Mathematical Aspects of Qualitative Modeling

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Qualitative Modeling

The Goal:
• Models that make the essential and possible distinctions only

Yes!
“Small” is beautiful ...

Reflects
• Goals
  • What has to be distinguished?
• Input
  • Observations, hypotheses
  • What can be distinguished?
  • Partially specified information
  • Noise
  • Limited knowledge
For Example

- Position of the accelerator pedal is above the idle threshold
- Increase in pressure increases negative acceleration of the wheel
- A leakage in the control pipe is not relevant to the turbo pressure

Reflects

- Goals
  - What has to be distinguished?
- Input
  - Observations, hypotheses
  - What can be distinguished?
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Everything is Real ...

- Position of the accelerator pedal is above the idle threshold
- Increase in pressure increases negative acceleration of the wheel
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One Possible View:

- There is a fine-grained (real-valued) model that describes the system accurately
- Base Representation:
  - $\text{DOM}_0(v_S)
  - $= \mathbb{R}^n$
- Model abstraction:
  - $\text{Model}_0 \rightarrow \text{Model}_1$
Relational Models

- Independently of the syntactical form:
- What set of states is allowed by the model?
- $R_S \subseteq \text{DOM}(v_S)$

A valid model of a behavior:
- $R_S$ covers all states of the behavior
- $\forall s \in \text{SIT} \; \text{Val}(v_S, v_{S,0}, s) \Rightarrow v_{S,0} \in R_S$
Domain Abstraction

- Position of the accelerator pedal is above the idle threshold
- Increase in pressure increases negative acceleration of the wheel
- A leakage in the control pipe is not relevant to the turbo pressure

Aggregate values leading to the same class of behaviors
- e.g. between “landmarks”: intervals
Domain Abstraction - Formally

General:
• \( \tau_i : \text{DOM}_0(v_i) \rightarrow \text{DOM}_1(v_i) \)

Aggregation of values:
• \( \tau_i : \text{DOM}_0(v_i) \rightarrow \text{DOM}_1(v_i) \subset \mathcal{P}(\text{DOM}_0(v_i)) \)

(Generalized) Intervals:
• \( \tau_i : \text{IR}_\infty \rightarrow \text{DOM}_1(v_i) \subset I(\text{IR}_\infty) \)

Real landmarks and intervals between them:
• \( L \subset \text{IR}_\infty \)
• \( \tau_i : \text{IR}_\infty \rightarrow \text{DOM}_1(v_i) \subset I_L(\text{IR}_\infty) \)
Symbolic Landmarks

Totally or partially ordered symbols:
- \((L, \leq_L) \subset \mathbb{IR}_\infty\)
- \(I_L: \{(l_i, l_j), (l_i, l_i) \mid l_i, l_j \in L\}\)

Interpretation over \(\mathbb{IR}_\infty\):
- \(\iota: L \rightarrow \mathbb{IR}_\infty\)
- \(\iota: L^2 \rightarrow I(\mathbb{IR}_\infty)\)
- \(l_i \leq_L l_j \rightarrow \iota(l_i) \leq_{IR} \iota(l_j)\)
Model Abstraction

- Domain abstraction
- $\tau: \text{DOM}_0(v_S) \rightarrow \text{DOM}_1(v_S)$
- induces model abstraction
- $R_S \subset \text{DOM}(v_S) \rightarrow \tau(R_S) \subset \text{DOM}_1(v_S)$

Theorem:
- If the base relation is a valid model of a behavior
- then so is its abstraction
- Important for consistency check
States and Behaviors

- $R_S$ and $\tau(R_S)$ describe the set of states
- What about behaviors?
- $=$ sequences of states

- Usually
- the constraints on state transitions are general
- i.e. apply to all models
- Reflecting continuity, integration, derivative relations

Theorem
- For two models model$_1$, model$_2$
- $\text{STATES}(\text{model}_1) = \text{STATES}(\text{model}_2)$
- iff $\text{BEHAVIORS}(\text{model}_1) = \text{BEHAVIORS}(\text{model}_2)$
Transformation of **Model Descriptions**

- But:
  - we do not want to compute $R_S$ and $\tau(R_S)$

**Instead**

- Transform e.g.
- (Ordinary differential) equations into model descriptions over $\tau(DOM(v_S))$ and solve them
- How are these solutions related to $\tau(R_S)$?
- Soundness and completeness?
A Simple Exercise: Qualitative Model of a Potentiometer

***Library***

- **Potentiometer**

<table>
<thead>
<tr>
<th>$pos_{pot}$</th>
<th>$i_{out}$</th>
<th>$v_{left}$</th>
<th>$v_{right}$</th>
<th>$v_{out}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>{gnd, betw, batt}</td>
<td>{gnd, betw, batt}</td>
<td>$= v_{left}$</td>
</tr>
<tr>
<td>$pos_{p, max}$</td>
<td>0</td>
<td>{gnd, betw, batt}</td>
<td>{gnd, betw, batt}</td>
<td>$= v_{right}$</td>
</tr>
<tr>
<td>(0, $pos_{p, max}$)</td>
<td>0</td>
<td>gnd</td>
<td>gnd</td>
<td>gnd</td>
</tr>
<tr>
<td>(0, $pos_{p, max}$)</td>
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<td>batt</td>
<td>{betw, batt}</td>
<td>betw</td>
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<tr>
<td>(0, $pos_{p, max}$)</td>
<td>0</td>
<td>batt</td>
<td>batt</td>
<td>batt</td>
</tr>
<tr>
<td>...</td>
<td></td>
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</table>
The Pedal Position Sensor

Library
• Switch

<table>
<thead>
<tr>
<th>pos_{switch}</th>
<th>state</th>
<th>V_{left}</th>
<th>V_{right}</th>
<th>V_{switch}</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, \text{pos}_{\text{med}}]</td>
<td>left</td>
<td>{gnd, betw, batt}</td>
<td>{gnd, betw, batt}</td>
<td>= V_{left}</td>
</tr>
<tr>
<td>(\text{pos}<em>{\text{med}}, \text{pos}</em>{\text{max}}]</td>
<td>right</td>
<td>{gnd, betw, batt}</td>
<td>{gnd, betw, batt}</td>
<td>= V_{left}</td>
</tr>
</tbody>
</table>
The Problem

• Generate a (qualitative) model that is appropriate for a particular device and task
Elements of the Context

- **Task and situation:**
  - requirements for distinctions on certain variables
  - possible distinctions
- **Structure and behavior models:** induce distinctions on other variables
- Required: a “base model”
Model-based Qualitative Model Abstraction

- Task and situation:
  - requirements for distinctions on certain variables
  - possible distinctions
- Structure and behavior models: induce distinctions on other variables
- Required: a “base model”
“Small” is Beautiful

- But difficult!