSketch's initial spatial relations to form a complete HHR for each image.

Structure-Mapping Engine (SME)

SME is a computational model of comparison based on structure-mapping theory. It operates on structured representations organized as predicate calculus statements (e.g., (**rightOf** Object-A Object-B)). Given two representations, it aligns their common structure to compute a mapping between them. A mapping consists of: 1) a set of correspondences, 2) a similarity score, and 3) candidate inferences based on carrying over unmatched structure. For this work, we used a normalized similarity score, ranging from 0 to 1. Note that SME is domain-general—it operates on visual representations as easily as more abstract conceptual representations.

One important feature in SME is the ability to specify *match constraints*, rules that affect what can be matched with what. For example, a user may specify that entities with a particular attribute can only match with other entities that share that attribute. In the visual domain, one can imagine a variety of possible match constraints (only allow objects with the same color, or size, or shape to match, etc). To support creative reasoning and comparison, our model uses no match constraints in its initial comparisons. As described below, it dynamically adds constraints when necessary as part of the problem-solving process.

Model Operations

The RPM model builds on a set of operations which have previously been used to solve other visual problem-solving tasks. Figure 12 illustrates the operations used at each step in the process (compare with Figure 10). Below, we describe each operation, including its input, its output, and any additional options available when performing the operation (e.g., the option of

representing either a contrastive or descriptive pattern of variance). In the following section, we will discuss the key strategic decisions made regarding these options at each step.

See the Appendix for additional details on how each operation is implemented.

Encode

Given an image, this produces a qualitative, structural representation at a particular level in the spatial hierarchy (Edges, Objects, or Groups). The representation contains a list of elements and a list of symbolic expressions. For example, an object-level representation would list the objects in an image and include relations like (**rightOf** Object-A Object-B). An edge-level representation would list the edges in an object and include relations like (**parallel** Edge-A Edge-B). A group-level representation would include any groups that could be formed (e.g., a row of identical shapes could be grouped together) but otherwise be identical to the object level. Note that objects may include closed shapes (e.g., a circle), open shapes (e.g., an 'X'), texture patches (a grouping of parallel edges), and negative space. See the Appendix for more details.

Additional Options. The level of the desired representation (Edges, Objects, or Groups) can be specified. Note that the RPM model, in keeping with HHRs, begins by representing at the highest level, Groups, because representations are sparser and easier to compare at this level.

Compare Shapes

This operation takes two elements and computes a transformation between them by comparing their parts—that is, it compares the edges in two objects or the objects in two groups. Like the other comparison operations (see below), it compares qualitative, structural representations using SME.

Compare Shapes can produce shape transformations, shape *deformations*, and group transformations. The shape transformations are rotations, reflections, and changes in scale. The

shape deformations are *lengthening*, where two edges along a central axis grow longer (Figure 13A); *part-lengthening*, where two edges along a part of the object grow longer (13B); *part-addition*, where a new part is added to the object (13C), and the *subshape* deformation, where two objects are identical, except that one has extra edges (used to trigger basic perceptual reorganization, as in Figure 8).

The group transformations are *identical groups*, *larger group* (similar to *subshape* where two groups are identical except that one has more objects), *different groups* (where groups have different arrangements of the same objects), and *object to group* (where a single object maps to a group of similar objects). As with *subshape*, the *larger group* and *object to group* transformations may serve as triggers for basic perceptual reorganization.

Recognize Shape

This operation assigns a shape category label to an object. While the model has no pre-existing knowledge of shape types (e.g., "square"), it can learn categories for the shapes found within a particular problem. Shapes are grouped into the same category if there is a valid transformation between them (scaling + rotation or reflection). When a new element is encountered, it is recognized by comparing it to an exemplar from each shape category. If it does not match any category, a new category is created.

Objects may be assigned arbitrary labels for their shape categories, which apply only in a particular problem context (e.g., all squares might be assigned the category "Type-1" within the context of solving a problem).

Compare Images

Given two image representations, *Compare Images* performs top-down comparison. First, it compares the image representations with SME to find the corresponding groups or

objects. Then, it calls *Compare Shapes* on those corresponding elements, in order to compute any shape transformations. It returns: a) a similarity score for the two images, computed by SME but modified based on whether corresponding elements are the same shape; b) a set of corresponding elements; c) a set of commonalities, based on those parts of the representations that aligned during the SME mapping; d) a set of differences.

There are three types of differences: 1) changes in the relational structure, found by SME, e.g., a change from (**above** Object-A Object-B) to (**rightOf** Object-A Object-B); 2) additions or removals of elements between the images (i.e., when an object is present in one image but absent in the next); 3) shape transformations between the images (see *Compare Shapes* above for a list of possible shape transformations). In some cases, an object may change shape entirely, as when a square maps to a circle. In this case, the transformation is encoded as a change between the two shapes' category labels.

Find Differences

This operation compares a sequence of images, using *Compare Images*, and produces a *pattern of variance*, a structural representation of the differences between them (see the previous section for a list of possible difference types). For example, suppose *Find Differences* was called on the top row in problem A. It would compare adjacent images (the left and middle images, and the middle and right images), identifying the corresponding objects. Here, the circle shapes correspond and the arrow shapes correspond. It would then encode the changes between adjacent images, using the differences from *Compare Images*. Between the first two images, the circle shape changes from being **right** of the arrow to **containing** the arrow. Also, the arrow rotates 90°. Between the next two images, the arrow moves to the **right** of the circle and rotates again.

Additional Options. As discussed above, much of the strategy in RPM relates to rows of images: how they should be compared, and how their differences should be represented. *Find Differences* supports several strategic choices regarding how images are compared:

1) Basic perceptual reorganization can be triggered, based on one object (or group) being a subshape of another. For the top row of problem E, the middle object is a subshape of the right object, so the right object can be segmented into objects, one identical to the middle object and one identical to the left object.

2) Complex perceptual reorganization can be triggered, based on two objects (or groups) sharing common parts. This is used in problem G.

3) *First-to-last comparison* can be performed. That is, the first and last images can be compared, even though they are not adjacent. For the top row in problem H, this allows one to see that the curved edge in the left image matches the curved edge in the right image.

4) *Strict shape matching* can be enforced for some or all shape categories. This means the SME mapping is constrained to only allow identically-shaped objects to match each other. This is paired with first-to-last comparison. In problem H, it would ensure that the curved edge in the left image doesn't map to anything in the middle image. Thus, *Find Differences* would determine that there is an object present in the first and last image, but not the middle image.

Find Differences also supports two strategic decisions for how a row's pattern of variance is represented, once it has been computed.

1) A pattern can be *contrastive* or *descriptive*. A contrastive pattern represents the differences between each adjacent pair of images, while a description pattern describes each image, abstracting out features that are common across the images. Shape labels from *Recognize*

Shape are used to capture information such as "This row contains a square, a circle, and a diamond."

2) A pattern can be *holistic* or *component*. A holistic pattern represents the row in terms of images, including what changes between images (contrastive) or what is present in each image (descriptive). Alternatively, a component pattern ignores the overall images and represents how each individual object changes (contrastive) or what objects are present in the row (descriptive).

If every image contains only a single object, then a component descriptive pattern breaks the objects down into their features and represents those separately. In problem C, each row contains a parallelogram, a circle, and a triangle; and a black object, a white object, and a gray object.

A component pattern of variance is the most abstract kind, since it abstracts out the image itself. Two things necessarily following from this: 1) ordering is not constrained, as there are no images to order; 2) spatial relations are not represented, as objects are no longer tied together in an image.

Generalize

Like *Find Differences*, this operation compares two or more items via SME. However, instead of encoding the differences, it encodes the commonalities, i.e., the attributes and relations that successfully align. It returns: a) a *similarity score*, computed by SME, and b) a new representation, containing the commonalities in the compared representation, i.e., the expressions that successfully aligned.

Infer Shape

This operation applies a shape transformation to an object to produce a novel object. It provides a way of solving geometric analogy problems ("A is to B as C is to…?") wherein we

have a transformation between A and B, and we want to apply it to C to infer D. For example, in Figure 14A we have a reflection over the x-axis between shapes A and B. Applying this to shape C produces D. Suppose instead we have Figure 14B, where this is no transformation between A and B. Normally, if there is no transformation, the operation cannot complete. However, in this case the operation exploits a feature of analogy problems: "A is to B as C is to D" is equivalent to "A is to C as B is to D" (Grudin, 1980). If there is no valid transformation between A and B, the operation checks for a transformation between A and C.

Infer Shape also makes changes to an object's color or texture. For example, suppose that in the A->B comparison, the fill color is changed, or a texture gets added or removed. The operation will similarly change the fill or texture of C to produce D.

Infer Shape also works on shape deformations (see the Appendix).

Infer Image

This operation applies a complete row's pattern of variance to an incomplete row to infer the missing image. It produces a qualitative, structural image representation, along with a list of the elements (objects and groups) in the image. This information is sufficient to support top-down comparison between the inferred image and existing images to select the best answer.

Infer Image works in two steps: 1) Compare the complete row to the incomplete row, identifying the corresponding elements. 2) Apply the differences from the complete row to the corresponding elements in the incomplete row, inferring the missing image. For example, consider problem A. The operation compares the top row to the bottom row (after computing a pattern of variance for each). The circle maps to the trapezoid and the arrow maps to the rectangle because in each case the smaller shape rotates and moves inside the larger shape. The

model takes the differences between the top row's second and third images and applies them to the bottom row, inferring that the rectangle should rotate and be to the **right** of the trapezoid.

There are two ways that *Infer Image* can fail. Firstly, it may be unable to apply a shape transformation. For example, there might be a *part removal* deformation, but the target object might lack extra parts to be removed. Secondly, the operation may find that there is insufficient information to compute the incomplete bottom row's pattern of variance. This happens on problems involving perceptual reorganization (e.g., problem G). Here, the model sees that in the top row, the first and second images were both reorganized, suggesting that information from the third image was necessary to reorganize them. Because there is no third image in the bottom row, the model does not attempt to complete the operation.

Detect Texture

This operation implements the texture completion strategy for non-matrix problems (e.g., problem I). Unlike the other operations, it does not use HHRs—instead, it prints every object in an image to a bitmap and operates directly on that bitmap. Given an image, and the location of a corridor along that image (e.g., the gray rectangles in Figure 11A), it scans along the corridor, looking for a repeating pattern.

Additional Options.

1) *Detect Texture* can be directed to insert a second image into the first image at a certain point. For example, it can insert one of the answer images into the hole (the object labeled "Answer") and evaluate how well that completes the texture.

2) *Detect Textures* can be directed to only consider textures at a particular frequency. After finding a repeating texture at a particular frequency on the top part of Figure 11A, it can be directed to seek out an answer that produces a texture at the same frequency in the bottom part.

Strategic Decisions

We now consider the strategic decisions made at each step in the problem-solving process.

1. Encoding each image

As described above, images are always encoded at the highest level possible, Groups, meaning similar objects are grouped together. This ensures a sparse, simple representation for problem-solving.

2, 3. Computing a pattern of variance for each row

If there are fewer differences between images, then patterns of variance will be more concise, and thus easier to store in memory. Thus, when computing a row's pattern of variance, the model tries to minimize differences between corresponding objects. It attempts to meet the following constraints:

A) *Identicality*: It's always best if corresponding objects are identical.

B) *Relatability:* If corresponding objects aren't identical, there should be at least some valid transformation or deformation between them.

C) *Correspondence*: Whenever possible, an object in one image should at least correspond to *something* in another.

If relatability is violated, the pattern of variance must describe a total shape change. If correspondence is violated, the pattern must describe an object addition. It is assumed that either of these makes the pattern more complex and more difficult to store in memory. Below, we refer to violations of relatability or correspondence as *bad shape matches*.

Figure 15 describes how the model pursues the above constraints. Essentially, the model calls the *Find Differences* operation repeatedly. Each call produces a pattern of variance for the

row. The model evaluates the results, and if certain conditions (specified in the figure) are met, it calls an updated *Find Differences* operation, specifying changes to the way images are represented or compared.

Step I (in Figure 15) is the initial call to *Find Differences*. Steps II and III support basic and complex perceptual reorganization. Basic perceptual reorganization is helpful for problem E, a figure addition problem. Because the second image in each row is a subshape of the third image, the third image is segmented into two objects. These objects are identical to the first and second image, allowing the model to detect figure addition.

Complex perceptual reorganization is helpful for problem G. Consider the top row. In the initial pattern of variance (step I), the corresponding objects are all different shapes. Because they contain similar edges, they are broken down into their edges (step III), with each edge treated as a separate object.

Step IV checks whether there are matching objects in the first and last images of a row. Note that *Find Differences* only performs this check when there are bad object matches in the first or last image. If there are, it compares the first and last images and checks whether this places any such bad objects into correspondence with identical objects.

In the second row of problem H, the curved edges are bad object matches—neither aligns with a similar shape in the middle image. Therefore, the first and last images are compared, and the model discovers that the curved edges match perfectly. This triggers step IV, in which the first and last images in the row are compared as part of the pattern of variance. At this step, the model further requires that curved edges can only match to other objects with identical shapes. This can be specified via an SME matching constraint. It ensures that the circle in the second

image won't match the curved edge in the first image. Thus, in the resulting pattern of variance, we find that the curved edge is present in only the first and last images.

Note that step IV also applies to problem E. Here, each row's third image contains a leftover object. When the third and first images are compared, the model discovers a perfect match to this leftover object. Thus, it finds that the third image contains both the objects found in the previous two images.

Step V implements the second half of complex perceptual reorganization: objects with the same correspondence pattern are grouped back together. In the top row of problem G, the edges forming the squared-off horizontal hourglass shape are found in all three images. Thus, these are grouped together to form a single object in each image. Similarly, in the second row, the edges forming the squared-off vertical hourglass are grouped together. Now each row contains one object that stays the same, along with the following changes: the first image has two vertical edges, the second image has two horizontal edges, and the third image has neither.

Step VI contains an additional heuristic for improving patterns of variance. If there are mismatched objects (violating relatability) and SME has found a lower-scoring, alternative image mapping that places better-matched objects into correspondence, the model switches to the alternative mapping.

Step VII finalizes the first-to-last comparison strategy. If this strategy was previously implemented in step IV, the model now requires that *every* object only match to other identical objects (i.e., objects with the same shape type). This makes the conservative assumption that if we are dealing with complex correspondences (between the first and last images), we will not also be dealing with shape transformations (where non-identical shapes correspond). While this

is the correct approach for the Standard Progressive Matrices test, future tests may challenge this assumption.

4. Comparing the patterns of variance for the top two rows

There are multiple ways to represent variance across a row of images. Carpenter et al.'s model has five rule types for describing variation. In contrast, the present model makes two strategic decisions when representing each row's pattern of variance. It evaluates these decisions by comparing the top two rows' patterns. If the patterns are highly similar, this indicates that the same differences have been detected in each row. If the patterns are *not* similar, this suggests that the rows are being represented incorrectly.

Recall that the decisions are *descriptive*/*contrastive* and *holistic*/*component*. A contrastive type represents the differences between images, while a descriptive type describes what is found in each image. A holistic type represents how images vary, while a component type represents how objects vary, independent of the image in which they are found.

The model iterates over the strategies in the following order: holistic contrastive, holistic descriptive, component descriptive, component contrastive. It evaluates each strategy by building a pattern of variance for the two rows and comparing them. It stops once a strategy is *sufficient*. If no strategy is sufficient, it picks the highest-scoring *valid* strategy. The criteria for sufficiency and validity depend on the strategy:

Holistic Contrastive: This default strategy is always valid. It is sufficient if the similarity of the two rows, as computed by SME, is above a threshold. For our simulation, we use a threshold of 0.80, where 1.0 would indicate a perfect match. However, a sensitivity analysis found that this threshold could vary from .67 to .87 and the results would be the same,

both for the simulation reported here and for our geometric analogy simulation (Lovett & Forbus, 2012).

Holistic Descriptive: This is the most concrete approach, since it represents what holds in each image, rather than differences between images. No leeway is allowed in the comparison. The strategy is valid and sufficient only if there are no differences detected between the rows.

Component: It is easy for component patterns of variance to appear similar—because spatial relations are abstracted out, only object attributes and transformations must align to produce a perfect match. Further restrictions must be applied, or component patterns will often override other, more informative patterns. Thus, they are restricted to only being valid when they produce an otherwise impossible mapping, e.g., one where two top-row objects in the same image map to two middle-row objects in different images.

5/6. Solving via visual inference or second-order comparison

After computing a generalized pattern of variance from the top two rows, the model applies the pattern to the bottom row to infer an answer image representation. First, the model builds a pattern of variance for the first two images in the bottom row. It applies any strategic shifts made on the first two rows, e.g., *component* vs. *descriptive* patterns, *holistic* vs. *component* patterns, or constraining matches to only be between identical shapes. Next, it generates the answer representation using the *Infer Image* operation (described above). *Infer Image* may fail if:

1. A shape transformation cannot be applied.

2) There is insufficient information for perceptual reorganization.

If visual inference succeeds, the model takes the inferred image representation and finds the best match among the answers. When the model compares the inferred image to a possible answer image, it can again use perceptual reorganization to improve the match.

If visual inference fails, the model reverts to second-order comparison. It iterates over the answers. For each answer, it inserts it into the bottom row and computes a pattern of variance, again using the strategic decisions from the top two rows. It compares each answer's pattern of variance to the generalized pattern from the above rows. The answer producing the most similar pattern is chosen.

Evaluation

The model was evaluated on the 60-item Standard Progressive Matrices (SPM) test. The test problems were recreated in PowerPoint and imported into the CogSketch sketch understanding system, as described above.

We also gathered human performance data, for comparison. SPM has been studied extensively in the past. However, we built a computerized version of the test, allowing access to data not typically available from paper versions, such as response times and answers test-takers considered before making their final answer.

In conducting this evaluation, our goal was to explore the representations, processes, and strategic shifts made by the model—to what extent are they sufficient for performing at a human level, and to what extent are they useful in explaining why a problem is easy or difficult for humans. For example, is a problem more difficult if it requires complex perceptual reorganization? Our goal was not to test the particular order in which the model performs operations and makes strategic decisions. We suspect humans vary a great deal in the order in which they perform operations.

For example, in studying a related geometric analogy task, Bethell-Fox, Lohman, and Snow (1984) found that higher-performing individuals preferred to solve directly for the answer (similar to our model's visual inference strategy), while others were more likely to try out each possible answer (similar to our model's second-order comparison strategy). In contrast, our model always attempts visual inference first. While our approach offers the possibility of explaining individual differences through separate models that match different population groups (Lovett & Forbus, 2011a), we view this as future work for the present domain.

Below, we describe the behavioral study. We then present the model simulation.

Behavioral Study

The SPM consists of five 12-problem sections. Section A uses 1x1 matrices, section B uses 2x2 matrices, and the remaining sections use 3x3 matrices. Only the 3x3 matrices were used in the behavioral study, as our emphasis is on explaining how people solve these more difficult problems. It is possible that participants would do worse on these problems without having first solved the easier 2x2 matrices. Therefore, participants were given two 2x2 matrices for practice.

Methods

Participants. The test was administered to 42 Northwestern University students. There were 16 males and 26 females. All students ranged from 18 to 22 years old, with a mean age of 18.8 and a median age of 18.

Materials. The experiment was run in CogSketch. CogSketch has a built-in harness to support behavioral experiments. Within CogSketch, participants viewed scanned-in images of each SPM problem, including the list of possible answers. When they clicked on an answer, a

square appeared around that answer. However, participants were still free to consider the chosen answer and potentially change their mind.[6]

Overall, the test used the 36 3x3 problems from SPM sections C-E, as well as two 2x2 problems from SPM section B: B3 and B9. The problems from section B were used for practice. We included an early B section problem and a later B section problem to provide greater range, as problems are designed to get progressively more difficult within each section.

Procedure. Participants began with a brief training period. Instructions within CogSketch described how matrix problems work and told participants that they had up to an hour to finish the problems, but that they might finish in less time. Participants solved two 2x2 matrices from section B of the SPM (B3 and B9). Participants were given feedback after each response.

Participants then answered 36 problems, sections C-E of the test, in order. Participants answered each problem by clicking on their chosen answer, causing a square to appear around it, and then hitting the "Next" button. CogSketch recorded their final answer for each question, as well as any previously selected answers. Timing information was also recorded. After each question, participants were given the option of taking a break and told to hit "Next" when they were ready to continue.

Results

Two participants were removed from the analysis because they fell more than two standard deviations below the mean. Among the remaining 40 participants, the mean performance was 30.0/36. Splitting by sections, the mean scores were:

Section C: 10.9/12

[6] This addresses a request by Raven, Raven, and Court (2000b), who asked that any computerized version of the test allow participants to view their chosen answer before committing to it.

Section D: 10.7/12

Section E: 8.5/12

An individual with these scores would typically score a perfect 12/12 on the two easier sections (Raven, Raven, & Court, 2000b, Table SPM2), resulting in an overall 54/60. This score is in the 61st percentile for American adults, according to the 1993 norms (Table SPM13).

The mean response time (considering only correct responses) was 21.1 s, and the median (by item) was 15.9 s. Across the 36 problems, there was a remarkably high correlation between mean accuracy and mean response time (r = -.89). Thus, participants were more likely to fail on problems that took longer to solve.

Discussion

The participants performed above average, according to the 1993 US norms. However, these norms are fairly old, and RPM scores are known to increase in a population over time (Raven, Raven, & Court, 2000b). Thus it may be helpful to consider more recent data. In 2000, a large-scale examination of Estonians found that 18-year-olds averaged 29.6/36 on the same sections (Lynn, Allik, & Irwing, 2000). In addition to achieving the same mean score, the Estonians showed similar error rates across the 36 problems (r = .96, comparing the percentage of correct responses on each problem).

It may be surprising that students at a major university would perform the same as average 18-year-olds. However, there are at least two reasons the Northwestern students' scores might be deflated: 1) They were taking a computerized version of the test, or 2) They were taking the test as a psychological experiment, rather than as an intelligence test. Previous studies (Williams & McCord, 2006; Arce-Ferrer & Guzmán, 2009) have suggested that results are comparable on computerized and paper versions of RPM. Therefore, we suspect the students

were less motivated because they saw the test as a psychological experiment rather than an intelligence test. The data on each problem's relative difficulty should still be valid, given the high correlation between Northwestern and Estonian error rates.

Simulation

We ran the computational model on the full, 60-problem SPM test. Each problem was recreated as accurately as possible, with two exceptions: 1) On one problem dashed lines were replaced with gray-colored lines. Presently, CogSketch lacks the perceptual ability to recognize dashed lines. 2) The final problem (E12) proved difficult to reconstruct, and it requires a strategy outside the bounds of this model (or any other model we know of): computing an arithmetic equation to relate the number of object parts in each image. Thus, it was counted as an incorrect response and left out of further testing. As described above, CogSketch requires that the user manually segment each image into separate objects, although the model can then perform grouping and segmentation operations.

Given the problems, the model automatically constructed image representations using the *Encode* operation. *Encode* generated the exact same representations as in our geometric analogy model (Lovett & Forbus, 2012) and nearly the same as in our oddity task model[7] (Lovett & Forbus, 2011a). Thus, we can evaluate the generality of our encoding scheme. Note that the initial representations produced by *Encode* were updated and modified during perceptual reorganization.

[7] As previously mentioned, the published oddity task model contained one change to its representations: it did not represent object size attributes. However, we have developed an unpublished version of the model that includes the size attributes and produces comparable results.

Results & Analysis

In analyzing the model's performance, three questions were asked: 1) How many problems can the model solve on each test section? If the model performs as well as human adults, this indicates the model's representations and processes are *sufficient* for the task. 2) Are the problems that are hard for the model also difficult for humans? If the model's error patterns match human patterns, this further suggests that the model's representations and processes are *similar* to those of humans. 3) Can the model help explain what determines a problem's difficulty? To evaluate the model's *explanation* ability, each problem was coded for whether it required certain model operations, and for its working memory load. A multiple linear regression determined whether these factors explained human performance.

Sufficiency. The model solved 56/60 problems, placing it in the 75th percentile for American adults, by the 1993 norms. This also placed it above the college-age participants, whose performance translated to a 54/60. Thus, the model appears at least as effective as the average adult.

Among the 56 solved problems, the strategy breakdown is as follows:

Texture Completion: 4

Visual Inference: 43

Second-Order Comparison: 9

The model solved most problems using visual inference. Only nine problems required reverting to second-order comparison. Note that four of these are from section A, the 1x1 matrices. Section A problems are conceptually easy, but they sometimes present perceptual problems for the model. In most cases, the model fell back on second-order comparison because of an error in its shape or image representations.

Among the 3x3 matrix problems, the model solved 28 via visual inference and four via second-order comparison. These four problems will be considered in the analysis below.

Similarity. Breaking the score down by section, the model achieved:

A: 12/12

B: 12/12

C: 12/12

D: 10/12 (failed on D11, D12)

E: 10/12 (failed on E11, E12)

The model answered all questions correctly on the easier 1x1 and 2x2 sections. It missed problems in only two sections. Within sections D and E, it missed the final two problems. RPM is designed to get progressively harder, so one would expect the final problems of each section to be the hardest.

According to the 1993 norms, the six hardest SPM problems were (from easiest to hardest): D11, C12, E10, E12, D12, E11 (Raven, Raven, & Court, 2000a, Table RS3C3). Thus, the four problems where the model failed were among the six hardest problems. According to the present behavioral experiment, the hardest SPM problems were: C12, E9, D11, D12, E10, E12, E11 (we list the seven hardest and mention C12 as it will be discussed below). Thus, the four failed problems were among the five hardest.

Problems where the model fails are among the hardest for human participants. This supports the argument that the model is using humanlike representations and processes. However, the model correctly solved some problems that are quite difficult for people (C12, E10). The next question is: can the model explain why those problems are difficult?

Explanation. We begin by discussing C12, an anomalous problem that is surprisingly difficult for people. We then present a series of linear models that converge on an explanation for problem difficulty in RPM.

Problem C12 (see Figure 16 for an analogous problem) is one of the most difficult for human test-takers. This is surprising because it is in one of the easier sections (C is the first to use 3x3 matrices), and there is nothing obviously difficult about it. The model solves it using the default strategy (a holistic contrastive pattern of variance, without comparing the first and last images). What makes this problem so challenging for humans?

van der Ven and Ellis (2000) have suggested some C problems are difficult due to a conflict between the *perceptual appearance* of the problem images and the answer images. For example, in C12, the first two rows show an object becoming longer. In the bottom row, and only in the bottom row, becoming longer causes the object to overlap another object. Thus, the answer contains a unique perceptual feature: overlapping objects (Figure 16, answer 5).

The model confirms this conflict in perceptual appearance for C12. Overall, the model solves 28 of the 3x3 matrices via visual inference. For 24 problems, the model infers an answer identical to the answer image. For the other four, the inferred answer is non-identical. On C12, the model fails to infer an **overlapping** relation between the two texture patches, so when it compares its inferred answer to the correct answer, it doesn't get a perfect match.

But one question remains. The model also fails to get a perfect match on three other problems (all from section E), all of which are much easier for humans. Why is C12 specifically so hard? Perhaps it is because there is no example of overlapping textures anywhere in the problem matrix. Unlike the three other problems, here the missing **overlapping** relation is

unique to the answer image. This may particularly cause confusion for human participants, as van der Ven and Ellis supposed.

In the further analysis, we focus on the 3x3 matrix problems. We exclude problems the model failed to solve, as well as C12 (discussed above). This leaves 31 problems: C1-C11, D1-D10, and E1-E10. Our analysis focuses on explaining two values for each problem: its difficulty, and the average response time to solve it. For difficulty, we use estimates (in logits) for each problem based on the 1993 norms. We use these values rather than difficulty estimates from our own data because it is difficult to produce reliable difficulty estimates from a sample size of 40 participants. Note, however, that the important results reported below come out the same when the difficulty measure is instead the percentage of participants in our study who correctly solved the problem.

For response time, we use the natural log of the average time required by our participants to solve each problem. Only response times for correct responses are included.

We have suggested that problem difficulty relates to correspondence-finding and representation. Problems may be more difficult if they require comparing the first and last images in a row, performing perceptual reorganization, or representing patterns of variance in a particular way. The 31 problems were coded for these factors by ablating the model: removing the model's ability to perform an operation and noting which problems it now failed to solve. In total, five factors were evaluated:

First-to-Last: Comparing the first and last images in the row.

Basic-Reorg: Basic perceptual reorganization.

Complex-Reorg: Complex perceptual reorganization.

Descriptive: Using a descriptive pattern of variance.

Component: Using a component pattern of variance.

Each factor could be 1 (indicating that a problem required this operation) or 0. The final two factors determine which pattern of variance types can be encoded. We presupposed that holistic contrastive pattern might be the most natural (e.g., example problem A), and thus tested whether problems requiring component patterns or descriptive patterns were more difficult. Note that the *descriptive* pattern of variance is actually more concrete than the contrastive pattern, as it simply describes what is true in each image. In contrast, the *component* pattern of variance is more abstract, as it breaks an image down into its individual objects.

Participant accuracy was modeled via a multiple linear regression on these factors. Table 1 shows the linear model. For each factor, this table indicates the factor's predicted contribution to item difficulty (B), the factor's unique contribution to the model's predictive power (ΔR), and the statistical significance of the factor's contribution. Overall, the model explains .48 of the variance in human performance. Two factors contribute significantly to the model: First-to-Last and Complex-Reorg. Thus, it appears that problems were more difficult when participants had to compare the first and last images in a row or perform complex perceptual reorganization.

This analysis found a far greater cost for complex reorganization than for basic reorganization. This confirms the model's prediction that complex reorganization is more cognitively demanding. Consider problems E and G, which require basic and complex perceptual reorganization, respectively. The finding that problems like G are far more difficult suggests that comparison is indeed tied to our initial division of a scene into objects, and that real effort is required to reorganize the objects in one's representation.

The analysis did not find any significant costs based on the pattern of variance type. However, there was a trend towards component patterns being more difficult. This may indicate

there is a cost for using more abstract patterns. On the other hand, descriptive patterns were easier, although this was far from significant.

Descriptive patterns of variance are required for most of section D, the section involving Carpenter's distribution-of-three rule. Overall, these problems are quite easy: participants averaged over .90 on all but the last two problems (D11, D12), the ones where the model failed. In contrast, problems involving contrastive patterns of variance (sections C and E) vary far more in their difficulty.

One possible reason for the increased variability is working memory load. Perhaps representing abstract differences is more cognitively demanding than representing what is true in each image, and thus working memory load is more of a factor for contrastive patterns than for descriptive patterns. This would match the results in geometric analogy (Lovett & Forbus, 2012), where working memory load mattered for the abstract differences more than for the individual images.

To test this hypothesis, Descriptive was replaced with a new factor: Diff-Elements. This factor describes the number of elements (objects or groups) in the model's generalized pattern of variance for the top two rows of a problem. Because working memory effects can be non-linear (Mulholland, Pellegrino, & Glaser, 1980), we follow our geometric analogy analysis in discounting the first two elements and only counting the number of elements beyond two. Furthermore, the number is set to zero any time a descriptive pattern of variance is used. Thus, this is working memory load for contrastive patterns of variance only.

Table 2 shows the resulting linear model. This is a much stronger model: it accounts for .63 of the variance in human performance. Furthermore, all factors except Basic-Reorg now contribute significantly. This suggests that working memory is a factor when one considers

abstract differences between images. It also indicates that there is indeed a cost for using a more abstract *component* pattern of variance.

We now consider the model's final strategic shift: reverting from visual inference to second-order comparison. Recall that the model makes this shift on four of the 31 problems. In one case (C6), it is unable to apply a shape deformation during visual inference. The other three cases involve perceptual reorganization, either basic (C11) or complex (E8, E9). In each case, the model cannot infer the answer because it is unsure how to organize the objects in the bottom row's first two images. Interestingly, while C11 requires only basic reorganization, the accuracy on this problem was lower than on other problems requiring basic reorganization.

A third linear model was built, replacing Basic-Reorg and Complex-Reorg with Disruptive-Reorg. This refers to any reorganization that disrupts visual inference, forcing one to revert to second-order comparison. Table 3 shows the results. This model accounts for .67 of the variance in human performance. Thus, it performs better than the above model with one less factor.

Response Times. It is important to ask whether the present model can also explain human response times on the SPM. As explained above, it is not our goal to capture the exact sequence of operations performed by problem-solvers. However, given the high correlation between mean accuracy and mean response time on the SPM problems (r = -.89), we would expect the above factors to explain much of the variance in response times as well.

Table 4 lists two linear models built to explain response times on the problems (compare to Tables 2 and 3 for problem difficulty). The results were quite similar to the difficulty results. There was only one notable difference: Complex-Reorg, despite having a high cost to response time, was only a marginally significant contributor. This may be due to the fact that only a small

number (2/31) of the problems required complex reorganization. Note that Disruptive-Reorg, which refers to any reorganization that results in a strategy shift to second-order comparison, was a significant contributor in the right table.

Failures. It is also useful to consider the four problems on which the model failed: D11, D12, E11, and E12. Problems D11 and D12 can be solved using a descriptive pattern of variance if one represents the appropriate features for each image. However, they depend on complex, often quantitative features such as the number of edges in an object, the number of objects in a group, and whether the objects in a group contain straight or curved edges. Figure 17 displays an analog for problem D12. Here, each row contains a group of squares, a group of triangles, and a group of "L" shapes; and each row has a group of two, a group of three, and a group of four. The correct answer is 8, a group of four curved triangles.

According to hierarchical hybrid representations, people rely on abstract qualitative features when possible. These problems require abandoning the default representation and performing an exhaustive search through the range of possible features in an image. This feature search can be seen as another form of re-representation. However, it is outside the bounds of the model and also quite difficult for the human test-takers.

Problem E11 can be solved via complex perceptual reorganization. However, the model fails to recognize the clues that perceptual reorganization is an option. Thus, it correctly predicts that this will be an especially difficult problem for human participants. On the other hand, problem E12 cannot be solved via any strategy used by the model (nor can other computational models solve it, to our knowledge). It requires counting the number of external loops in each shape, and subtracting the number of internal loops. This bizarrely unique strategy explains why E12 is so difficult for human participants.

While the model effectively predicts the problems on which participants will fail, it does not predict the incorrect answers participants give on these problems. To do so, the model would need to incorporate the guessing strategies employed by participants when they simply don't know what's going on, e.g., copying over an adjacent cell of the matrix (Vodegel Matzen, van der Molen, & Dudink, 1994). In many cases, these guessing strategies may give rise to a divergence of responses. For example, on problems D11 and D12, there were four different answers that were each selected by at least 10% of participants in our study. On problem E11, there were five.

Discussion

We may draw several conclusions from the analysis above. Firstly, there is a cost for abstract representations. When participants had to represent *differences* between images, rather than what held true in each image, they became slower and less accurate as the number of objects increased. Participants were also less effective when they had to represent objects separately from their images (component patterns of variance). This cost might also involve working memory load, as it may be more difficult to remember each object when they aren't tied together in images.

Secondly, perceptual reorganization presents a significant challenge. Problems are harder when participants must reorganize the objects in each image, particularly when complex reorganization is required. Given the current results, there are at least two points when a cost may be incurred: a) When participants compare the images in each of the top two rows, they must reorganize their representations to determine what is changing between the images. b) When participants attempt to solve for the answer in the bottom row, they may have difficulty performing a reorganization without knowing the final image, forcing them instead to consider

each possible answer. We suspect that both of these are sources of difficulty. Note that in the analysis, Disruptive-Reorg, which referred to a basic or complex reorganization that forced the model to consider each possible answer, was a particularly effective predictor of problem difficulty and response time.

Finally, the analysis shows a consistent cost for the first-to-last image comparison strategy shift. Recall that this involves: a) comparing the first and last images in the row, and b) constraining mappings to only align identically-shaped objects. This result suggests there is a cost when object correspondences become more complex.

Related Work

Re-Representation

Hofstadter's Fluid Analogies Research Group shares our interest in perception and re-representation, and their critical roles in analogical reasoning (Hofstadter, 1995). However, whereas we see these as separate processes that can occur in sequence, FARG models interleave perception and mapping by stochastically choosing at each timestep from a wide range of possible micro-operations. For example, Copycat (Mitchell, 1993) solves letter string analogies such as "**abc** is to **abd** as **iijjkk** is to…?". In solving this problem, Copycat would tend to group the i's, j's, and k's together based on similarity, at the same time as it was computing a mapping between **abc** and **iijjkk**. The answer chosen would vary depending on the timing and success of the grouping operations.

The research at FARG was pioneering in its focus on perception and re-representation. However, the work is limited due to 1) the focus on microdomains like letter string analogies; 2) the highly stochastic nature of the models, with some of the most interesting solutions being reached only rarely; 3) the lack of psychological support for the models. Though the group

argued that perception and analogical mapping are necessarily interleaved, we believe the present work shows that perception, mapping, and re-representation can work in sequence to solve large-scale problems.

Raven's Matrices

Hunt (1974) proposed two models for solving a subset of the Advanced Progressive Matrices (APM). While the first was a concrete Gestalt model, the second was an abstract analytic model. The analytic model had a pre-existing set of operations (e.g., constancy, expansion, addition, composition). Given a row or column (and assuming it knew the representations and correspondences), it would check whether each operation described the changes between images. While Hunt's model was never fully implemented, the model's operations and strategies match several of the rules Carpenter et al. would later define.

Carpenter, Just, and Shell (1990) presented the first automated computational models. As described above, their models incrementally identified instances of five hard-coded rule categories. Their FAIRAVEN model performed as well as typical college student on the APM, while their BETTERAVEN model performed as well as the best students. Their models were limited in that they used hand-coded input representations, they possessed a simplified correspondence-finding strategy, and they did not perform re-representation.

Note that the Carpenter models tested for five types of rules (e.g., Quantitative Pairwise Progression), whereas the present model uses four types of patterns of variance (e.g., holistic contrastive). Nonetheless, the present model can solve problems involving all five Carpenter rules, as well as some problems (e.g., problem G) that don't appear to match any Carpenter rule.

Rasmussen and Eliasmith (2011) presented a spiking neuron model designed to solve the APM. The model relied on the user to hand-code both the symbolic image representations and

the object correspondences between images. Given the appropriate representations, it learned rules describing how objects vary. No results were presented for the model, and it was unclear how it would handle problems like problem B (where the present model uses descriptive patterns of variance, and Carpenter et al.'s model uses a distribution of three rule).

Cirillo and Ström's (2010) model for solving SPM shared several features with the present model. Representations were automatically generated from vector graphical descriptions (similar to sketches, but more abstract) and organized hierarchically. The model moved down the hierarchy, going from the most concrete to the most abstract as needed during problem-solving. However, the model had two limitations: 1) It did not perform shape comparison; rather, it recognized five shape types (e.g., ellipse, rectangle) and encoded each object's rotation from the upright position. Thus, it could not handle complex shapes and shape deformations. 2) Rather than building up differences from image comparisons, the model was entirely top-down: it possessed seven pattern-matching strategies, each with its own rules about correspondence-finding (most searched exhaustively over all combinations). Overall, the model solved 28/36 problems from SPM sections C, D, and E, suffering particularly in section C, which has more perceptual shape-based problems.

Kunda and colleagues (Kunda, McGreggor, & Goel, 2013; McGreggor, Kunda, & Goel, 2014) have developed two related models that operate directly on scanned images of the problems, requiring no symbolic representations. These models compare images by computing mathematical transformations between their pixels. The *affine* model computes a single transformation between images, composing operations such rotation and scaling. It is effective for problems involve shape transformations or figure additions, but it fails on descriptive pattern

of variance problems (e.g., example problem B), a problem type that is relatively easy for human test-takers.

In contrast, the *fractal* model computes many transformations between partitions of each image. Using a second-order comparison approach, it performs impressively well on the APM, matching the 75th percentile for American adults. Surprisingly, it performs less well on the SPM, achieving 50/60 (compared to the present model's 56/60), which is the 41st percentile. It is unclear what the model's performance can tell us about visual comparison and problem-solving in humans. Its approach of representing images at the pixel level and performing exhaustive searches for optimal transformations does not appear to be consistent with what we know about human perception and reasoning.

Kunda and colleagues view their approach as complementary to the more symbolic, propositional approaches described above. This distinction is similar to the qualitative/quantitative distinction that motivated the present model, with the Kunda models falling on the quantitative side. We believe our approach is important in that it uses both qualitative representations for comparison and reasoning and quantitative information for computing shape transformations.

Future Work

There are several important directions for future work on modeling visual problem-solving.

Solving the Advanced Progressing Matrices

The present model solves problems from the Standard Progressive Matrices test (SPM), whereas some previous models have been built for the Advanced Progressive Matrices test (APM) (Carpenter, Just, & Shell, 1990; Kunda, McGreggor, & Goel, 2012). We chose to focus

on the SPM for two reasons: 1) The SPM has a good difficulty level for capturing typical human intelligence, as the average adult can solve most SPM problems (Raven, Raven, & Court, 2000b). 2) The APM presents challenges not addressed by the current theory and model.

In particular, many APM problems require performing a search through a space of non-obvious visual features—in contrast, only a few SPM problems require such a feature search (e.g., problem D12, see Figure 17 for an analog). Feature search is a form of re-representation, but it is less straightforward than the perceptual reorganization performed by the present model. To model it effectively, we believe it is necessary to build a *visual routines* system in which basic operations like curve-tracing, scanning, and counting can be flexibly combined to encode different visual features (Ullman, 1984). Notably, none of the current models which solve APM problems possess such a system—either they use hand-coded inputs (Carpenter, Just, & Shell, 1990), or they use no symbolic representations at all (Kunda, McGreggor, & Goel, 2012).

Learning Patterns of Variance

The present model has four pattern of variance types built into it, based on the contrastive/descriptive and holistic/component distinctions. In contrast, humans may learn ways to represent differences as they are taking the test. We believe it would be immensely valuable to model this learning process. However, the great majority of psychological studies have focused on individual problem difficulties, and not how problem-solving strategies are progressively learned across problems. We believe more human data is required before a learning process may be captured effectively in a computational model.

Modeling Working Memory Limitations

The present work indicates that working memory may be an important factor in determining problem difficulty, but the model has no built-in working memory limitations.

Currently, we believe it is an open question how working memory interacts with analogical thinking. For example, must relational structure be stored entirely in working memory during mapping? According to the long-term working memory theory (Ericsson & Kintsch, 1995), structured information may be offloaded to long-term memory and called up only as needed during reasoning. Note that other analogical reasoning models have addressed the memory issue directly by incorporating working memory constraints (Hummel & Holyoak, 1997; Doumas, Hummel, & Sandhofer, 2008).

Modeling Individual Differences

This work aims to model a typical problem-solving strategy, used by the average test-taker. However, individuals will vary in the strategies they employ. In a previous study (Lovett & Forbus, 2011a), we modeled cultural differences by showing that ablating certain operations in the model caused it to behave more like one group or the other. With RPM, we would like to study individual differences in problem-solving strategy. We have already mentioned that test-takers might vary in using visual inference or second-order comparison, but they might also vary in focusing on rows or columns, in their ability to perform shape transformations (Janssen & Geiser, 2010), or in their ability to use small-scale vs. large-scale representations in the spatial hierarchy (Lovett & Forbus, 2011a).

Unlike the case of cultural differences, with individual differences there are no a priori groups into which to divide participants. Therefore, the limited set of problems in the SPM may not provide sufficient statistical power for this analysis. By designing new problems which, according to the model, require particular operations and giving these to test-takers, we may gain new insights into how people vary in their visual problem-solving abilities.

Evaluating the Qualitative Vocabulary

This and other modeling work has led to the development of a qualitative vocabulary, a list of qualitative attributes and relations which capture the 2D visual features we claim are most salient to people (Forbus & Lovett, 2011b). It is important that this vocabulary be tested on humans. We propose to do so using a same/different paradigm, where participants compare two images and respond "same" if they are identical, or "different" if they spot any difference (Farell, 1985). A common finding with this paradigm is that participants spot differences more easily if they cross a qualitative or categorical boundary—e.g., it is easier to distinguish blue from green than to distinguish two shades of blue (Bornstein & Korda, 1984). While this effect, termed *categorical perception* has been demonstrated extensively for object features like color, there is far less evidence regarding qualitative relations *between* objects (but see Kim & Biederman, 2012). We are currently testing for categorical perception of topological relations: **touching**, **overlapping**, and **contains**.

Conclusion

We have argued that visual problem-solving requires analogical thinking. Visual scenes are represented symbolically and compared via structural alignment. Representations are iteratively updated and refined, as one searches through the space of hierarchical hybrid representations (HHRs) for the key commonalities and differences. Importantly, re-representation is directly driven by the comparison process, rather than any top-down search through a space of possible representations. The commonalities or differences identified during comparison can be reified and used in further reasoning, including future comparisons.

These claims are supported by our computational model. The model, which implements HHRs and structure-mapping, meets three criteria: 1) Sufficiency: The model performs as well as

American adults. 2) Similarity: Problems that are difficult for the model are also difficult for people. 3) Explanation: The model can help explain what makes one problem harder than another.

Overall, the computational model stands out above previous problem-solving models for several reasons: 1) It uses the Structure-Mapping Engine, an existing model of analogy, to perform all comparisons. Thus, it demonstrates that structure-mapping can play a ubiquitous role in the problem-solving process, not only for identifying commonalities and differences, but also for computing shape transformations and determining when re-representation is necessary. 2) It builds on a set of representations and processes that have been used to model several other problem-solving tasks. Thus, it allows us to test the generality of the problem-solving approach, rather than tying our claims purely to Raven's Matrices. 3) It supports ablation analysis, in which components of the model are broken and the resulting error patterns are examined. Thus, beyond supporting the initial claims, the model can generate hypotheses about visual problem-solving in humans.

The ablation analysis suggests three key factors underlying effective performance on difficult RPM problems:

1. One needs a high working memory capacity. Problems were more difficult when the model represented contrastive differences across more objects.
2. One must be able to think abstractly about concrete stimuli. Problems were more difficult when they required a component pattern of variance, where one abstracts away the images and thinks about each object in isolation.
3. One must be able to flexibly re-represent stimuli to support comparisons. Problems were more difficult when they required complex perceptual reorganization.

Because we have linked visual problem-solving to analogical reasoning, we can generalize these factors to intelligent thought more broadly. Working memory is already a well-established factor which affects problem-solving performance on Raven's (Carpenter, Just, & Shell, 1990; Unsworth & Engle, 2005) and in other tasks (e.g., Johnstone & El-Banna, 1986; Andersson, 2007; Engle, 1994).

Abstraction plays an important role in analogical retrieval and inference—when we view a new problem or scenario, we can retrieve an analogous example more effectively if the example was represented and stored in a more abstract way. Education researchers often find that when students are taught an abstract version of a concept, they are better able to transfer it to new stimuli, or even to a new domain (e.g., Kaminski, Sloutsky, & Heckler, 2008; Goldstone & Son, 2005). If a student can form her own abstractions from concrete examples, she should be better able to learn and apply new concepts.

Re-representation is critical because analogies are slaves to their symbolic representations. If two cases happen to be represented with different relational structure, they will fail to align, and the only way to complete the analogy will be to change the structure. Importantly, re-representation depends on the other two factors because a) working memory is needed to keep multiple representations in mind, and b) abstraction is an effective form of re-representation. However, it may also depends on a certain fluidity—a willingness to abandon one's particular point of view and search for a new perspective.

Interestingly, one can turn the above claims around and say that effective re-representation strategies increase one's apparent working memory capacity. This is because if one can discover more concise ways to represent a problem, an image, or even a single shape, one frees up cognitive resources. Taking an anecdote from the Carpenter, Just, & Shell (1990)

Raven's paper, participants who saw a row as containing an upward-facing arrow, then a rightward-facing arrow, then a downward facing arrow tended to be less effective problem-solvers than participants who saw a row as containing an arrow that was rotating clockwise. Participants with the second, more concise representation would need to remember fewer details while solving a problem. Similarly, Taatgen (2014) has argued that effective representational strategies could allow participants to perform better even on tests designed specifically to test working memory capacity.

We began this paper with the claim that analogy is the cornerstone of human intelligence. Perhaps representational flexibility is a shining tower, a key feature separating great thinkers from good thinkers, allowing them to draw insightful connections, solve complex problems, and make significant discoveries.

References

Andersson, U. (2007). The contribution of working memory to children's mathematical word problem solving. *Applied Cognitive Psychology, 21*(9), 1201-1216.

Arce-Ferrer, A. J., & Guzmán E. M. (2009). Studying the equivalence of computer-delivered and paper-based administrations of the Raven Standard Progressive Matrices Test. *Educational and Psychological Measurement, 69*, 855-867.

Bethell-Fox, C. E., Lohman, D. F., & Snow, R. E. (1984). Adaptive reasoning: Componential and eye movement analysis of geometric analogy performance. *Intelligence, 8*, 205-238.

Biederman, I. (1987). Recognition-by-components: A theory of human image understanding. *Psychological Review, 94*, 115-147.

Bornstein, M. H., & Korda, N. O. (1984). Discrimination and matching within and between hues measured by reaction times: Some implications for categorical perception and levels of information processing. *Psychological Research, 46*, 207-222.

Burke, H. R., & Bingham, W. C. (1969). Raven's Progressive Matrices: More on construct validity. *Journal of Psychology: Interdisciplinary and Applied, 72*(2), 247-251.

Carpenter, P. A., Just, M. A., & Shell, P. (1990). What one intelligence test measures: A theoretical account of the processing in the Raven Progressive Matrices test. *Psychological Review, 97*(3), 404-431.

Cattell, R. B. (1963). Theory of fluid and crystallized intelligence: A critical experiment. *Journal of Educational Psychology, 54*, 1-22.

Cirillo, S., & Ström, V. (2010). *An anthropomorphic solver for Raven's Progressive Matrices* (No. 2010:096). Goteborg, Sweden: Chalmers University of Technology.

Dehaene, S., Izard, V., Pica, P., & Spelke, E. (2006). Core knowledge of geometry in an Amazonian indigene group. *Science, 311*, 381-384.

Doumas, L. A. A. & Hummel, J. E. (2013). Comparison and mapping facilitate relation discovery and predication. *PLOS One, 8 (6),* e63889. doi:10.1371/ journal.pone.0063889.

Doumas, L. A. A., Hummel, J. E., & Sandhofer, C. M. (2008). A theory of the discovery and predication of relational concepts. *Psychological Review, 115*, 1-43.

Engle, R. W. (1994). Memory. In R. Sternberg (Ed.), *Encyclopedia of Intelligence* (pp. 700-704). New York: MacMillan.

Ericsson, K. A., & Kintsch, W. (1995). Long-term working memory. *Psychological Review, 102*(2), 211-245.

Falkenhainer, B., Forbus, K., & Gentner, D. (1989). The structure mapping engine: Algorithm and examples. *Artificial Intelligence, 41*, 1-63.

Farell, B. (1985). "Same"-"different" judgments: A review of current controversies in perceptual comparisons. *Psychological Bulletin, 98*(3), 419-456.

Ferguson, R. W., Aminoff, A., & Gentner, D. (1996). Modeling qualitative differences in symmetry judgments. *Proceedings of the 18th Annual Meeting of the Cognitive Science Society*.

Forbus, K. (2001). Exploring analogy in the large. In D. Gentner, K. Holyoak, & B. Kokinov, (Eds.) *Analogy: Perspectives from Cognitive Science*. Cambridge, MA: MIT Press.

Forbus, K. (1983). Qualitative reasoning about space and motion. In D. Gentner & A. Stevens (Eds.), *Mental Models*. West Orange, NJ: LEA Associates, Inc.

Forbus, K., Nielsen, P., & Faltings, B. (1991). Qualitative spatial reasoning: The CLOCK project. *Artificial Intelligence, 51*(1-3), 417-471.

Forbus, K., Usher, J., Lovett, A., Lockwood, K., & Wetzel, J. (2011). CogSketch: Sketch understanding for cognitive science research and for education. *Topics in Cognitive Science, 3*(4), 648-666.

Gentner, D. (2010). Bootstrapping the mind: Analogical processes and symbol systems. *Cognitive Science, 34*(5), 752-775.

Gentner, D. (2003). Why we're so smart. In D. Gentner and S. Goldin-Meadow (Eds.), *Language in Mind: Advances in the Study of Language and Thought* (pp.195-235). Cambridge, MA: MIT Press.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science, 7*, 155-170.

Gentner, D., Brem, S., Ferguson, R. W., Markman, A. B., Levidow, B. B., Wolff, P., & Forbus, K. (1997). Analogical reasoning and conceptual change: A case study of Johannes Kepler. *The Journal of the Learning Sciences, 6*(1), 3-40.

Gentner, D., & Markman, A. B. (1994). Structural alignment in comparison: No difference without similarity. *Psychological Science*, *5*(3), 152-158.

Gentner, D., & Markman, A. B. (1997). Structure mapping in analogy and similarity. *American Psychologist, 52*, 45-56.

Gentner, D., & Smith, L. A. (2013). Analogical learning and reasoning. In D. Reisberg (Ed.), *The Oxford Handbook of Cognitive Psychology* (pp. 668-681). New York, NY: Oxford University Press.

Glick, M. L., & Holyoak, K. J. (1983). Schema induction and analogical transfer. *Cognitive Psychology, 15*, 1-38.

Goldstone, R. L., & Medin, D. L. (1994). Time course of comparison. *Journal of Experimental Psychology: Learning, Memory and Cognition, 20*(1), 29-50.

Goldstone, R. L., Medin, D. L., & Gentner, D. (1991). Relational similarity and the non-independence of features in similarity judgments. *Cognitive Psychology, 23,* 222-264.

Goldstone, R.L., & Son, J.Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *Journal of the Learning Sciences, 14*, 69-114.

Grudin, J. (1980). Processes in verbal analogy solution. *Journal of Experimental Psychology: Human Perception and Performance, 6*(1), 67-74.

Hahn, U., Chater, N., & Richardson, L. B. (2003). Similarity as transformation. *Cognition, 87*, 1-32.

Hobeika, L., Diard-Detoeuf, C., Garcin, B., Levy, R., & Volle, E. (2016). General and specialized brain correlates for analogical reasoning: A meta-analysis of functional imaging studies. *Human Brain Mapping, 37*(5), 1953-1969.

Hochstein, S., & Ahissar, M. (2002). View from the top: Hierarchies and reverse hierarchies in the visual system. *Neuron, 36*, 791-804.

Hoffman, D. D., & Richards, W. A. (1984). Parts of recognition. *Cognition, 8*(1–3), 65–96.

Hofstadter, D., & Sander, E. (2013). *Surfaces and Essences: Analogy as the Fuel and Fire of Thinking*. New York, NY: Basic Books.

Hughes, H. C., Norzawa, G., & Kitterle, F. (1996). Global precedence, spatial frequency channels, and the statistics of natural images. *Journal of Cognitive Neuroscience, 8*, 187-230.

Hulleman, J., te Winkel, W., & Boselie, F. (2000). Concavities as basic features in visual search: Evidence from search asymmetries. *Perception & Psychophysics, 62*(1), 162-174.

Hummel, J. E., & Biederman, I. (1992). Dynamic binding in a neural network for shape recognition. *Psychological Review, 99*, 480-517.

Hummel, J. E., & Holyoak, K. J. (1997). Distributed representations of structure: A theory of analogical access and mapping. *Psychological Review, 104,* 427-466.

Hummel, J. E., & Stankiewicz, B. J. (1996). An architecture for rapid, hierarchical structural description. In T. Inui and J. McClelland (Eds.), *Attention and Performance XVI: Information Integration in Perception and Communication* (pp. 93-121). Cambridge, MA: MIT Press.

Huttenlocher, J., Hedges, L. V., & Duncan, S. (1991). Categories and particulars: Prototype effects in estimating spatial location. *Psychological Review, 98*(3), 352-376.

Janssen, A. B., & Geiser, C. (2010). On the relationship between solution strategies in two mental rotation tasks. *Learning and Individual Differences, 20*(5), 473-478.

Johnstone, A. H., & El-Banna, H. (1986). Capacities, demands and processes – A predictive model for science education. *Education in Chemistry, 23*, 80-84.

Julesz, B. (1984). A brief outline of the texton theory of human vision. *Trends in Neurosciences, 7*(2), 41-45.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The advantage of abstract examples in learning math. *Science*, *320*(5875), 454.

Kim, J. G., & Biederman, I. (2012). Greater sensitivity to nonaccidental than metric changes in the relations between simple shapes in the lateral occipital cortex. *NeuroImage, 63*, 1818-1826.

Kinchla, R. A., & Wolfe, J. M. (1979). The order of visual processing: "Top-down," "bottom-up," or "middle-out." *Perception and Psychophysics, 25*, 225-231.

Kokinov, B., & French, R. M. (2003). Computational models of analogy-making. In L. Nadel (Ed.), *Encyclopedia of Cognitive Science, Vol. 1* (pp.113 - 118). London: Nature Publishing Group.

Kosslyn, S. M. (1996). *Image and Brain: The Resolution of the Imagery Debate*. Cambridge, MA: MIT Press.

Kosslyn, S. M., Koenig, O., Barrett, A., Cave, C. B., Tang, J., & Gabrieli, J. D. E. (1989). Evidence for two types of spatial representations: hemispheric specialization for categorical and coordinate relations. *Journal of Experimental Psychology: Human Perception and Performance, 15*, 723-735.

Kroger, J. K., Holyoak, K. J., & Hummel, J. E. (2004). Varieties of sameness: The impact of relational complexity on perceptual comparisons. *Cognitive Science, 28*, 335-358.

Kubricht, J., Lu, H., & Holyoak, K. J. (2015). Animation facilitates source understanding and spontaneous analogical transfer. *Proceedings of the 37th Annual Conference of the Cognitive Science Society.*

Kuehne, S., Forbus, K., Gentner, D. and Quinn, B. (2000). SEQL: Category learning as progressive abstraction using structure mapping. *Proceedings of the 22nd Annual Meeting of the Cognitive Science Society.*

Kunda, M., McGreggor, K., & Goel, A. K. (2013). A computational model for solving problems from the Raven's Progressive Matrices intelligence test using iconic visual representations. *Cognitive Systems Research, 22-23*, 47-66.

LaGasse, L. (1993). Effects of good form and spatial frequency on global precedence. *Perception and Psychophysics, 53*, 89-105.

Larkey, L., & Love, B. (2003). CAB: Connectionist Analogy Builder. *Cognitive Science,* 27*,* 781-794.

Love, B. C., Rouder, J. N., Wisniewski, E. J. (1999). A structural account of global and local processing. *Cognitive Psychology, 38*, 291-316.

Lovett, A., & Forbus, K. (2012). Modeling multiple strategies for solving geometric analogy problems. *Proceedings of the 33rd Annual Meeting of the Cognitive Science Society.*

Lovett, A., & Forbus, K. (2011a). Cultural commonalities and differences in spatial problem-solving: A computational analysis. *Cognition, 121*(2), 281-287.

Lovett, A. & Forbus, K. (2011b). Organizing and representing space for visual problem-solving. *Proceedings of the 25th International Workshop on Qualitative Reasoning*.

Lovett, A., Gentner, D., Forbus, K., & Sagi, E. (2009a). Using analogical mapping to simulate time-course phenomena in perceptual similarity. *Cognitive Systems Research 10*(3): Special Issue on Analogies - Integrating Cognitive Abilities, 216-228.

Lovett, A., Tomai, E., Forbus, K., & Usher, J. (2009b). Solving geometric analogy problems through two-stage analogical mapping. *Cognitive Science 33*(7), 1192-1231.

Lynn, R., Allik, J., & Irwing, P. (2004). Sex differences on three factors identified in Raven's Standard Progressive Matrices. *Intelligence, 32*, 411-424.

Maki, R. H. (1982). Why do categorization effects occur in comparative judgment tasks? *Memory & Cognition, 10*(3), 252-264.

Markman, A. B., & Gentner, D. (1993). Structural alignment during similarity comparisons. *Cognitive Psychology, 25*, 431-467.

Markman, A. B., & Gentner, D. (1996). Commonalities and differences in similarity comparisons. *Memory & Cognition, 24*(2), 235-249.

Marr, D., & Nishihara, H. K. (1978). Representation and recognition of the spatial organization of three-dimensional shapes. *Philosophical Transactions of the Royal Society of London, Series B, Biological Science, 200*(1140), 269-294.

McGreggor, K., Kunda, M., & Goel, A. K. (2014). Fractals and Ravens. *Artificial Intelligence, 215*, 1-23.

McLure, M. D., Friedman, S. E., Lovett, A., & Forbus, K. D. (2011). Edge-cycles: A qualitative sketch representation to support recognition. *Proceedings of the 25th International Workshop on Qualitative Reasoning*.

Medin, D. L., Goldstone, R. L., & Gentner, D. (1993). Respects for similarity. *Psychology Review, 100*(2), 254-178.

Mulholland, T. M., Pellegrino, J. W., & Glaser, R. (1980). Components of geometric analogy solution. *Cognitive Psychology, 12*, 252-284.

Navon, D. (1977). Forest before trees: The precedence of global features in visual perception. *Cognitive Psychology, 9*(3), 353–383.

Nothdurft, H. C. (1991). Texture segmentation and pop-out from orientation contrast. *Vision Research, 31*(6), 1073-1078.

Oppenheim, A. V., & Schafer, R. W. (2009). *Discrete-Time Signal Processing, Third Edition*. Upper Saddle River, NJ: Prentice Hall.

Palmer, S. E. (1977). Hierarchical structure in perceptual representation. *Cognitive Psychology, 9*, 441-474.

Palmer, S., & Rock, I. (1994). Rethinking perceptual organization: The role of uniform connectedness. *Psychonomic Bulletin & Review, 1*(1), 29-55.

Penn, D. C., Holyoak, K. J., & Povinelli, D. J. (2008). Darwin's mistake: Explaining the discontinuity between human and nonhuman minds. *Behavioral and Brain Sciences, 31*(2): 109-130.

Primi, R. (2001). Complexity of geometric inductive reasoning tasks: Contribution to the understanding of fluid intelligence. *Intelligence, 30*, 41-70.

Rasmussen, D., & Eliasmith, C. (2011). A neural model of rule generation in inductive reasoning. *Topics in Cognitive Science, 3*(1), 140-153.

Raven, J., Raven, J. C., & Court, J. H. (2000a). *Manual for Raven's Progressive Matrices and Vocabulary Scales: Research Supplement 3. Neuropsychological Applications*. Oxford: Oxford Psychologists Press.

Raven, J., Raven, J. C., & Court, J. H. (2000b). *Manual for Raven's Progressive Matrices and Vocabulary Scales: Section 3. Standard Progressive Matrices.* Oxford: Oxford Psychologists Press.

Raven, J., Raven, J. C., & Court, J. H. (1998). *Manual for Raven's Progressive Matrices and Vocabulary Scales: Section 1. General Overview.* Oxford: Oxford Psychologists Press.

Rosielle, L. J., & Cooper, E. E. (2001). Categorical perception of relative orientation in visual object recognition. *Memory & Cognition, 29*(1), 68-82.

Sagi, E., Gentner, D., & Lovett, A. (2012). What difference reveals about similarity. *Cognitive Science, 36*(6), 1019-1050.

Schwering, A., Gust, H., Kühnberger, K., & Krumnack, U. (2009). Solving geometric proportional analogies with the analogy model HDTP. *Proceedings of the 31st Annual Meeting of the Cognitive Science Society.*

Shepard, R. N., & Cooper, L. A. (1982). *Mental Images and Their Transformations*. Cambridge, MA: MIT Press.

Shepard, R. N., & Metzler, J. (1971). Mental rotation of three-dimensional objects. *Science, 171*, 701-703.

Snow, R. E., Kyllonen, P. C., & Marshalek, B. (1984). The topography of learning and ability correlations. In R. J. Sternberg (Ed.), *Advances in the Psychology of Human Intelligence* (vol. 2, pp. 47–103). Hillsdale, NJ: Lawrence Erlbaum Associates.

Snow, R.E., & Lohman, D.F. (1989). Implications of cognitive psychology for educational measurement. In R. Linn (Ed.), *Educational Measurement, 3rd ed.* (pp. 263–331). New York, NY: Macmillan.

Sternberg, R. J. (1977). *Intelligence, Information Processing, and Analogical Reasoning*. Hillsdale, NJ: Erlbaum.

Stevens, K. A. (1981). The information content of texture gradients. *Biological Cybernetics, 42*, 95-105.

Taatgen, N. (2014). Between architecture and model: Strategies for cognitive control. *Biologically Inspired Cognitive Architectures, 8*, 132-139.

Ullman, S. (1984). Visual routines. *Cognition, 18*, 97-159.

Unsworth, N., & Engle, R. W. (2005). Working memory capacity and fluid abilities: Examining the correlation between operation span and Raven. *Intelligence, 33*(1), 67-81.

van der Ven, A. H. G. S., Ellis, J. L. (2000). A Rasch analysis of Raven's standard progressive matrices. *Peronsality and Individual Differences, 29*, 45-64.

Vendetti, M., Wu, A., & Holyoak, K. J. (2014). Far out thinking: Generating solutions to distant analogies promotes relational thinking. *Psychological Science*, *25*, 928-933.

Vodegel Matzen, L. B. L., van der Molen, M. W., & Dudink, A. C. M. (1994). Error analysis of Raven test performance. *Personality and Individual Differences, 16*(3), 433-445.

Wertheimer, M. (1938). Laws of organization in perceptual forms. In W. D. Ellis (Ed.), *A Source Book of Gestalt Psychology*. London: Kegan Paul, Trench, Trubner & Co.

Williams, J. E., & McCord, D. M. (2006). Equivalence of standard and computerized versions of the Raven Progressive Matrices Test. *Computers in Human Behavior, 22*, 791-800.

Yan, J., Forbus, K., & Gentner, D. (2003). A theory of rerepresentation in analogical matching. *Proceedings of the 25th Annual Meeting of the Cognitive Science Society*.

Zagar, R., Arbit, J. & Friedland, J. (1980). Structure of a psychodiagnostic test battery for children. *Journal of Clinical Psychology, 36*, 313 - 318.

Appendix A

Model Operations

Encode

Encode supports representations at three levels: Edges, Objects, and Groups. *Encode* begins with the objects, since these are provided by CogSketch. It can segment one object into edges, or join several objects into a group. At each level, a set of qualitative attributes and relations describe the elements.

Note that our goal is not to model the details of how the encoding process works; encoding likely does not begin at the object level (but see Palmer & Rock, 1994). Our goal is to model the resulting representation. The qualitative attributes and relations in our representations were selected based on a) psychological evidence; and b) the constraints imposed by RPM and other problem-solving tasks. *Encode* produces representations that work consistently across tasks. Our geometric analogy model (Lovett & Forbus, 2012) uses identical representations to the present model, while the oddity task model's (Lovett & Forbus, 2011a) representations differ in one minor respect—they lack attributes for an object's relative size.

Below, we describe representations at the Edge, Object, and Group levels. We briefly summarize the qualitative features encoded at each level. For the full set of qualitative terms, and more details on interactions between the levels, see Lovett and Forbus (2011b).

Edges

Given an object made of one or more ink strokes, *Encode* identifies an object's meaningful edges by joining together ink strokes with common endpoints and segmenting ink strokes at corners. Corners are detected as discontinuities in curvature, or locations where three or more edges meet. See (Lovett et al., 2009b) for details on the edge-segmentation algorithm.

This produces a list of edges, and a list of junctions where the edges meet. For example, a square would have four edges with junctions between them.

Encode's edge-level representation describes the edges along an object's contour (Hoffman & Richards, 1984), i.e., the object's shape. Internal edges are not represented at this level. For example, the house in Figure 18 would be represented as five edges. This representation scheme is effective for comparing two-dimensional shapes, but further detail is required to represent three-dimensional objects (Biederman, 1987; McLure et al., 2011).

If an object is a closed shape, the representation describes the cycle of edges going around that shape. It indicates adjacent corners along that cycle and classifies them as **convex** or **concave**. It also describes **parallel** and **perpendicular** edges. Note that because these features are orientation-invariant, two shapes at different orientations will have essentially the same representation. Thus, the model can determine that Figures 18B and 18C are the same shape, and it can compute transformations between them (see below). (Representations for *open* shapes are similar but contain fewer details.)

Optionally, *Encode* can produce an orientation-specific representation, describing the relative locations of an object's edges. This more concrete representation is less useful for computing shape transformations, but it does provide greater detail. It is used by the model when computing shape deformations (see *Shape Comparison* below).

Objects

Objects are provided directly by CogSketch, so *Encode* does not need to identify them. However, it does possess some basic grouping abilities. If several parallel lines have been drawn as separate objects, it will group them together to form a texture patch (Figure 19A). This texture patch can be associated with some other object that forms its contour, e.g., the square in

Figure 19B. The square in Figure 19C contains two overlapping texture patches. In addition, if an object contains one or more other objects such that they create a single, convex piece of negative space, this negative space will represented as a separate object, e.g., a square containing a black triangle creates the perception of a complementary white triangle in Figure 19D.

The object-level representation includes attributes for color, texture, closure, curvedness, symmetry, and size (based on the overall distribution of sizes in the problem). There are relations for relative position, topology, and shape transformations (i.e., rotations, reflections, and changes in scale between objects in an image—see the next section on *Compare Shapes*).

Groups

Encode groups objects together based on the Gestalt grouping rules of similarity, proximity, and good continuation (Wertheimer, 1938). For several objects to be grouped together, they must be of similar size and shape (based on their object-level attributes); they must be equidistant from each other; and they must be aligned with the grouping's orientation. In Figures 20A and 20C, the circles would be joined into a single group. In 20B there would be two groups, based on proximity. In 20D the triangles would be grouped, but in 20E they would not because they don't align with the grouping's orientation. In 20F, the image's topology breaks the circles into two groups.

The group-level representation describes any groups, as well as any individual objects that failed to group. For example, Figure 20F would contain two groups and one individual object: the rectangle. In many cases a group-level representation will contain no groups at all, and so it will be identical to the object level. The qualitative relations at the group level are similar to those at the object level.

Compare Shapes

Shape transformations are computed using the shape comparison strategy described previously: 1) Compare edge-level representations with SME, get corresponding edges. 2) Take one pair of corresponding edges and compute a quantitative transformation between them, consisting of a rotation or reflection, a size change, and a translation. 3) Apply this transformation to the *quantitative* edges in the first object, checking whether the transformed edges match those in the second object, in terms of their orientation, length, location, and curvedness. Note that a rotation of approximately 0°, combined with a scaling factor of about 1.0, indicates that the objects are identical.

For shape transformations, there is a single quantitative transformation that transforms all edges in the first shape into their corresponding edges in the second shape. For shape *deformations*, however, the model computes separate transformations for each pair of corresponding edges, as follows: For *lengthening* (Figure 13A), two edges about a central axis become longer. Other edges change (translating or becoming longer or shorter) to accommodate the changes in the lengths of the two central edges. For *part-lengthening* (Figure 13B), two edges along the bounds of one part become longer (parts are defined by concave corners, Hoffman & Richards, 1984). Again, other edges change to accommodate. For *part-addition* (Figure 13C), new edges are added to the shape to produce an additional part (again, defined by concave corners), and other edges change to accommodate. Finally, for *subshape* (Figure 8), there is a perfect identity match between the corresponding edges in two shapes, but one shape has additional edges.

Within the current model, the subshape deformation is never encoded as a difference between two images (e.g., when representing a row of differences with *Find Differences*). It is only used to guide basic perceptual reorganization.

For group transformations, the two groups' object-level representations are compared to identity corresponding objects. Then, one pair of corresponding objects are compared to check if they are the same shape. Group transformations are only encoded for groups whose objects are the same shape (e.g., two groups of circles).

For groups, rotations, reflections, and deformations are not considered. The only possibilities are: 1) There is a perfect identity match between corresponding objects within the groups (indicates *identical groups*, a *larger group*, or *object to group*, depending on the number of objects in each group. 2) There is *not* an identity match between corresponding objects (indicates a *different groups* transformation).

Recognize Shape

Shape categories are stored in a *shape registry*. Each category is stored as an arbitrary label (e.g., *Type-1*), a prototype (simply the first example encountered), and a list of all known exemplars. When a new object is encountered, it is compared to the most similarly-sized exemplar in each category. If a valid shape transformations is found, the object is added to that category. If no match is found, a new shape category is created.

An object's shape may be referenced in either of two ways: 1) Using the shape's category label, or 2) Using the category label *and* the transformation from the prototype. For an example of this, see the next section.

Compare Images

This operation compares two images using SME to find the corresponding objects or groups. Then it compares these using *Compare Shapes*. It returns: a) a similarity score for the two images; b) a set of corresponding elements; c) a set of commonalities; and d) a set of differences.

For example, suppose the images in Figure 21 were compared (the letters are included as object labels, and are not part of the images). The initial SME mapping would align A with D, B with E, and C with F, based on each object's shape attributes, and on the spatial relations between objects. The model would call *Compare Shapes* on each pair, determining that there is a 90° rotation between A and D, that B and E are identical, and that E and F are entirely different shapes. It computes its output as follows:

Similarity score. The similarity score is computed by SME. The initial mapping's score is likely to be heavily influenced by attributes, since *Encode* produces a large set of shape attributes. To simplify the score, the model updates the image representations, removing all shape attributes and replacing them with a single attribute for each object, referring to its exact shape (this is distinct from the category labels described above, as here the two shapes must be identical to share the same attribute). In Figure 21, B and E will share the same shape attribute, while other pairs will have different attributes. The model then reruns SME and takes SME's similarity score. Thus, the similarity score describes whether the spatial relations and the shapes are the same in the two images.

Corresponding elements. This is the list of corresponding elements in the images.

Commonalities. This is a list of symbolic expressions which SME aligned during the mapping. In Figure 21, one commonality is that C/F is **right** of B/E.

Differences. The differences consist of: 1) Changes in the spatial relations, found by SME. In Figure 21A, B is **right** of A, but in 21B, E is **below** D. 2) Additions or removals of elements between images. 3) Transformations and shape changes. Transformations are represented qualitatively. For example, rotations are rounded to the nearest 90°, so there is a qualitative **RotatatedShape-90** attribute for the A/D pair.

Shape changes (such as the C/F pair) are trickier to represent. Ideally, one would represent "a square changes to a circle." However, the model has no foreknowledge of the shapes it will encounter, so it cannot have names for them all. Instead, it uses the category labels computed by *Recognize Shape*. The model can account for transformations from the category prototypes. For example, suppose the category prototypes for square and circle shapes were established based on the shapes in Figure 21. The square and circle shapes in Figure 22 are smaller instances of those categories. Because they are both smaller instances, there is no perceived change in size from 22A to 22B, and the model would represent the shape change as merely a change between the two shape categories.

Note that the model similarly recognizes textures so that it can encode changes to an object's texture. Because texture patches are groupings of parallel lines, a change in texture means the orientation or thickness of the lines has changed.

Resolving ambiguity. Sometimes the initial comparison finds multiple equally good mappings between two images. The model employs two strategies for resolving ambiguity to pick the preferred mapping. Firstly, consider Figures 23A and 23B. Based purely on object-level shape features, the object in 23A maps equally well to either object in 23B. In this case, the model performs shape comparisons between the 23A object and the two 23B objects, selecting the mapping that produces the best shape match.

Secondly, consider Figures 23C and 23D. Here, all shapes match equally well, and the mapping truly is ambiguous. The model imposes a left->right and top->bottom ordering in such cases. Here, the circle in 23C maps to the leftmost circle in 23D. We do not claim this is a better mapping than the alternative. It simply ensures the model is consistent.

Find Differences

The details on this operation are fully given in the main text.

Generalize

This operation computes a generalization over two or more representations. This works as follows (Kuehne et al., 2000): 1) Compare the first two items with SME. Abstract out any expressions that don't align, building a new structural representation with the commonalities in these two items. 2) For each additional item, compare it to the generalization, pruning off any expressions that don't match this item. This produces a single structural representation describing what is common to all the items.

The operation also returns a similarity score, based on the similarity SME finds between the items. If more than two items were compared, it returns the lowest similarity score from among the comparisons.

Infer Shape

Given a shape transformation between two object ('A' and 'B'), this operation applies that transformation to another object ('C') to generate a novel object ('D'). Recall that shape transformations include rotation, reflections, and changes and scale. These transformations can be easily applied to the edges in object C to produce the edges for the new object D (transforming the location, orientation, length, and curvedness of the edges).

Inferring shape *deformations* is more difficult because there is a separate quantitative transformation between each pair of edges in the A/B deformation. Thus, the corresponding edges must be identified in object C because the transformations can be applied to them. For example, Figure 24A shows a deformation in which a part is added between A and B. Here, the leftmost edge in shape A grows twice as long, becoming an edge of the newly added part. To infer a deformation, *Infer-Shapes* first compares shape A to shape C. It finds a reflection over the x-axis between them. It then applies the appropriate quantitative transformation to each edge; here, again, the leftmost edge grows to twice as long. It adds the new part after applying any appropriate A->C transformations; in this case, it reflects the part over the x-axis before adding it. The resulting shape is D, which has a part added at the top, instead of a part added at the bottom.

Sometimes, there is no valid mapping between the edges of A and C. For example, in Figure 24B, there is no mapping between the rectangle (A), and the double-headed arrow (C). In this case, the operation must make its best guess at the appropriate edges for the deformation. In Figure 24B, the operation lengthens the horizontal edges, while in Figure 24C, the operation lengthens the left part.

Infer Image

This operation applies a complete row's pattern of variance to an incomplete row, to infer what the final image in the row should be. For a contrastive pattern of variance, the differences come in three forms: additions/removals of spatial relations, shape transformations, and additions/removals of objects. The operation uses all of these to infer the missing image. It adds or removes relations to the corresponding elements in the incomplete bottom row. For example, on the bottom row of problem A, it removes a **contains** relation and adds a **rightOf** relation,

inferring that the rectangle should now be right of the trapezoid. Next, it takes the shape transformations and applies them to the corresponding elements in the bottom row, using *Infer Shape*. Finally, it removes any objects that should be removed. The operation can also handle additions of new objects. However, in RPM, it is never necessary to infer the shape of an object that exists only in the missing image.

For a descriptive pattern of variance, the differences are not explicitly encoded. Instead, the pattern encodes what is found in each image or for each object. For example, problem D is solved using a component descriptive pattern of variance, where each image has been broken up into objects. In this case, the operation can simply compare the top and bottom rows, identify which top-image elements were not found in the bottom row, and place them in the final, missing image. Here, the bottom row lacks a horizontal line and a circle, so these are selected for the missing image. Note that because component patterns do not encode spatial relations between objects, the operation will not generate the spatial relationship between the line and the circle. Essentially, it determines only, "There should be a line and a circle in the missing image."

Detect Texture

This operation detects textures using a simplified representation. Given an image, it prints all the objects in the image to a bitmap—a 2x2 array of points which makes no distinction between one object and another. It then scans along a corridor of the bitmap (Figure 11) to see if there is a repeating texture.

Detect-Texture uses autocorrelation (Oppenheim & Schafer, 2009), a common approach from signal processing, to detect repeating patterns in a corridor. The idea is to divide the signal into windows and check whether consecutive windows correlate with each other. One varies the

window length to find the best correlation. Here, each window is a portion of the corridor's total length.

Detect Texture computes two signals from a window: rows and columns. For rows, it takes each row of the window and adds up the pixels that are "on," i.e., those that are not empty space. Similarly, it takes each column of the window and adds up the pixels that are "on." In computing the correlation between two windows, *Detect Texture* takes the lower of the correlation between rows and the correlation between columns.

A row's overall score is the minimum correlation value for all adjacent windows. If this score exceeds a threshold (0.85), the operation is a success. It returns the score and the window size that produced this score.

Autocorrelation is a useful tool for detecting repeating patterns in an image. However, we view this as only an approximation of human texture detection—accurately modeling low-level texture detection is not a priority for this work. For a more in-depth model of texture processing, see (Stevens, 1981).

Table 1

Linear Model for Participant Accuracy

(R^2 = .48, F(5,25) = 4.69, p = .004)

Factor	B	ΔR^2	*t* value
Intercept	-1.09		
First-to-Last	1.80	.16	2.83*
Basic-Reorg	0.69	.03	1.19
Complex-Reorg	2.68	.20	3.15*
Descriptive	-0.46	.01	0.74
Component	1.03	.05	1.61

* p < .05

Table 2

Linear Model for Participant Accuracy, Using Number of Elements

(R^2 = .63, F(5,25) = 8.62, p < .001)

Factor	**B**	**ΔR^2**	***t* value**
Intercept	-1.49		
First-to-Last	1.63	.14	3.08*
Basic-Reorg	0.40	.01	0.82
Complex-Reorg	1.79	.08	2.32*
Diff-Elements	0.52	.16	3.30*
Component	1.04	.07	2.17*

* p < .05

Table 3

Linear Model for Participant Accuracy, Using Disruptive Reorganization (R^2 = .67, F(4,26) = 13.19, p < .001)

Factor	**B**	**ΔR^2**	***t* value**
Intercept	-1.49		
First-to-Last	1.74	.16	3.54*
Disruptive-Reorg	1.83	.12	3.05*
Diff-Elements	0.49	.15	3.43*
Component	1.11	.08	2.49*

* p < .05

Table 4

Linear Models for Participant Response Times (R^2 = .73, F(5,25) = 13.35, p < .001)

Factor	B	ΔR^2	*t* value
Intercept	2.35		
First-to-Last	0.48	.10	3.03*
Basic-Reorg	0.12	.01	0.83
Complex-Reorg	0.46	.04	1.96ǂ
Diff-Elements	0.24	.27	5.00*
Component	0.47	.12	3.28*

(R^2 = .73, F(4,26) = 17.42, p < .001)

Factor	B	ΔR^2	*t* value
Intercept	2.36		
First-to-Last	0.50	.11	3.21*
Disruptive-Reorg	0.40	.05	2.08*
Diff-Elements	0.24	.28	5.21*
Component	0.49	.13	3.46*

* p < .05 ǂ p < .07

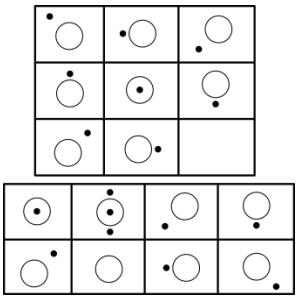


Figure 1. Raven's Matrix problem. To protect the security of the test, all the problems presented here were constructed by the authors. Many are analogous to actual problems.

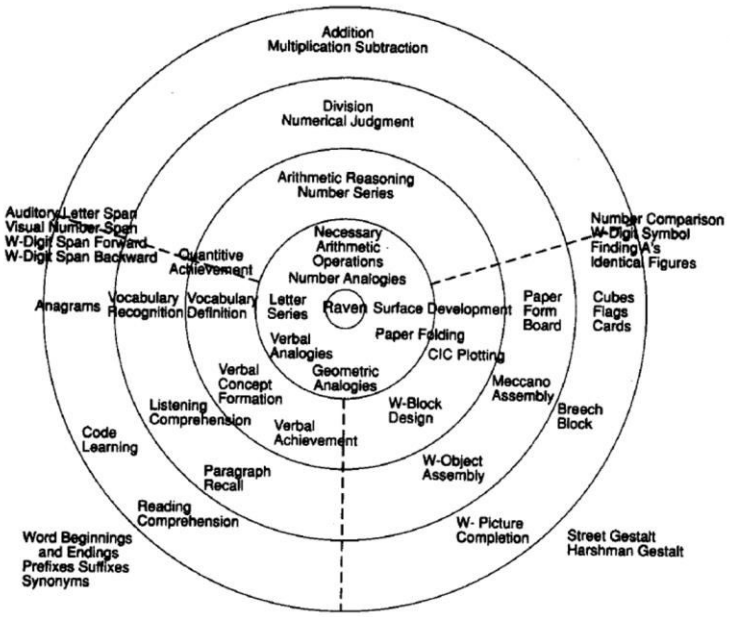
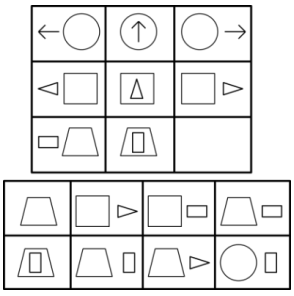
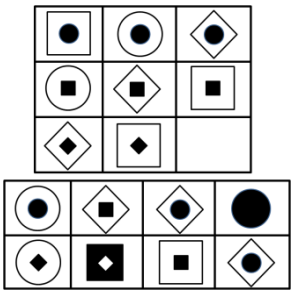
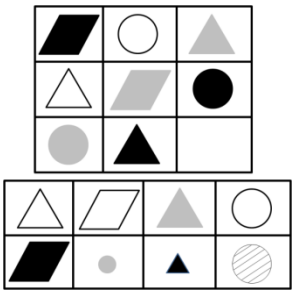


Figure 2. Scaling analysis of ability tests based on intercorrelations. Reprinted from Snow, Kyllonen, & Marshalek, 1984.

	A	B	C
			
Carpenter Rules	Quantitative Pairwise Progression, Constant in a Row	Distribution of Three, Constant in a Row	Distribution of Three (applies twice)
Our Model's Strategy	Holistic Contrastive	Holistic Descriptive	Component Descriptive
Answer	4	5	2

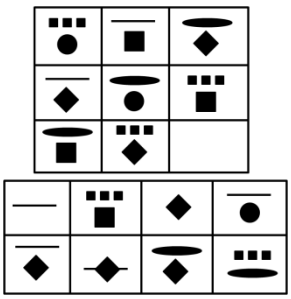
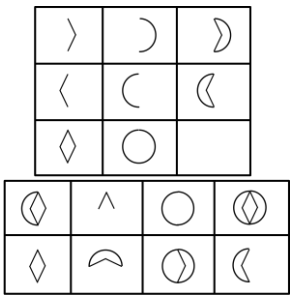
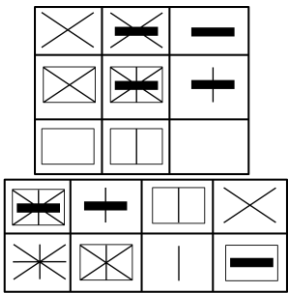
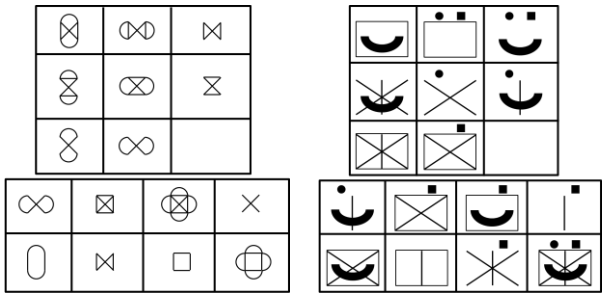
	D	E	F
			
Carpenter Rules	Distribution of Three (applies twice)	Figure Addition	Distribution of Two
Our Model's Strategy	Component Descriptive	Holistic Contrastive	Holistic Contrastive
Abstraction Operation	None	Basic Perceptual Reorganization	Basic Perceptual Reorganization
Answer	4	4	7

Figure 3. Example problems for various Carpenter et al. (1990) rules, and the strategy and abstraction operation our model would use to solve each problem.

G **H**



Carpenter Rules	???	Distribution of Two (applies twice)
Our Model's Strategy	Holistic Contrastive	Component Contrastive
Abstraction Operation	Complex Perceptual Reorganization	Basic Perceptual Reorganization
Answer	4	4

Figure 4. Two particularly difficult Raven's problems.

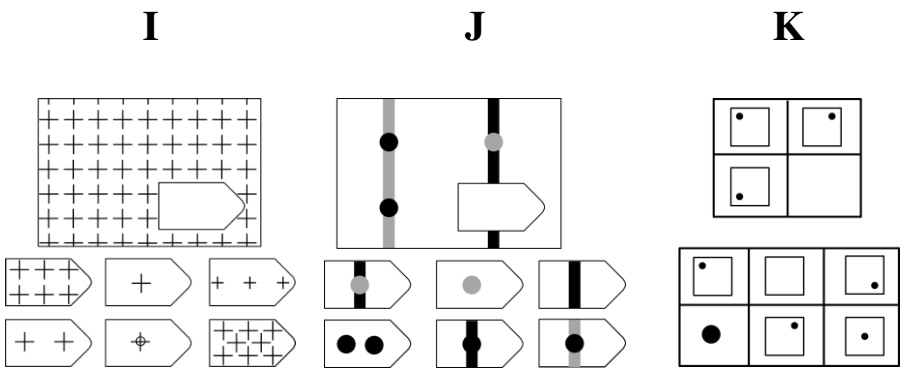


Figure 5. Three simpler Raven's problems.

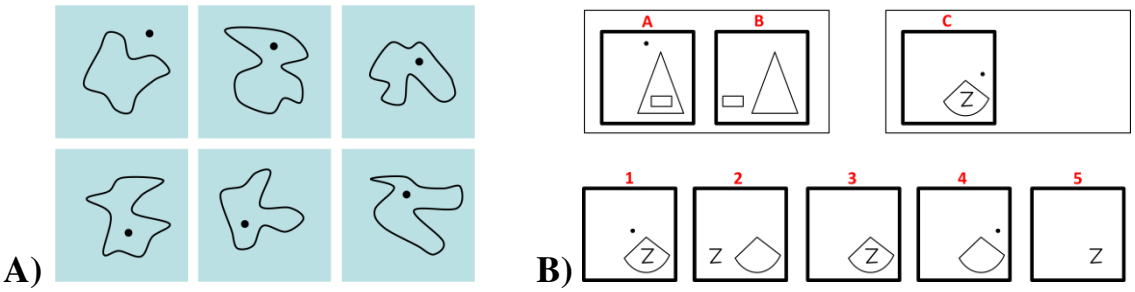


Figure 6. A) Oddity task problem from Dehaene et al. (2006). Pick the image that doesn't belong. B) Geometric analogy problem from Lovett et al. (2009b).

Figure 7. Images that might be compared structurally.

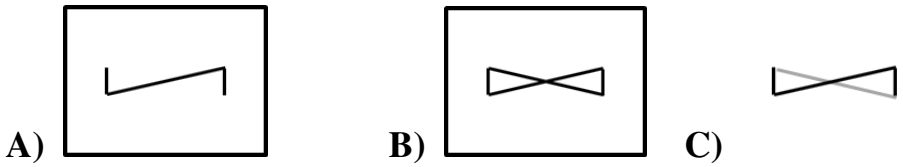


Figure 8. Basic perceptual reorganization facilitates this comparison.

Figure 9. Complex perceptual reorganization facilitates this comparison.

1) **All Rows:** Encode a visual representation for each image.
2) **Top Row**: Compare adjacent images, find corresponding objects, generate a *pattern of variance* to describe how these objects change.
3) **Middle Row**: Compare adjacent images, generate a pattern of variance.
4) **Top & Middle Row**: Compare two patterns to produce a generalized pattern.
5) **Bottom Row**: Project differences from rows above to corresponding objects in the bottom row to infer the answer image.

OR

6) **Bottom Row, Answers**: Plug each answer into the bottom row to generate a pattern of variance, and compare this to the generalized pattern.

Figure 10. Overview of the Raven's Matrices model.

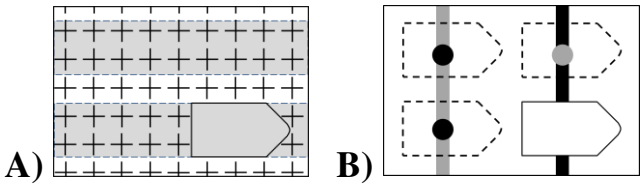


Figure 11. Strategies for solving problems I and J.

1) **All Rows:** *Encode*

2) **Top Row**: *Find Differences*
   - *Compare Images*
     - *Compare Shapes*
   - *Recognize Shape*
     - *Compare Shapes*

3) **Middle Row**: *Find Differences*
   - *Compare Images*
     - *Compare Shapes*
   - *Recognize Shape*
     - *Compare Shapes*

4) **Top & Middle Row**: *Generalize*

5) **Bottom Row**: *Infer Image*
   - *Infer Shape*

OR

6) **Bottom Row, Answers**: *Find Differences*
   - *Compare Images*
     - *Compare Shapes*
   - *Recognize Shape*
     - *Compare Shapes*

Figure 12. Overview of the Raven's Matrices model with operations.

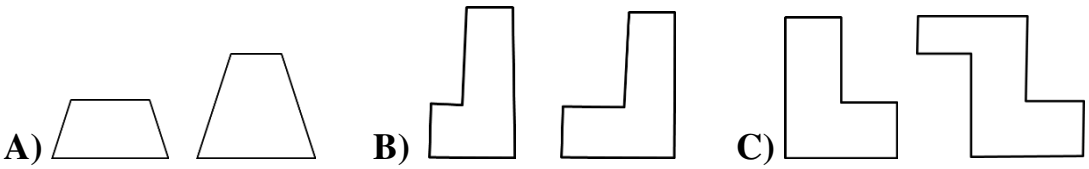


Figure 13. Examples of shape deformations.

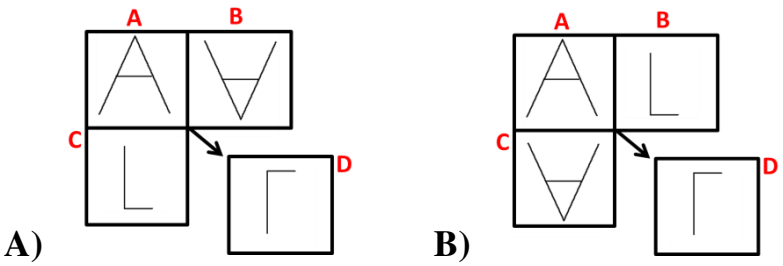


Figure 14. Examples of *Infer Shape* with reflections.

	Test	Directives to *Find-Differences*	Supported Strategy
I)	*None*	*None*	
II)	**If** one shape is a subshape of another	Break down subshapes	*Basic perceptual reorganization*
III)	**While** there are mismatched shapes with common parts	Break shapes down into parts	*Complex perceptual reorganization*
IV)	**If** shapes in the first and last image match	Compare the first and last images	*First-to-last comparison*
V)	**If** corresponding parts can be grouped back together into shapes	Groups parts together into shapes	*Complex perceptual reorganization*
VI)	**If** alternate image mappings give better shape matches	Use alternate image mappings	*Enforcing shape relatability*
VII)	**If** the first and last images are being compared	Require strict shape matches	*First-to-last comparison*

Figure 15. Steps for computing a row's pattern of variance (each step is a call to *Find Differences*).

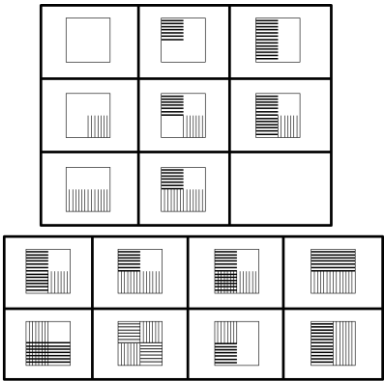


Figure 16. Analog for SPM problem C12.

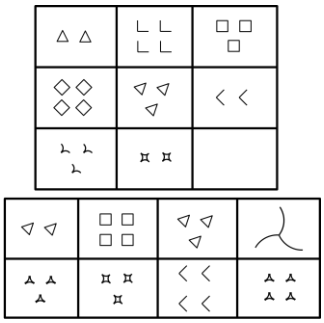


Figure 17. Analog for SPM problem D12.

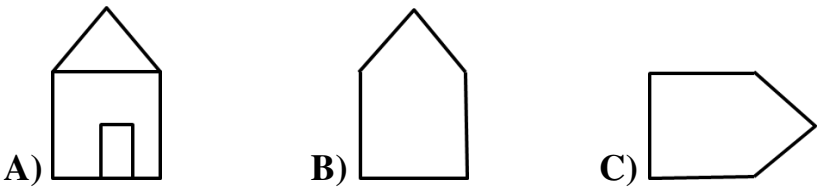


Figure 18. A: A house. B: The house's contour. C: The contour, rotated.

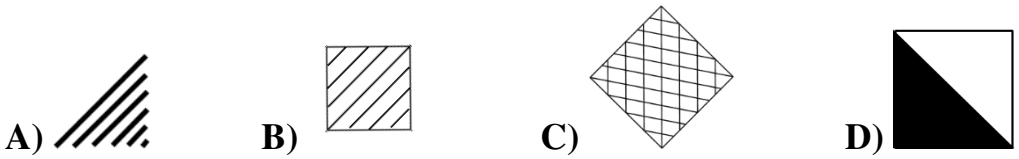


Figure 19. Examples of texture patches and negative space.

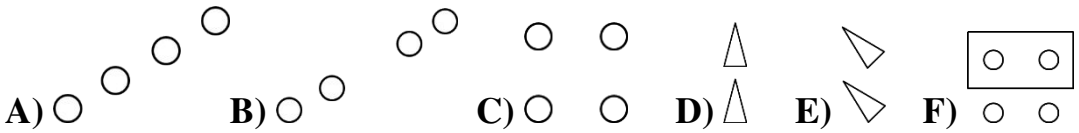


Figure 20. Grouping examples.

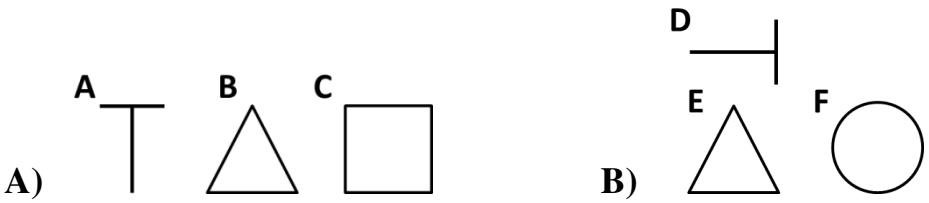


Figure 21. Two images with differences in their shapes and spatial relations.

A)  **B)** 

Figure 22. Smaller versions of the C/F pair from Figure 21.

Figure 23. Two ambiguous mappings.

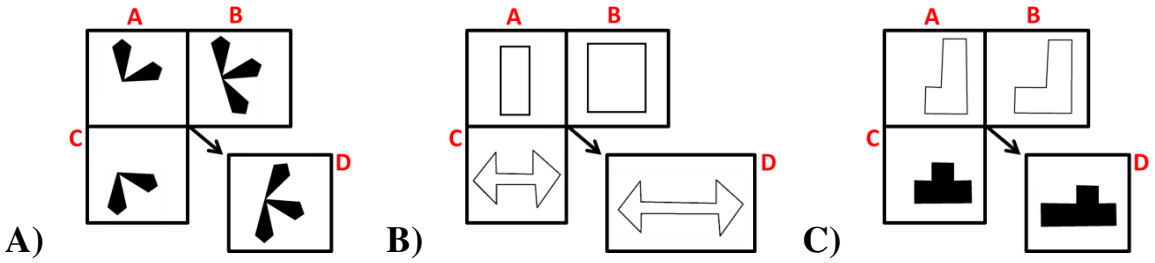


Figure 24. Examples of *Infer-Shape* with deformations.