# AN ALGORITHMIC VIEW AT CAUSAL ORDERING

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## ABSTRACT

Causal Ordering has been much considered in Qualitative Physics to try and help Constraints Satisfaction Problems and to provide with an intuitive explanation of a device behavior. We focus on two approaches (the first defined by Iwasaki&Simon and the second by De Kleer&Brown) which have been widely debated in [Iwasaki 86] and [deKleer 86].Here we will consider those two problems from an alogorithmic viewpoint.

In [Iwasaki 86] Iwasaki&Simon define a "Causal Ordering" between variables of a physical device which can be computed from the set of equations modelling the device. We present a low order polynomial algorithm which computes this "Causal Ordering" without any algebraic manipulation and using graph theory, when the system is non-degenerated.

In [deKleer 86], DeKleer and Brown define "mythical causality" "which describes the trajectory of non-equilibrium states the device goes through before it reachieves a situation where the quasi-static models are valid". Using "Causal Heuristics", they trace the instantiation order of the variables while satisfying the constraints (modelling the device), which determines "causal orderings". We consider computational tractability of determining "mythical causality". The complexity of the problem is likely to depend on the expressiveness of the considered qualitative calculus. We show that the determination of "mythical causality" in a non-linear and general version of qualitative calculus is NP-hard.

Finally we give some plausible directions for dealing with incomplete information in order to extend Iwasaki & Simon's approach.

We believe this paper will clarify the notion of Causal Ordering at least on a computational viewpoint.

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## 1. Introduction

Causality is an important concept as far as Qualitative Physics and Common Sense reasoning are concerned. This special relevance is due to the fact that one of the ambitions of Qualitative Physics is to provide "causal accounts of device behavior" while carrying out qualitative treatment [De Kleer 84]. In an ordinary language concept, causality is assumed to qualify the relation between some events. From a mechanistic viewpoint, causality is the origin of any phenomenon.

In the engineering context that prevails here, phenomena are constituted by the evolution of some well-described physical devices. The devices are supposed to be characterized by the value of some variables. The kind of events we are interested in, are variations of the value of the variables. It is the relation between those kind of events that will be qualified as causal. Using (abusing) familiar notation, the causal relation between variables variations, will be considered as a causal relation between the variables themselves.

The whole July 86 issue of the Artificial Intelligence Journal [Iwasaki 86a], [De Kleer 86] and [Iwasaki 86b] was dedicated to the comparison of two approaches to causality : the Causal ordering of Iwasaki & Simon, and the mythical causality of De Kleer & Brown.

Both approaches consider steady states of physical devices described by quantitative equations or qualitative differential equations. They both adhere to the framework of classical physics. Both consider that a good device description is structural in a certain sense. The determination of causal relations between variables aims in both cases to give a functional account of the device and to describe its behavior. Iwasaki & Simon use structural equations, each standing for a single mechanism describing the system. De Kleer & Brown use equations related to components and to interactions between topologically related components.

The approaches then diverge, this may not only be due to the fact that Iwasaki & Simon borrowed concepts from economics while De Kleer & Brown's approach originates from electrical engineering.

The "Causal ordering" approach of Iwasaki & Simon is stated in a rather algebraic form. We give in section 2, arguments showing that not only the problem statement is algebraic but also its solution can be easily but not trivially grasped using algebraic or rather combinatorial concepts.

On the other hand, De Kleer & Brown's viewpoint about mythical causality is better stated in a logical or problem-solving perspective. They consider that certain kinds of proof on the fact that a device can reach an equilibrium state after a perturbation, are causal explanations of possible behaviors. They claim that causal relations between variables can be determined from those causal explanations. In section 3, we discuss the computational complexity of establishing causal relations in this logically-oriented framework. Considering a hard version of the qualitative calculus of De Kleer & Brown, we show that computing in this framework is closely related with mythical causality determination. In that context, mythical causality determination seems as hard as hard constraint satisfaction problems like graph-coloring.

Apart from their quite different statements, the two views at causality differ in several ways [Iwasaki 86a], [De Kleer 86],[Iwasaki 86b]. Starting from the same structural description of a device, they give different causal accounts. The causal relations do not have the same properties. Causal ordering is a quasi-order, mythical causality provides with a partial order. Some of their divergences (for example treatment of feedback loops) are more physical than computational. We try here to develop a purely algorithmic comparison of the approaches considered by Iwasaki & Simon and De Kleer & Brown and have compared the complexity of the two problems.

Finally, in section 4, we consider another difference between the two views at causality : the eventual inability of causal ordering to cope with underdetermined systems. Our algorithm for computing causal ordering can operate on an underdetermined system and determinate the seed of all causal orderings derived from completions of the system.

This algorithmic viewpoint clarifies the debate between the two "generally consistent analysis" by magnifying the distinctions between the two approaches.

## 2. An efficient algorithm to determine causal ordering

### 2.1. Definition of Causal ordering [Iwasaki 86a]

The behavior of physical devices is described through equations systems. Those systems are supposed to be **non-degenerated**, involving as many variables as equations. Causal ordering is a relation between the variables of the system. That relation is a **quasi-order** : it is reflexive and transitive. Causal ordering may not be antisymmetrical [Iwasaki 86a] (p.19), that is why causal ordering is not necessarily an order, but only a quasi-order.

The quasi-order derives from a decomposition of the equation system into **Minimal Complete Subsystems** .

**Definition** 1 : A subsystem of equations is complete or self-contained iff it contains as many variables as equations

A subsystem $S_1$ i said to be contained in the subsystem $S_2$ iff every equation from $S_1$ also belongs to $S_2$.

**Definition** 2 : A complete subsystem is minimal if it does not strictly include an equation set that is complete too.

A straightforward remark indicates that two distinct Minimal Complete Subsystems (M.C.S.) deriving from a non-degenerated system are disjoint : no variable appears in both M.C.S. It is noteworthy to remark that two distincts M.C.S. can be solved independently, if we are handling equations over some powerful structure like a field. Determining the M.C.S. of an equation system, can be a preprocessing step in equation solving and be of some help in applying a "divide and conquer" method. But the goal of Iwasaki & Simon [Iwasaki 86a] [Iwasaki 86b] is not to solve systems but to use their decomposition to determine causal ordering. We will see that this process does not require any algebraic manipulation, and so can deal with non-linear equations.

Using those definitions the system decomposition is carried out in the following way. Let $S_0$ denote the initial system and $C_0 = C_1 \cup C_2 \cup \ldots C_p$ be the union of its minimal complete subsystems. $C_0$ is self-contained. Starting from the rest $S_0/C_0$, we build $S_1$, a self-contained system by removing all variables from $C_0$ (as they could be computed from $C_0$ alone, they can be regarded as parameters in $S_1$). Every M.C.S. determines a class of variables. For every equation e from $S_0/C_0$, if e contains a variable occurring in $C_i$, e is labelled by i. This determines a labelling for $S_1$. The process is iterated. M.C.S.s from $S_1$ are computed ($C_1 = C_{p+1} \cup \ldots \cup C_q$). If any equation from $C_r$ (with $p+1 \le r \le q$) is labelled by i, then a causal link is drawn from every $C_i$ variable to every $C_r$ variable .

The labels from equations in $S_1/C_1$ are augmented in the same manner as during the first phase. The second rest $S_2$ derives from $S_1/C_1$ by removing variables occurring in $C_1$. The process is iterated until the rest is empty.

If some initial or intermediate M.C.S. contains more than one variable (i.e. there is a feedback loop), causal ordering is a quasi-order.

Determining causal ordering is equivalent to computing the M.C.S. of the successive rests. The intuitive procedure to compute the M.C.S. requires time exponential in the size of the largest M.C.S. It cannot be considered as satisfactory. The following exposes an efficient algorithm for computing causal ordering based on graph theoretical concepts.

**2.2. A case study**

A simplified condenser from a thermal power plant [Gallanti 86] illustrates our approach.

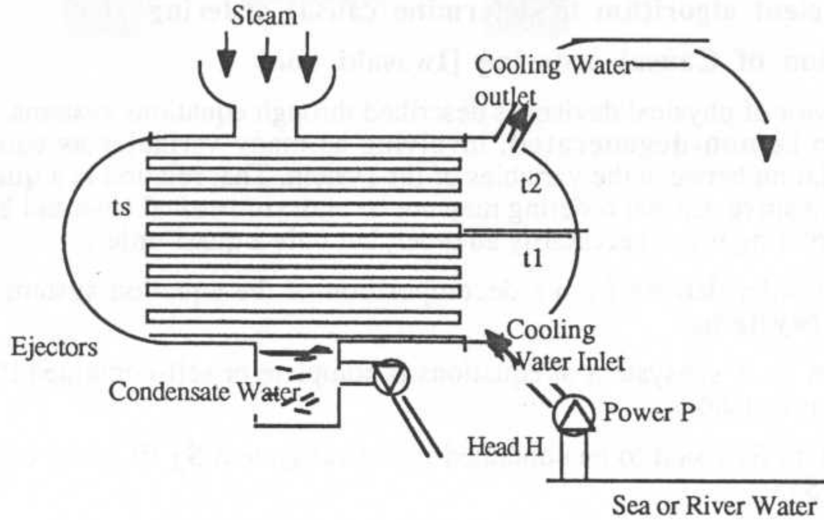



Fig 1 : Condenser from a Thermal Power Plant (Esprit Project P256).

"The role of the condenser in the water-steam cycle of a thermal power plant is to cool the steam coming from the turbine to bring it back to the liquid state" [Gallanti 86] .

"Condensation is obtained using surface condensers (i.e., without mixing cooling water with the steam) in order to keep the chemical characteristics of the cycle water unaffected. The cooling fluid is open-loop water taken from rivers or from the sea. In order to increase cycle efficiency, fluid condensation occurs in depressurized conditions, i.e. the pressure inside the condenser is kept lower than the outside atmospheric pressure. The cooling water is circulated by a water pump through a tube bundle placed inside the condenser, where steam condensation takes place. The condensate water is gathered at the basis of the condenser , from which it is removed by the condenser pumps" [Gallanti 86] .

Two main processes take place in the condenser : heat exchange between the two fluids and cooling water circulation (hydraulic process).

The condenser is described by a non-degenerated set of simplified equations given below. These equations involve :

- Process variables (pressure, temperature, mass flow ...)

- Design parameters (determined during design phase, such as the material, geometric characteristics of the circuit ....).

Moreover, we can distinguish between dependent and independent variables. Independent variables have been called "exogenous" variables.They are the cause of the device behavior. Some process variables are exogenous as their value is imposed from the outside of the system (the temperature of the cooling water, the quantity of heat flowing). Some design parameters have imposed values (because of design decision, because of fault or malfunction).

Here are the exogenous variables (and their unities) of the simplified condenser :

* Design parameters (which can be affected by fault or malfunction) :

- n : number of tubes
- delta : average thickness of a tube m

- lambda : thermal conductivity of the tube material $m^2s^{-1}$
- s : thermal transmission coefficient between external wall and steam. $ms^{-1}$

*External parameters :

- t1 : temperature of the sea water (upon which we cannot act) C
- P : Power of the pump (upon which we can act) W

Here are some parameters (and their unities) :

D : outer diameter of a tube m
l : length of a tube m
f : proportionnality factor depending on the motion type $Jm^{-4}s^2$

Here are the remaing variables (process and design variables and their unities) :

d : Inner diameter of a tube m
A : Cross section $m^2$
G : Flow rate $m^3s^{-1}$
H : Head provided by the pump $Jm^{-3}$
Dp : Load loss of cooling water through the tube bundle $Jm^{-3}$
v : velocity of the water $ms^{-1}$
Q : Amount of heat removed by cooling water $Cm^3s^{-1}$
S : Exchange heat surface $m^2$
t2 : Output water temperature C
ts : Temperature in the condenser C
K : Thermal exchange coefficient $ms^{-1}$
w : Thermal transmission coefficient between internal tube wall and the cooling water $ms^{-1}$
ps : pressure inside the condenser $Nm^{-2}$

Here are the simplified equations describing the condenser's behavior (where M+ stands for a monotonically increasing function) :

EQ1 : D = d + delta
EQ2 : $A = n \prod d^2$
EQ3 : G = v x A
EQ4 : P = G x H
EQ5 : H = Dp + Cte
EQ6 : $Dp = f \times l \times v^2 / n \times d$
EQ7 : Q = G (t2 - t1)
EQ8 : $S = n \prod l \times D$
EQ9 : K x S / G = log [(ts - t1) / (ts - t2)]
EQ10 : 1 / K = 1 / s + delta / lambda + 1 / w
EQ11 : w = cte x w
EQ12 : Q = M+(ts)
EQ13 : ps = M+(ts)

Equations 1 to 6 are related to the hydraulic process, while equations 7 to 13 are related to the heat exchange process. The two processes are linked through G (flowrate) and v (water velocity). Some equations describe the geometric aspects of the condenser (Equations 1, 2, 3) while the other ones describe a physical process (for example the flow rate G is determined by the water speed v and the cross section A). In this sense, it seems that these equations can be called "structural". Consequently, we can state that this description is consitent w.r.t. Iwasaki &Simon's approach. Moreover, the "no-function-in-structure" principle of de Kleer & Brown is respected : the fuctionning of the device is not implicitely given in the equations.

## 2.3. Graphical representation of equation systems

We have chosen to use graph techniques to express our problem. We will assess the link between M.C.S. we are looking for and Bipartite Elementary Subgraphs being built thanks to a perfect matching. We reason by induction on the order of M.C.S., modifying the representation after each step.

The causal ordering Problem may be stated using graph concepts because causal ordering depends solely on the occurrences of variables in the equations, neither on coefficients values nor on the kind of involved expressions. We show that the M.C.S. are associated with a peculiar kind of graphs. Finally we give an efficient algorithm that partitions the graph representing an equation set and by this way provides with the causal ordering.

### 2.3.1. Representation of equation systems through bipartite graphs

The problem is to represent the relations "appears in" between variables and equations from a system.

Every set of equations is associated with a graph $G = (\mathcal{E} \cup X, E)$. Every point from $\mathcal{E}$ (resp. X) stands for an equation (resp. a variable). (e, x) from $\mathcal{E} \times X$ belongs to E (the set of lines) iff x stands for a variable occurring in the equation represented by e . Every subsystem (M.C.S., rest, ...) induces a subgraph in a natural way. By design choice, G defines a bipartite graph (or bigraph).

A graph G = (U, E) is **bipartite** iff the point set U may be partitioned into two subsets such as there is no line from E between points from a same subset.

For every subset Y from $\mathcal{E}$ , $\Gamma_E(Y)$ denotes the **neighborhood** of Y : $\{x | x \in X, \exists y \in Y, \{x, y\} \in E\}$.

Following Berge [Berge 70] : "A matching M in G(U, E) is a subset of E such that every point is incident to at most one line of M". M is perfect iff every point is incident to exactly one line in M. König-Hall's theorem [Berge 70 p128] provides with a necessary and sufficient condition for the existence of perfect matchings : a bipartite graph $G = (\mathcal{E} \cup X, E)$ has a perfect matching iff for every subset Y from $\mathcal{E}$, $|\Gamma_E(Y)| \geq |Y|$.

**Claim** : The bipartite graph $G = (\mathcal{E} \cup X, E)$, associated with a non-degenerated system has a perfect matching.

**Proof** : As the system is non-degenerated, $|\mathcal{E}| = |X|$, and any subset Y from $\mathcal{E}$ is associated with a set of equations which have more than $|Y|$ variables, so $|\Gamma_E(Y)| \geq |Y|$.

In figure 2, we give the representation of the equation set of the condenser through a bipartite graph.

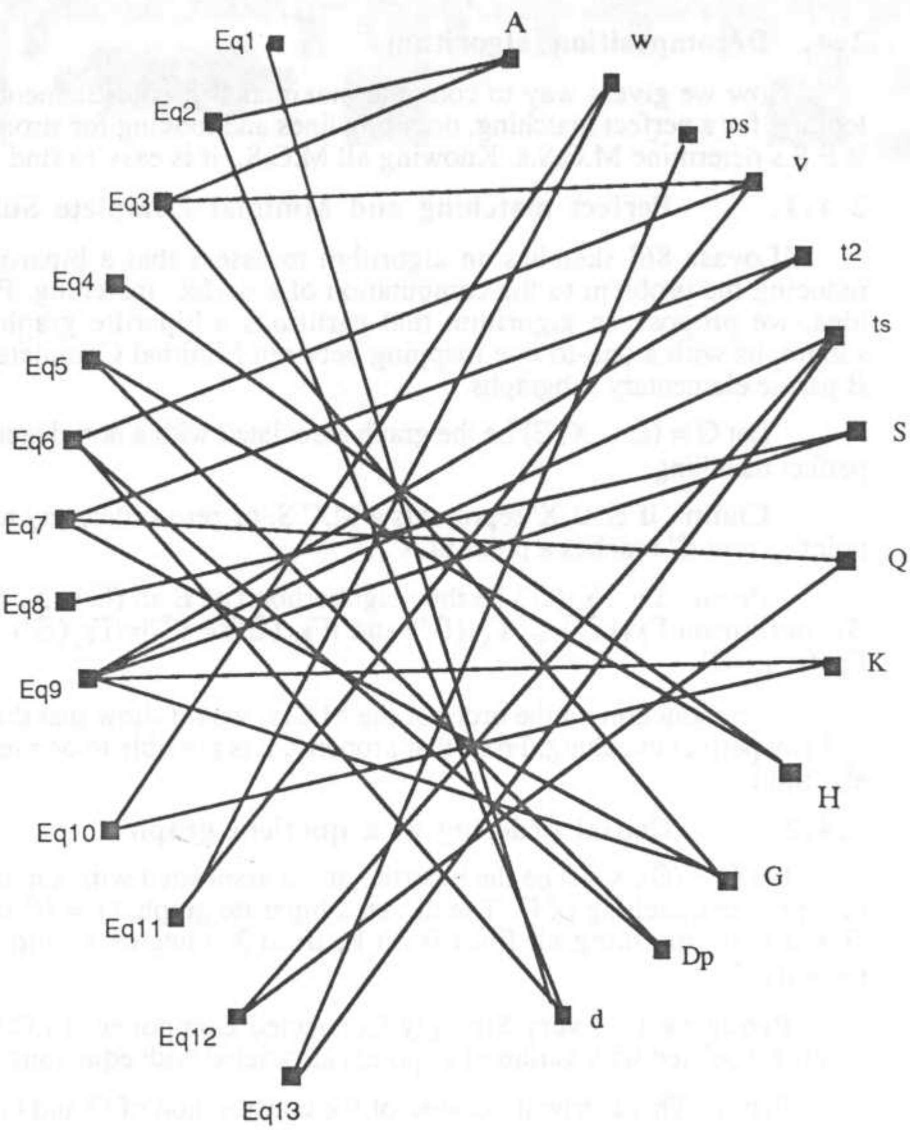



**Fig 2 Condensor : Bipartite graph associated with the equations set**

### 2.3.2. Minimal Complete Subsystems and Bipartite Elementary subgraphs

**Definition** : A bipartite graph is elementary iff it is connected and every line occurs in some perfect matching.

[Lovasz 86] states properties of Bipartite Elementary Graphs of special relevance to our problem. A bipartite graph $G = (\mathcal{E} \cup X, E)$ is elementary iff $|\mathcal{E}| = |X|$ and for every non empty proper subset Y of $\mathcal{E}$, $|\Gamma_E(Y)| > |Y|$.

**Property** : Every M.C.S. induces a maximal bipartite elementary subgraph.

**Claim** : The bipartite subgraph associated with a first order M.C.S. is elementary.

**Proof** : Let A be a M.C.S. and G a subgraph $(\mathcal{E} \cup X, E)$ induced by A. A is complete,so $|\mathcal{E}| = |X|$. For every non-empty proper subset B from A, as A is non-degenerated,we have $|\Gamma_E(B)| \geq |B|$, as A is minimal we cannot have $|\Gamma_E(B)| = |B|$ ; consequently $|\Gamma_E(B)| > |B|$ and G is elementary.

Moreover, by induction on the order of the M.C.S., we can conclude that every M.C.S. can be associated with a maximal Bipartite Elementary Subgraph.

## 2.4. Decomposition algorithm

Now we give a way to compute maximal Bipartite Elementary Subgraphs (B.E.S.) by looking for a perfect matching, orienting lines and looking for strongly connected components. B.E.S.s determine M.C.S.s. Knowing all M.C.S., it is easy to find the causal ordering.

### 2.4.1. Perfect matching and Minimal Complete Subsystems

[Lovasz 86] sketches an algorithm to assess that a bipartite graph is elementary by reducing the problem to the computation of a perfect matching. Following the same kind of idea, we propose an algorithm that partitions a bipartite graph into bipartite elementary subgraphs with a one-to-one mapping between Minimal Complete Subsystems and Maximal Bipartite elementary subgraphs.

Let $G = (\mathcal{E} \cup X, E)$ be the graph associated with a non-degenerated system and M be a perfect matching.

**Claim** : If $\mathcal{E}' \cup X'$ represents a M.C.S. of zero-order, any matching line issued from a point e from $\mathcal{E}'$ reaches a point in X'.

**Proof** : Let $\Gamma_M(\mathcal{E}')$ be the neighborhood of E' in $(\mathcal{E} \cup X, M)$.
By definition $\Gamma_M(\mathcal{E}') \subseteq \Gamma_G(\mathcal{E}')$ and $|\Gamma_M(\mathcal{E}')| = |\mathcal{E}'| = |\Gamma_G(\mathcal{E}')| = |X'|$ so $\Gamma_M(\mathcal{E}') = X'$ and $\Gamma_M(X') = \mathcal{E}'$.

An induction on the order of the M.C.S. would show that the claim holds for any order and any perfect matching. From that property, it is possible to derive an efficient decomposition algorithm.

### 2.4.2. Causal ordering as a quotient graph

Let $G = (\mathcal{E} \cup X, E)$ be the bipartite graph associated with a non-degenerated system and C be a perfect matching of G. The directed bipartite graph $G' = (\mathcal{E}' \cup X', E')$ is constructed from G and C by orienting all lines from E, from X towards $\mathcal{E}$ and all lines from C, from $\mathcal{E}$ towards X .

**Property** 1 : Every Strongly Connected Component (S.C.C.) in G contains as many points associated with variables as points associated with equations.

**Proof** : This is trivial because of the construction of G and G'.

**Property** 2 : Every maximal bipartite elementary subgraph (associated with a M.C.S.) in G induces an S.C.C. in G'.

**Proof** : Let A induce a bipartite elementary subgraph in G. A is strongly connected. There is a directed path from every point e from $A \cap \mathcal{E}$ to every point x in $A \cap X$. Let $X_j \subseteq X$ (resp $\mathcal{E}_j \subseteq \mathcal{E}$) be the set of points reachable from X, by a path of length less than 2j+1 (resp. 2j). By definition $X_0 = \Gamma^+_{G'}(\{x\})$ $\mathcal{E}_0 = \{e\}$.

$X_j$ and $\mathcal{E}_j$ are growing sequences for every k $X_{|\mathcal{E}|+k} = X_{|\mathcal{E}|}$ and $\mathcal{E}_{|\mathcal{E}|+k} = \mathcal{E}_{|\mathcal{E}|}$. Moreover as $|\mathcal{E}_{j+1}| = |X_j|$ by construction of G'(induction on j), we have $|X_{|\mathcal{E}|}| = |\mathcal{E}_{|\mathcal{E}|}|$. By construction of G', $\mathcal{E}_{|\mathcal{E}|} \subseteq A \cap \mathcal{E}$ and $X_{|\mathcal{E}|} \subseteq A \cap X$ and strict inclusions would contradict the minimality of bipartite elementary subgraph. As every point in $A \cap X$ has a neighbor in $A \cap X$ every point in A is reachable from every other point from A of G'.

**Property** 3 : If A induces a S.C.C. in G', A induces a maximal bipartite elementary subgraph in G (A is associated with a M.C.S.).

**Proof** : By property 1, $|A \cap \mathcal{E}| = |A \cap X|$ and A induces a Bipartite Elementary Subgraph (B.E.S.) in G. The restriction of C to A is a perfect matching of A. For every $B \subseteq A \cap X$, $|\Gamma_G(B) \cap A| \geq |B|$. If A induced a maximal B.E.S in G or if it was strictly contained in a B.E.S of G, A would not induce a S.C.C. in G' by property 2.

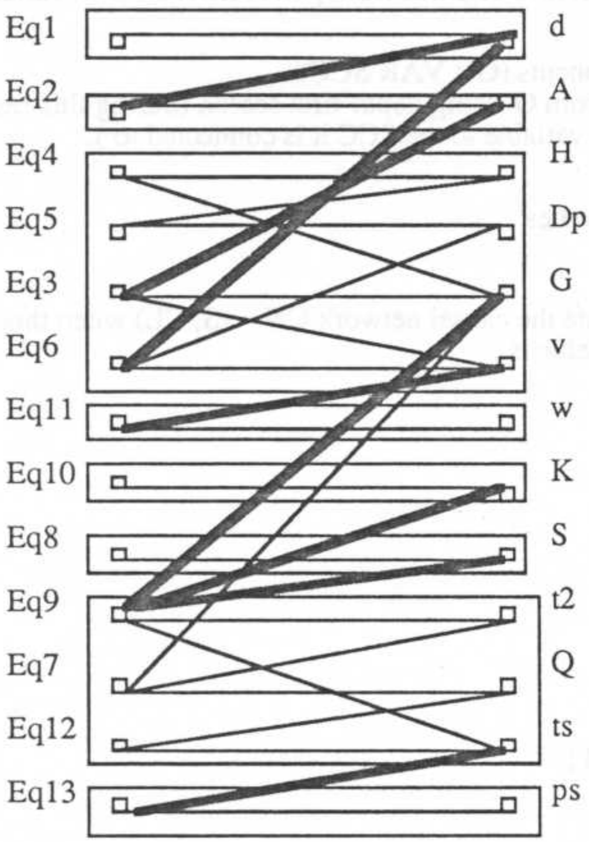
Given a digraph G', the quotient graph G" is a directed acyclic graph where the points represent the S.C.C. from G'. A line joins a point x to a point y in G" iff a line from one point of the S.C.C. associated with x joins a point from the S.C.C associated with y.

**Property** 4 : Every line in the quotient graph of G' represents a causal link between the variables of the associated M.C.S.

**Property** 5 : Every non-symmetrical causal link is associated with a link of the transitive closure of the quotient graph.

**Proof** : Property 4 is trivial. Property 5 is proved by induction on the order of the M.C.S.

The decomposition of the graph associated with the description of the condenser is shown in figure 3.



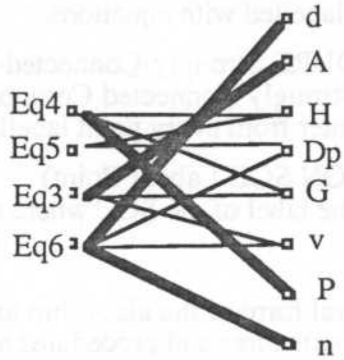




**FIG 3** : Left, graphic representation of the decomposition into bipartite elementary subgraphs (boxed). For clarity sake, the lines have not been oriented, and exogenous variables are not indicated. Horizontal lines stand for lines from the perfect matching. Bold lines stand for lines which will appear in the Causal Ordering.

Right, detail of a minimal complete subsystem of fourth order. Bold lines join equations with variables which do not belong to the M.C.S.(here exogenous variables have been represented).

### 2.4.4. Algorithm and Complexity

Let $\Sigma = (E, X)$ a non-degenerated system be the **Input.**

Let CN = (S, CL) the directed acyclic graph standing for the Causal Network be the **Output**.

Before describing the algorithm, we define some variables, give some possible data structure and sketch some procedures.

G : the labelled bipartite graph associated to $\Sigma$. It should be implemented using adjacency lists, since we deal with sparse graphs.

C : a list of edges from G, representing the perfect matching (C stands for coupling).

G' : The labelled bipartite directed graph built from G and C.

SCC : The strongly conneceted components from G'. The set of strongly connected components should be represented using an array. This may be computed by simple inspection of the directed edges from the points labelled with equations to points labelled with variables in G'.

PROCEDURE Perfect-matching (G ; VAR C)
C := Computation of a perfect matching (using Dinic's algorithm, for example).

PROCEDURE Directed-graph (G C ; VAR G')
G' := Graph built from G whose lines have been oriented from points labelled with equations to points labelled with variables and the lines from C oriented from points labelled with variables to points labelled with equations.

PROCEDURE Strongly-Connected-Components (G' ; VAR SCC)
SCC := Strongly Connected Components from G' using depth-first search (during this search, add a pointer from every point labelled by a variable to the SCC it is connected to ).

FUNCTION SCC-Label ( Point)
Returns the label of the SCC where the point lies.

The general form of the algorithm to compute the causal network CN = (S, CL) when those auxiliary structures and procedures are available is :

```
GLOBAL ALGORITHM (G , CN)
S:= ø;
CL:= ø;

C:= ø;
G':= ø;
SCC:= ø;
BEGIN

Perfect-matching (G C) ;
Directed-Graph (G C G') ;
Strongly-Connected-Components (G' SCC) ;

/*Creation of the points S from CN*/

FOR s in SCC DO
        CREATE a point s labelled by s-label ;
        S:= S ∪ {s} ;
ENDDO;

/*Creation of lines CL for CN*/

FOR (u,v) in EDGES of G DO
        IF  (NOT (SCC-Label (v) = SCC-Label (u))
                AND
            (NOT (SCC-Label (v) , SCC-Label (u)) in CL)

        THEN
                CL:= CL ∪ {(SCC-Label (v) , SCC-Label (u))};
ENDDO;

END
```

Here is the complexity of each step of the previously described algorithm:

The problem size $|\Sigma|$ is the number of variable occurrences in the equation system, it equals the graph size, i.e. the number of lines in the graph. The graph order is twice the number of variables : n.

- Building the bipartite graph associated with the system. This step has a O $(|\Sigma| + n)$ complexity.

- Determining a perfect matching in the bipartite graph by using network flow techniques like the Dinic algorithm [Even 1979 p97]. This step has a O $(n^{0.5} \times |\Sigma|)$ complexity.

-Building the bipartite directed graph with appropriate data structures, the representation of the bipartite graph may be used, so the complexity may be O (n). The digraph size is $O(|\Sigma| + n)$.

- Computation of the digraph strongly connected components. Using depth-first-search [Even 1979 p53], the complexity is O $(|\Sigma| + n)$.

- Computation of quotient graph which requires an examination of lines from $\mathcal{E}$ to X. The time complexity of that step is O $(|\Sigma|)$.

Consequently, as we may assume that $|\Sigma| \geq n$, the total time-complexity is O $(n^{0.5} \times |\Sigma|)$. (If the transitive closure of the digraph is requested, it becomes O $(n^3)$ ).

The following graph is the output of the previously described algorithm when the input is the set of equations describing the condenser.

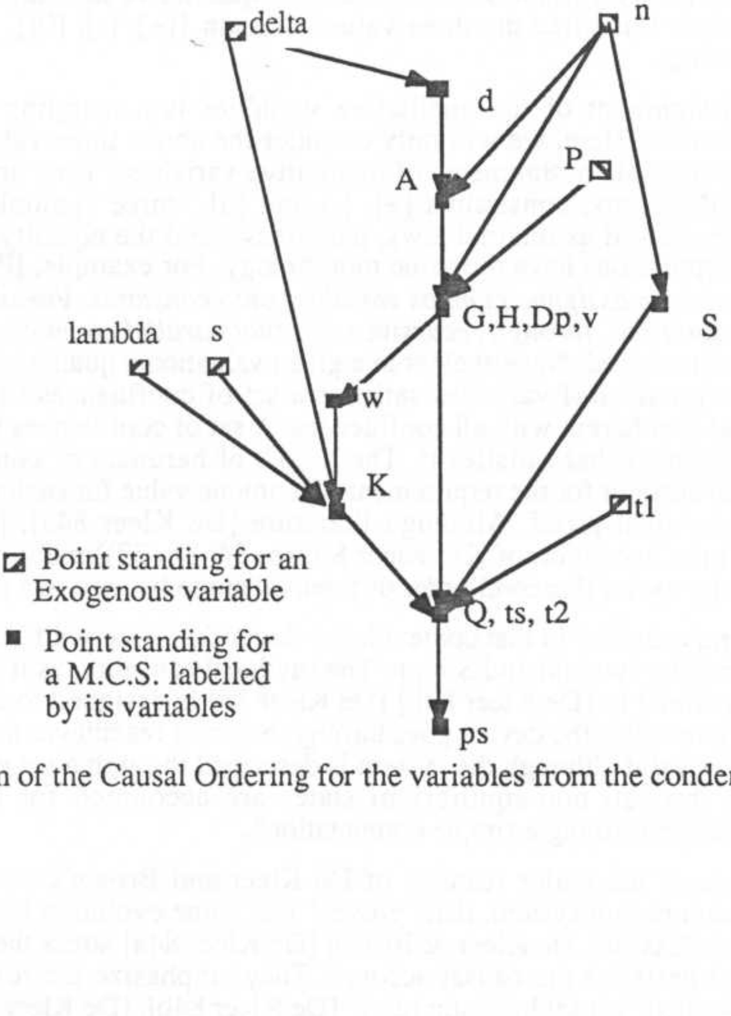



**FIG 4** : Diagram of the Causal Ordering for the variables from the condenser.

## 3. Complexity of "mythical causality" computation

Determining Causal ordering is easy, what can be said about the determination of mythical causality ?

### 3.1. "Mythical causality" : qualitative issues.

On the one hand, the causal ordering approach described by Iwasaki & Simon derives a quasi-order on the variables by using combinatorial properties of equation systems. The use of those properties is allowed by the properties of the domain of variables : it is a field. On the other hand De Kleer and Brown have focused attention on qualitative aspects of reasoning in physical engineering.

Physical systems are modelled using a quite different background. As well as in the classical approach of Iwasaki and Simon, a good description has to be structural. But here the emphasis is not on the association between equations with basic mechanisms, but on the association between qualitative variables with components of the system, relations between those variables in a component and relations between two components. The distinction between those two kinds of structural descriptions does not seem computational in essence.

What is certainly of great computational relevance are properties of the domain of values of qualitative variables, the meaning of qualitative constraints, and the computations that are used to determine **mythical causality**.

### 3.2. Formalization of solving confluences, relation with mythical causality determination

The qualitative variables take values in a qualitative domain, this domain is finite and linearly ordered. It is often the three-valued domain {[+], [-], [0]}, where each value has its intuitive meaning.

An assignment of the qualitative variables is a mapping from variables into the qualitative domain. Here, we will only consider the above three valued domain. Confluences constrain the possible assignments of qualitative variables. They are described by variables, denoted by identifiers, constraints [+], [-] and [0], three symbols "+", "-" and "." which should not be viewed as internal laws, parenthesis and the equality symbol =. Arithmetic & Qualitative expressions have the same morphology. For example, [P][V]=[T] and $\delta P+\delta V=\delta T$ are confluences. *An assignment maps variables onto constants. Variable assignment constrains expression valuations through recursive rules that parallel computing equivalents in Analysis.* [+]+[-] is undetermined. Nevertheless in a given valuation a qualitative expression has a unique value. An assignment of variables satisfies a set of confluences iff it generates at least one valuation that is coherent with all confluences. A set of confluences is satisfiable iff there is at least an assignment that satisfies it. The proofs of hardness of confluence solving we have seen, failed to account for the requirement of a unique value for each expression, this has led us to develop our own proof. Although literature [De Kleer 84a], [De Kleer 84b] does not explicitly (at the exception of [De Kleer 86] (p. 34, 36, 37)) mention non-linear confluences they seem quite useful if second-order differential are to be used in a device description.

Deriving causality in that context has a flavor that is very different from the derivation of causal ordering by Iwasaki and Simon. The mythical causality, as it is suggested by [De Kleer 84a] and explained in [De Kleer 84b] [De Kleer 86] is designed to describe "the trajectory of non-equilibrium states the device goes through before it reachieves a situation where the quasi-static model is valid although the system is described through a set of equilibrium constraints. Trajectories through non-equilibrium states are accounted for by viewing the device's components as performing a simple computation".

It is one of the major features of De Kleer and Brown's work to model the physical system as a computing system, that "proves" that some evolution from an equilibrium state to another one is feasible. De Kleer & Brown [De Kleer 84a] stress the fact that not all "proofs" provide good basis for the causal account. They emphasize the role of proofs that are built using the so-called "causal heuristic rules" [De Kleer 84b], [De Kleer 86].

Whatever the causal heuristics may be, an algorithm that is capable of computing a causal account of any device behavior provides with a tool to solve the confluences satisfaction problem. This may not seem obvious.

Any algorithm that computes mythical causality is able, starting from an equilibrium state, after a disturbance originating from a single point (for example concerning a single variable) to find another equilibrium state.

Let $\Sigma$ be a system of confluences, the question is : is $\Sigma$ satisfiable or not satisfiable ?

Let x' be a fresh variable that does not appear in $\Sigma$, every equation in $\Sigma$ : Ei ßi = ki where ßi and ki are qualitative expressions is associated with the equation E'i : x' (ßi - ki) = [0]. Let $\Sigma'$ be the resulting system. Every assignment in which x'=[0] satisfies $\Sigma'$. To find a satisfying assignment for $\Sigma$, it is enough to introduce the single disturbance x'=[+].

This trivial and polynomial reduction of the confluence satisfiability problem to the mythical causality determination problem works only because we have put no restriction on the morphology of the confluence system, although the indications given by De Kleer & Brown about what is a good structural description may have stringent morphological consequences. If the locality principle of De Kleer & Brown is respected, it is hard to imagine how a single variable like the fresh variable x' could appear in all confluences describing a device.

If the morphological consequences of De Kleer and Brown's modelling principles can be clarified, it remains to check if confluence solving remains reducible to mythical causality determination in that restricted context.

In the following, we will give another reason to consider restricted versions of confluence solving. Embedding mythical causality determination in general confluence solving is not likely to provide with feasible solutions, since it is likely to be intractable.

### 3.3. Computational status of mythical causality determination

The algorithms described by [De Kleer 86] (§2.1, §2.4) as a metaphorical illustration of mythical causality determination, do not completely state the problem difficulty. The algorithm sketched in §2.1 from [De Kleer 86] for computing mythical causality is compared with gaussian elimination ; it looks polynomial as it does not use any backtracking, and each step is polynomial. But the pivoting rule involves simplification and substitution rules provided by algebraic structures like groups or fields, and qualitative domains usually do not have such a structure. Replacement of the pivoting rule by causal heuristics cannot be done without allowing the exploration of an exponential-sized search-tree. The algorithm stated in [De Kleer 86] (§2.4) may be exponential in the size of the greatest feedback loop in the worst case since it is a blind exhaustive search. This is not enough to state that qualitative calculus is hard. It looks intermediate between equation solving (an easy problem) and hard constraint satisfaction problems like graph-coloring. We give here a proof that solving a set of non-linear confluences is at least as hard as determining whether an hypergraph is 2-colorable.

### 3.4. Reduction from hypergraph 2-colorability to non-linear confluences solving

Statement of the two problems :

**Hypergraph-2-colorability** [Garey 79]
Instance : Collection C of subsets (lines) of a finite set of points S.
Question : Is there a coloring of S with two colors such that no line in C is uniformly colored.

**Confluences satisfiability** :
Instance : Collection S of non-linear confluences on a finite set of qualitative variables .
Question : Is S satisfiable ?

**Reduction**

Each point $v_i$ from S is associated with a qualitative variable $x_i$ and a constraint (1) $x_i . x_i =[+]$ that ensures that $x_i$ takes values in {[+], [-]}.

Each line $\{v_{i1}, v_{i2} \dots v_{ij}\}$ from C is associated with $j^2$ auxilliary qualitative variables $p_{kl}^{i}$ with $1\leq k \leq j$ and $1\leq l \leq j$ and the constraints :

(2) $p_{k\,l}^{\ i} . p_{k\,l}^{\ i} = p_{k\,l}^{\ i}$, ensuring that every $p_{k\,l}^{\ i}$ takes values in {[0], [+]}

(3) $((\Sigma_{l=1}^{\ j} p_{k\,l}^{\ i})\ [+])=(\Sigma_{l=1}^{\ j} p_{k\,l}^{\ i})$, ensuring that for every k $\Sigma_{l=1}^{\ j} p_{k\,l}^{\ i}$ behaves like natural coefficients

(4) $(\Sigma_{l=1}^{\ j} \Sigma_{k=1}^{\ j} p_{k\,l}^{\ i})= [+]$, ensuring that at least one $p_{k\,l}^{\ i}$ takes value [+] .

(5) $\Sigma_{k=1}^{\ j} (\Sigma_{l=1}^{\ j} p_{k\,l}^{\ i}) .x_{ik}= [0]$

The transformation is polynomial since the resulting set of confluences has size $O(|S|+|C|^2)$. Assignments derive from 2-colorings by substituting value [+] (resp. [-]) for $x_i$ to color yellow (resp. red ) for $v_i$.

Every 2-coloring is associated with an assignment that satisfies the qualitative system, the first kind of constraint is always satisfied. Considering the $i^{th}$ line, let p be the number of yellow-colored points in it. For each such $v_{ik}$ set

$p_{k\,1}^{\ i} =[+]$ and $p_{k\,l}^{\ i} =[0]$ for $l>1$.

There must be one red-colored $v_{ik0}$ in the line,

let $p_{k0 l}^{\ i}$ be [+] if $l<p$ and [0] otherwise and let $p_{k\,l}^{\ i} =[0]$ if $ik\neq ik0$ and vik is red-colored.

Such an assignment generates a valuation that agrees with all kinds of constraints (induction on expression size).

Every assignment that satisfies the constraints is associated with a 2-coloring. The coloring of the $v_i$ is derived from the $x_i$'s assignment. Plugging constraints of kind (2), (3) & (4) into the corresponding constraint of kind (5) guarantees that not all terms can have a null value, that the terms have the same sign as the corresponding $x_{ik}$, entailing that $x_{ik}$'s do not have all the same value.

This just proves that confluences solvability is NP-hard, since we do not know whether the problem is in NP (it does not seem straightforward to verify that an assignment satisfies a given system).

This results raises several questions concerning its significance.

The first question concerns the relevance of the concept of non-linear confluences. Despite elliptic quotations ([De Kleer 86] p 34, 36 & 37), most of the confluence literature deals with linear confluences. Th preceding result does not rule out the possibility that linear confluence solving is solvable in polynomial time. But this would mean that non-linear qualitative calculus has a strictly greater descriptive power than linear qualitative calculus. If linear confluences were solvable in polynomial time and if there were a (polynomial-time) transformation that maps solvable non-linear confluences systems onto solvable linear confluences systems, the polynomial time algorithm for solving linear confluences could be used to build a polynomial-time algorithm for solving non-linear systems.

If non-linear confluences are more descriptive than linear confluences, it remains to determine whether this difference is of relevance to qualitative physics.

The second question concerns the very relevance of the intractability result with respect to the results of De Kleer and Brown. They expose (in [De Kleer 84b]) problems on which their heuristic rules work well, and which do not require any backtracking. There seems to exist a broad class of qualitative systems on which the mythical causality determination is feasible in linear time. This may be analyzed according to two (not incompatible) directions.

1° The availability of a satisfying assignment may be of great help. The confluence solving problem could still be hard for the class of confluences, but the availability of a solution could make the search for another solution easy.

2° The confluences solving problem restricted to that class of instances could be easy.

For example, constraint satisfaction problems that are easily solvable with backtrack-free search have been investigated in the literature [Freuder 82]. In those problems, where the constraints are just that some labels should be different like in graph coloring, there is no broader sufficient condition than "acyclicity of the constraints graph". Bringing such a condition in the field of confluences solving would forbid the description of any feedback loop. So it seems interesting to characterize sufficient conditions for backtrack-free solvability that does not involve acyclicity of the constraint graph but that exploits the special kind of constraints met in qualitative physics and the locality conditions exhibited by De Kleer & Brown.

We also ignore if it is hard to solve sets of confluences with the promise that the system has a unique solution (which may be quite rare in qualitative physics). [Valiant 86] shows that solving instances of the satisfiability problem with the promise that there is a unique solution is not easier than the whole problem under random reductions. It is also the same for hypergraph 2-coloring. Since our reduction does not preserve the number of solutions (is not parsimonious), we ignore if it is also the case for non-linear confluences solvability.

## 4. Coping with incomplete information

Causal ordering seems, at first glance to deal specifically with self-contained systems. This is quoted as a severe limitation in [De Kleer 86].

Nevertheless we may wonder wether some causal ordering may be determined in an incomplete system. This approach could be relevant to the study of large devices constituted by relatively independent components. Every component could be studied independently leaving variables describing interfaces with other components undetermined.

The system $\Sigma$ describing a device component is incomplete but non degenerated. It can be associated with a bipartite graph $(\mathcal{E} \cup X, E)$ where $|\mathcal{E}| < |X|$, but the König-Hall condition is still assessed when subparts from $\mathcal{E}$ are examined. Every possible completion of the system does not modify $\mathcal{E}$, but adds "virtual" equations associated with the remaining variables.

If a subsystem from $\Sigma$ is self-contained, the causal-ordering between the variables in the system will be the same in all completions of the system. The ordering between other variables from $\Sigma$ will depend on the completion (cf the different mythical causal links derived by different uses of the "causal heuristics" in [De Kleer 86]). A small adaptation of the algorithm stated in [Iwasaki 86] allows to determine the seed of causal ordering.

**Property** 1 : If $G = (\mathcal{E} \cup X, E)$ is a bipartite graph associated with an incomplete non-degenerated system and $G' = (\mathcal{E}' \cup X', E')$ is associated with a self-contained subsystem, then in any maximal matching of $G'$ no lines joins $\mathcal{E}'$ to $X-X'$.

**Proof** : By the König-Hall property any matching M in G saturates every point in $\mathcal{E}$ and hence in $\mathcal{E}'$. The following is trivial.

**Property** 2 : If $G=(\mathcal{E} \cup X,E)$ is the bipartite graph associated with an incomplete non-degenerated system and $(\mathcal{E}' \cup X', E)$ is associated with the non-empty maximal self-contained subsystem, then application of the algorithm stated in 2.4.4. partitions $\mathcal{E}' \cup X'$ into bipartite elementary graphs associated with the complete minimal subsystems of the maximal self-contained subsystem.

**Proof** : By property 1, no strongly connected component in the quotient graph can contain points from $(\mathcal{E} \cup X)/(\mathcal{E}' \cup X')$ and $\mathcal{E}' \cup X'$. The algorithm operates on the subgraph induced by $\mathcal{E}' \cup X'$ like on any bipartite graph with a perfect-matching.

The problem is that some strongly connected components provided by the decomposition algorithm are not associated with any M.C.S. of the self-contained subsystem. Those unuseful strongly connected components can be determined by a breadth-first-search of the quotient graph. The breadth-first-search should begin with zero-order M.C.S. (if the system contains a non-empty self-contained subsystem there must be some), for each strongly connected component which contains equation-associated points, determine the variables which value is necessary to solve the equation set associated with the strongly connected component. If all the variables in that set belong to the self-contained subsystem, the strongly connected is associated with a M.C.S. of the self-contained subsystem (the order does not matter), the variables associated with the strongly connected component should be marked as belonging to the self-contained subsystem. This computes the maximal self-contained subsystem. Correction of the procedure can be assessed by induction on the maximal depth of the quotient graph.

It is difficult to illustrate the capabilities given in this section with respect to incomplete data on the simplified condenser example, since this device cannot easily be relevantly decomposed in relatively independent components. Nevertheless we could assume that the relation between the device behavior and the ingoing water temperature is unknown, by removing Eq7 and Eq9 from the equation set. This incomplete causal ordering still brings some information about the device description, even though the links involving some variables like ps, ts, t2 & Q are not known.

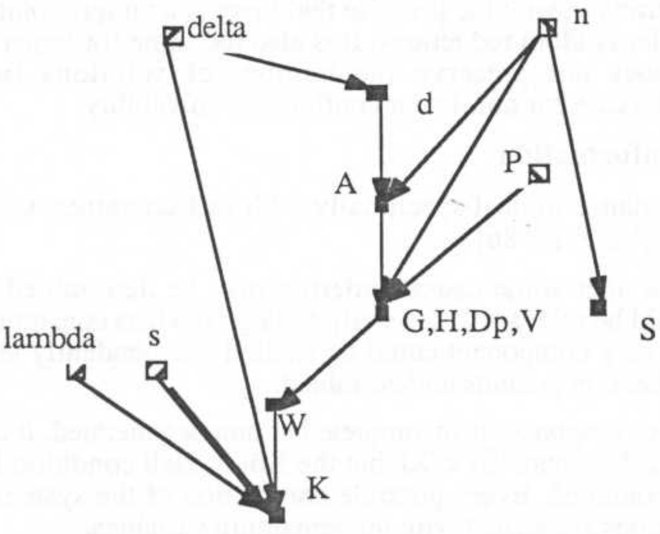



**FIG 5** : Incomplete causal ordering of the variables describing the condenser .Equations 7 and 9 have been removed from the description .

## 5. Conclusion

We have given an efficient algorithm to determine Causal Ordering in a set of equations that can be non-linear. We have proven that solving non-linear confluences is NP-hard and we would like to state apparently open problems :

The algorithmic scheme for computing the causal ordering may not be optimal. Computing the perfect matching is the bottleneck. The question is whether the determination of a perfect matching is linearly reducible to the decomposition into bipartite elementary graphs. An answer to that question would help in locating the very difficulty of computing causal ordering for any computing device.

Solving non-linear confluences seems difficult. But linear confluences may turn out to be easily solvable. A reduction of linear confluences solving to linear programming should be attempted. Polynomially solvable confluence classes should be searched for and compared with sets of confluences which describe problems declared easy by engineers. This would maybe bring some insights in the skills involved in qualitative reasoning.

[De Kleer 86] suggests that causal ordering could help in constraint propagation & consistent labelling problems. Causal ordering appears as a limiting case of finding a good instantiation order in a consistent labelling problem. [Freuder 85] presents very appealing ideas to determine the instantiation order in constraint propagation problems. It would be interesting to investigate possible applications of the kind of algorithms stated in 1 to such problems.

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