

Reasoning about Dynamic Systems and Bifurcations

Juan J. Flores

juanf@zeus.umich.mx

Division de Estudios de Postgrado

Facultad de Ingenieria Electrica

Universidad Michoacana

Morelia, Mexico

Andrzej Proskurowski

andrzej@cs.uoregon.edu

Computer Science Department

University of Oregon

Eugene, OR, USA

Abstract

A bifurcation occurs in a dynamic system when the structure of the system itself and therefore also its qualitative behavior change as a result of changes in one of the system's parameters. In most cases, an infinitesimal change in one of the parameters make the dynamic system exhibit dramatic changes. In this paper, we present a framework (QRBD) for performing qualitative analysis of dynamic systems exhibiting bifurcations. QRBD performs a simulation of the system with bifurcations, in the presence of perturbations, producing accounts for all events in the system, given a qualitative description of the changes it undergoes. In such a sequence of events, we include catastrophic changes due to perturbations and bifurcations, and hysteresis. QRBD currently works with first-order systems with only one varying parameter. We propose the qualitative representations and algorithm that enable us to reason about the changes a dynamic system undergoes when exhibiting bifurcations, in the presence of perturbations.

Introduction

When we think of dynamic systems, we think of Ordinary (perhaps Partial) Differential Equations, and their solutions with time. They may be linear or non-linear. By system dynamics we understand the set of qualitative features a system exhibits when excited properly. Several works define qualitative descriptions and algorithms that solve dynamic systems, and provide those solutions in qualitative terms (Kuipers 1986; Forbus 1984; de Kleer & Brown 1984).

Those works consider a non-changing dynamic system and provide solutions to its transient response with time. But dynamic systems depend not only on state variables and their derivatives, but on parameters, and those parameters may be functions of time. For instance, the mass of a rocket changes as it burns fuel, the characteristics of an electrical machine change as it ages, the load of an electrical power system changes during the day, etc.

Changes in the parameters, even if they are infinitesimal, may cause a dynamic system to exhibit totally different qualitative properties. For instance, a damped mass-spring system may stop oscillating if the damping

increases. Those changes in the topology of the phase space representation of the dynamic system are known as bifurcations, and the values of the parameters for which a bifurcation occurs are called bifurcation points.

The work presented in this paper concerns the determination of the behavior of a dynamic system exhibiting bifurcations. The analysis will take into account changes in parameters and perturbations to the system and will derive a sequence of events the system exhibits under those circumstances. All the analysis is accomplished at a qualitative level.

The paper is organized as follows: The following section provides a gentle introduction to dynamic systems, ordinary differential equations, solutions, phase portraits, and bifurcation diagrams; next, we define the problem to be solved; next section defines the qualitative representation for the different components involved in the process; after that, we propose a representation for events and the dynamics exhibited by the system under analysis; once we have defined the representation, we present a simulation algorithm that allows us to reason about dynamic systems and bifurcation diagrams; following that, we show some results, where simulations include qualitative plots and accounts for the different events present in a given scenario; we then propose directions that need to be explored; the final section summarizes our findings.

Dynamic Systems

The main tool for modeling dynamic systems is the differential calculus. Differential equations can roughly be divided in two types: ordinary and partial differential equations. This paper deals with dynamic systems that can be modeled by ordinary differential equations (ODEs).

An ODE is an equation of the form

$$F(t, x, \frac{dx}{dt}, \dots, \frac{d^n x}{dt^n}) = 0 \quad (1)$$

Function $f(t)$ is a solution to Equation (1) if

$$F(t, f(t), \frac{df(t)}{dt}, \dots, \frac{d^n f(t)}{dt^n}) = 0 \quad (2)$$

For some ODEs, the solution can be obtained analytically, in a closed form; for others (mainly non-linear

ODEs) there is no analytical solution and we need to resort to numerical integration to obtain a numeric expression of the behavior of a system under given initial conditions.

Whether analytical or numerical, a solution is a function of time. If we plot the solution and its first derivative, abstracting away the parameter time t , we obtain a trajectory on the so-called phase plane. A representative set of possible trajectories forms a phase portrait. A phase portrait includes all qualitative features that distinguish a dynamic system. Such characteristics include fixed points (attractors and repellers), nullclines, limit cycles, and for more complex (chaotic) systems, even strange attractors.

The reader is probably familiar with two-dimensional phase portraits, even with qualitative phase portraits (Lee & Kuipers 1993) and works like the non-intersecting trajectories filters for QSIM (Lee & Kuipers 1988). For first-order systems, phase portraits are unidimensional, and do not show as many features as in higher order systems.

Let us consider the nonlinear system represented by $\dot{x} = \sin(x)$. If we plot x against \dot{x} , we get a plot like that of Figure 1, where trajectories in the phase portrait flow or lie on the x axis. On the phase diagram, flow is to the right when $\dot{x} > 0$, and to the left when $\dot{x} < 0$, as indicated by the arrows. When $\dot{x} = 0$ there is no flow; the places x^* where $\dot{x} = 0$ are called *fixed points*. There are three kinds of fixed points: stable *attractors* (solid black dots), unstable *repellers* (open circles), and semistable¹ (half is open circle and half is solid).

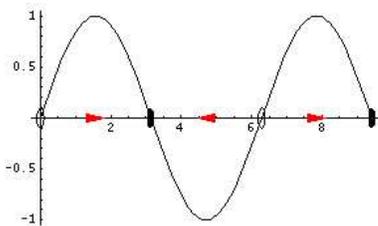


Figure 1: Unidimensional flow for first-order system

In all fixed points the derivative is zero, but there is a difference between stable and unstable equilibrium. We say that x^* is a stable fixed point if all trajectories that start near x^* approach it as $t \rightarrow \infty$. Think of a bowl and a marble inside it; if the marble is at rest and we push it a little, the marble will come back to its original equilibrium state (the bottom of the bowl). On the other hand, a fixed point x^* is unstable, if all trajectories that start near it are driven away from it. Now think of an upside-down bowl with a marble on top; if the marble is at rest and we push it a little, the marble will not come back to the equilibrium point, but will be driven away from it (by gravity, in this case).

¹not shown in Figure 1. Figure 6 shows some cases of semistable fixed points

First-order systems are very simple systems, but they exhibit interesting features when their parameters change with time. The qualitative structure of the phase portraits can change when we allow parameters to vary. Fixed points can be created or destroyed, or their stability can change. The qualitative changes in the topology of a phase portrait, due to the change in parameters are called *bifurcations*, and the value of the parameters where a bifurcation occurs are called *bifurcation points*.

Figure 2 shows the phase portraits for equation $\dot{x} = rx + x^3 - x^5$, for different values of parameter r , and Figure 3 shows its respective Bifurcation Diagram (BD). Note that arrows, indicating flow, have been removed from diagrams in Figure 2 for clarity. Also note that there is a region where the values of r make the linear and cubic terms dominate over the quintic term. For larger values of r , the quintic term dominates, yielding an interesting region where three fixed points coalesce into one, and the stable fixed point at the origin changes stability.

Bifurcation analysis has a large number of applications in science and engineering. Those applications range from radiation in lasers, outbreaks in insect populations, electronics, electrical power systems, etc., see (Strogatz 1994).

Problem Definition

We have defined bifurcations, bifurcation points, and bifurcation diagrams. The problem we address in this work is the following:

Given a qualitative description of a bifurcation diagram and the dynamics of the system with state x , determine the behavior of x as a result of perturbations as parameter r changes.

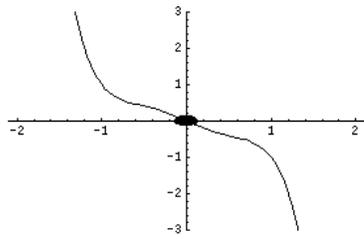
To address this problem, we rely in the following assumptions:

Continuity. All variables and functions involved in the process are continuous and continuously differentiable.

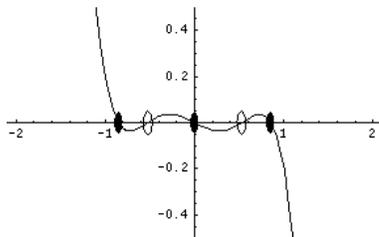
Two Time-Scales. In the process of simulation, two time scales will be considered. The variation of the system's parameters is much slower than the transient time to stabilize the system after a perturbation. That is, $t(\text{variation of } r) \gg t(\text{transients of } x)$.

Perturbations at Landmarks. Perturbations always occur at landmarks of r . (See the definition of landmarks in next section.) If we need a perturbation to occur in between two landmarks, we simply create another landmark between the two original ones, and let the perturbation occur at the new landmark.

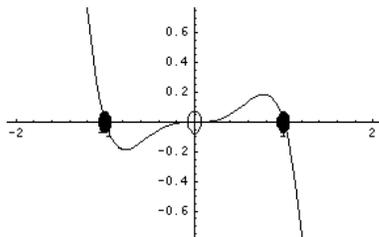
Small Perturbations. Perturbations are small enough so not to cross other fixed points. If we do not make this assumption, the behavior of the system would depend on the relative magnitude of the perturbation, with respect to the distance to the next fixed point in



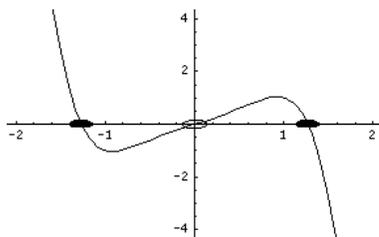
(a) $r = -1$



(b) $r = -0.2$



(c) $r = 0$



(d) $r = 1$

Figure 2: Behavior of $\dot{x} = rx + x^3 - x^5$ for different values of r

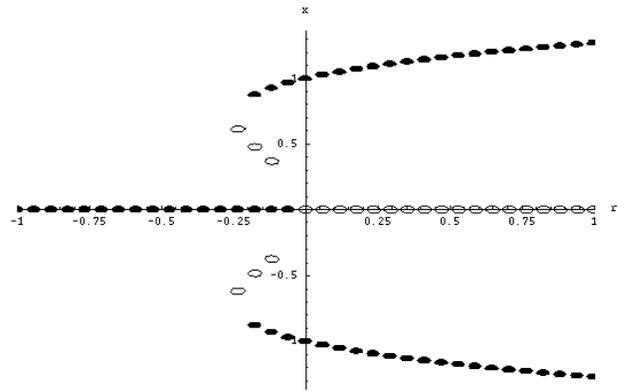


Figure 3: Bifurcation diagram for $\dot{x} = rx + x^3 - x^5$

the direction of the perturbation. This last assumption may not hold if a perturbation occurs infinitesimally close to a bifurcation point. Implementation of the system without this assumption, would lead to branching in the prediction of behavior. I.e. at any perturbation, we would need to consider cases where its magnitude would not reach, exactly meet, or pass each landmark in the direction of the perturbation.

Qualitative Representation

This section proposes the representations needed to accomplish the reasoning tasks we have in mind.

To reason about the dynamics of systems exhibiting bifurcations and under the influence of small perturbations, we need to start by computing the bifurcation diagram for such a system. Once we have a numerical description of the bifurcation diagram, we proceed to qualitatize it. (Keller 1987) presents several numerical methods to compute a Bifurcation Diagram. A good method for producing a qualitative description of a noisy signal is presented by Kay et al. in (Kay, Rinier, & Kuipers 2000). This reduction to a qualitative description is outside the scope of this paper, so we assume that such description has been produced.

We start with the main components of a qualitative description of a bifurcation diagram, and then define the representation for varying parameters and for perturbations. We borrow most of the notation from already developed representation schemes in the area of qualitative reasoning, mainly from QSIM (Kuipers 1994).

The main components of a dynamic system are variables, parameters, and constraints. Since we are starting from a bifurcation diagram, we do not need to represent the structure of the system (the constraints). A bifurcation diagram can thus be described in terms of the values of its state variables and parameters at different points in time.

At any point in time, the value of each variable and parameter is specified in terms of its relationship with a totally ordered set of *landmark values* and its direc-

tion of change. The set of landmark values is called the quantity space. Quantity spaces typically include $-\infty, 0$, and ∞ .

The qualitative value (or magnitude) of a variable can be a landmark or an open interval between two landmarks. In the latter case, we do not know anything about where in that interval the value of the variable lies; we just know it is within the interval. The qualitative direction of a variable is the sign of its derivative, in this case, the sign of the rate of change of x with respect to r . As opposed to Kuipers' representation, we decided to include ∞ as a possible value for qualitative direction. That is because at some points in a bifurcation diagram the rate of change of x wrt. r becomes ∞ .

In a bifurcation diagram, we plot x versus r , where neither r is a function of x nor vice versa. So we need to represent the bifurcation diagram as a relation (in the mathematical sense, a relation is a subset of the Cartesian product). In a BD representation, we need to include the qualitative states of the variable and the parameter, and the type of stability presented in the segment. So, a Bifurcation Diagram Segment (BDS) is a triple of the form $\langle qual(r), qstate(x), nature \rangle$, where *nature* can be any of $\{st, us, ls, rs\}$ (*st* =stable, *us* =unstable, *ls* =left-stable, *rs* =right-stable.) In this representation, a BDS is a subset of the bifurcation diagram where the qualitative values of x and r , and its stability remain unchanged in all that subset.

As an example, the variables involved, their quantitative spaces, and the qualitative representation of the bifurcation diagram of Figure 3 are shown in Figure 4.

Variables:
$\{r, x\}$
Quantity Spaces:
$QS(x) = (-\infty, x_4, x_3, 0, x_2, x_1, \infty)$
$QS(r) = (-\infty, r_0, r_1, 0, r_2, \infty)$
BD Representation:
$x \ R \ r$
$R = \{ \langle (-\infty, 0), (0, 0), s \rangle,$
$\langle (0, \infty), (0, 0), us \rangle,$
$\langle (r_1, \infty), ((x_2, x_1), +), s \rangle,$
$\langle r_1, (x_2, \infty), rs \rangle,$
$\langle (r_1, 0), ((0, x_2), -), us \rangle,$
$\langle 0, (0, \infty), us \rangle,$
$\langle (r_1, 0), ((x_3, 0), +), us \rangle,$
$\langle r_1, (x_3, \infty), ls \rangle,$
$\langle (r_1, \infty), ((x_4, x_3), -), s \rangle \}$

Figure 4: Qualitative representation of bifurcation diagram in Figure 3

Note that the qualitative representation of the bifurcation diagram does not provide enough information to draw curved lines like those of Figure 3. Figure 5 shows the qualitative representation of that bifurcation diagram. You can see that a BDS is a region where

the form of growth of x wrt r does not change, but the shape of that growth is unspecified. We decided to represent that growth as a straight line; the resulting BDS expresses our knowledge of the landmarks of x and r at the ends of the BDSs.

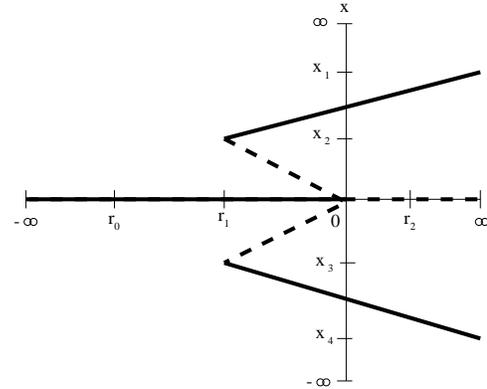


Figure 5: Qualitative bifurcation diagram for Figure 3

Dynamics and Events

According to Section *Problem Definition*, we need to provide a suitable representation for the dynamics of the system.

Since we are dealing with two time-scales, transient responses of the system (in other contexts referred to as the system's dynamics) are considered instantaneous when compared with the time taken for a parameter to change. On the other hand, when observing the transient responses of the system can be observed (from the same perspective in which QSIM predicts behavior, for example), we consider the parameters as constant.

With those considerations in mind, when we talk about *system's dynamics*, we are referring to the set of changes in parameters and perturbations present throughout the analysis of the system. We also consider as a part of the system's dynamics the changes of state variables of the system.

Parameter Changes

The changes in the parameters are represented as a sequence of qualitative states of each parameter. For the scope of the work reported in this paper, we are dealing only with one changing parameter, so the changes will be a list of qualitative states of parameter r . This list of changes may represent any qualitative continuous function of r with respect to time. Such function does not have to be monotonic, and may have as many extrema as necessary. Turning points of r in time (values for r where dr/dt is zero) establish landmarks for r .

Perturbations

The perturbations induced to the system can be represented as a list of single perturbations; a perturbation is a pair formed by the qualitative value of r where the

perturbation occurs, and the qualitative direction of the perturbation itself. The domain of the qualitative direction of a perturbation is limited to $\{+, -\}$.

Now, perturbations occur at a given instant of time, so we cannot associate them to a given value of r , but to a value of r at a point in time. In order to avoid the explicit introduction of time in our representation, we include perturbations in the description of the behavior of r , which implicitly contains time.

Parameter Changes

Our simulation will be performed at the slow time-scale (i.e. the time-scale when parameter r changes). In that time-scale, we need to specify the changes occurring to r . This description will not include time explicitly; we are not interested in the value of time when events happen, but in the order they happen.

Behavior of r , $B(r)$, is a qualitative description of how r changes with time; it is a sequence of qualitative states of r . A qualitative state contains qualitative magnitude and qualitative direction. The qualitative magnitude can be a point (landmark), or an interval, of two landmarks, not necessarily consecutive. The qualitative direction of r can be any of $\{-, 0, +\}$. States alternate between point and interval states.

Combining the behavior of r with the perturbations in x , we obtain a single descriptor of D . D is a list of pairs $(qstate_r, dir_x)$, indicating that at state $qstate_r$ there will be a perturbation to x with direction dir_x . The domain of the qualitative direction of a perturbation is limited to $\{+, -\}$.

Dynamics

In terms of the system's dynamics, a first-order system may exhibit a limited set of features. If the system starts at a stable fixed point, and a perturbation is induced, the system will return to the same stable fixed point (an attractor).

On the other hand, if the system starts at an unstable fixed point and a perturbation is produced, the system will be driven away from the fixed point, either to another stable fixed point or to infinity.

In order to establish a limited set of precise rules for predicting behavior of the system under the presence of perturbations, we need to state the following theorem.

Theorem 1 (Alternation of Fixed Points)

Consider the first-order ODE $\dot{x} = f(x, r)$, where f is a continuously differentiable function, and r is a parameter. For a given value of r , the nature of the corresponding fixed points alternate.

Proof: The proof is based on the continuity of f and on the nature of the roots of f , which are the fixed points of x .

Case 1. The nature of two consecutive roots of $f(x)$ can only be (st, us), or (us, st). See Figure 6(a).

Case 2. If f touches 0 at x^* , and f has no more roots, the nature of the fixed point can be determined by

the magnitude of $f(x^* - \epsilon)$ and $f(x^* + \epsilon)$. It is either ls or rs. See Figure 6(b).

Case 3. If $f(x)$ only touches 0, it may have one root on the left and/or one on the right. Let us consider the case where there exists a stable fixed point x_1^* to the left of x^* . Linear stability analysis indicates that $f'(x_1^*) < 0$ for stable fixed points (Strogatz 1994). This fact implies that $f(x^* - \epsilon) < 0$ and $f(x^* + \epsilon) < 0$ for any infinitesimal ϵ . Therefore, the fixed point at the touch will be right stable. See Figure 6(c). In the case where there exists an unstable fixed point to the left of x^* , $f'(x_1^*) > 0$, and $f(x^* - \epsilon) > 0$ and $f(x^* + \epsilon) < 0$. Therefore, the fixed point at the touch will be right stable. A similar argument applies if there exists a fixed point to the right of x^* .

Case 4. For consecutive touches (highly unlikely), all consecutive fixed points will be of the same nature: rs if there is an us fixed point at the right of the touches, ls otherwise. See Figure 6(d).

In all cases we observe an alternation on the nature of fixed points.

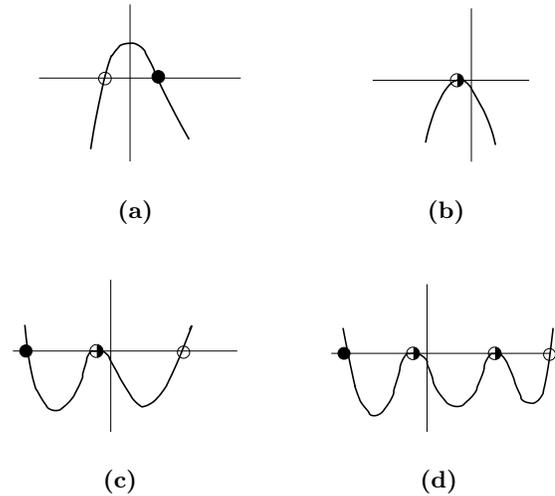


Figure 6: Cases for Theorem 1.

Based on Theorem 1, we can define the following cases for prediction of behavior in the presence of a perturbation. If the system is at an unstable fixed point, a positive perturbation occurs, and there is a stable fixed point above it, the system will stabilize at that fixed point; see Figure 7(a). If the system is at an unstable fixed point, a positive perturbation occurs, and there is no fixed point above it, the system will “blow up” (i.e. it will tend to infinity); see Figure 7(b). If the system is at a stable fixed point, and a positive perturbation occurs, the system will return to the same attracting fixed point no matter whether there is or there is not a fixed point above it; see Figures 7(c) and 7(d). Similar cases occur for negative perturbations. Sub-figures of Figure 7 show the state of the system before and after

the perturbation, and the symbols we will use for each event in graphical displays of the simulation.

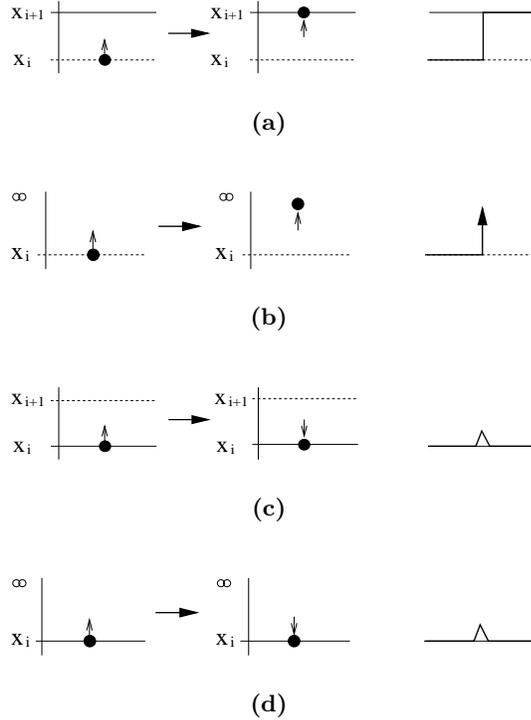


Figure 7: Predicting behavior caused by a perturbation

Other types of events that may occur are what we call *falling off* on a bifurcation diagram. When the system is at a stable fixed point, and parameter r varies, the system will travel along the BDS it is lying on at that point. If the BDS ends and there is no other BDS that places a fixed point at that state of the system, the system will be attracted by the nearest attracting fixed point.

Theorem 2 allows us to constrain the number of cases present in our simulation, and provides a more solid background for the development of the algorithm.

Theorem 2 (No ambiguity on fall-offs) *If a dynamic system is found in the conditions described as fall-off, then the system will transit into exactly one stable fixed point.*

Proof. Theorem 2 is a consequence of the property of determinism on dynamic systems, see (Strogatz 1994), chapter 4. If there were more than one (in which case it would be two for first-order systems; one above and one below), there would be two possible next states for the system, which would make the system non-deterministic.

Given Theorem 2, Figure 8 shows the possible cases for decreasing r . Other two mirror cases are possible

for increasing r . The thick line represents the behavior of x through time, moving along with r .

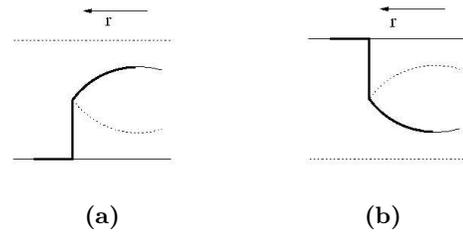


Figure 8: Predicting behavior caused by fall-offs

If the system undergoes perturbations while at an unstable state, or finds itself in a fall-off situation, the resulting changes in magnitude may be dramatic. Those are called catastrophes, since such changes in magnitude may take the system to states outside the operating region of the device. A catastrophe may make the device blow up, or if it is protected, to switch off. An example of such situation can be found in electrical systems, where changes (typically increases) in the load of a power system, may take it to a state that makes the voltage drop, currents increase, and protections operate, producing the well known black-outs.

There is another characteristic exhibited by dynamic systems near bifurcation points (specially nonlinear systems), known as hysteresis. When a catastrophic change occurs, the system moves to another stable state, and therefore is lying on a different BDS; this change is typically produced by a change in a parameter. After the catastrophe, reverting the changes in the parameter does not restore the system to its original state. This lack of reversibility as a result of the variation of a parameter is called hysteresis.

Section *Results* provides examples of all the events mentioned in this section.

Simulation Algorithm

Now that all the necessary representation has been defined, and all possible events and cases were presented, we are in possibility to present and explain the simulation algorithm.

The algorithm (see Figure 9) starts at initial state given by x_0 , and visits all landmarks in r 's trajectory, specified in D . At each landmark, if there exists a perturbation it computes the next state of the system. Also, at each landmark, it verifies if the BDS ends, producing a fall-off situation. If there is a fall-off, the next state is computed.

There are a few places in the algorithm that deserve more attention, specially at the time of implementation. function *pert* takes a state in the description of the changes of the system, and returns the sign of a perturbation, if one exists. In line 2, *pert* determines if the dynamics of the system include a perturbation at s_r . In

```

Simulation( $x_0, BD, D$ )
1: for all states  $s_r$  in  $D$  do
2:   if  $\exists$  pert( $s_r$ ) then
3:      $x = \text{nextFP}(x_0, \text{pert}(s_r), s_r)$ 
4:     record( $x_0, x, s_r, BDS(s_r)$ )
5:   end if
6:   if fallOff( $x, s_r$ ) then
7:      $x = \text{nextFP}(x_0, s_r)$ 
8:     record( $x_0, x, s_r, BDS(s_r)$ )
9:   end if
10:   $x_0 = x$ 
11: end for

```

Figure 9: Simulation algorithm

line 3, the algorithm needs to determine the next state after a perturbation occurs. This is the part where the cases of Figure 7 come to play. In order to determine what case applies to a given situation, we first need to search the BDS list for another BDS that coincide with the actual BDS at the present landmark, in the direction indicated by the perturbation. That landmark or interval is the new state. In line 6, it is necessary to determine if the current BDS, in the direction of change of r , ends at the present landmark. If that is the case, we need to determine if another BDS continues where the current one ends. If that is not the case, then we need to determine which BDS contains an attracting fixed point, either above or below the fall-off. In line 7, the implementation needs to determine where the system will continue after a fall-off. Function nextFP is being overloaded to determine the next fixed point in the simulation, produced by a perturbation or a fall-off.

For each perturbation or fall-off in the simulation, we record $(x_i, x_j, s_r, BDS(s_r))$; that is, the initial and final states, where the event occurred, the state of r when that happened, and the BD segment the system was at when the event occurred. (Lines 4 and 8 of the code.)

Line 10 updates the state of the system at each iteration on the simulation.

Results

This section presents one example of a simulation of a first-order system exhibiting bifurcations. This example contains all features and events, so it is representative of QRBD's capabilities.

The bifurcation diagram of Figure 3 can be represented in qualitative terms, as presented on Figure 4, on Section *Qualitative Representation*.

Let us assume that the system starts in state $(x(0) = 0, r(0) = r_0)$, with r changing according to $B(r) = ((r_0, +), ((r_0, r_3), +), (r_3, 0), ((0, r_3), -), (r_0, 0))$. Let us also assume that the following perturbations will be induced to the system: $P(x) = ((0, -), (r_3, +))$. These two components of the dynamics of the system are combined into a single descriptor

$$D = \{((r_0, +), \{\}), ((r_0, r_1), +), \{\}), ((r_1, +), \{-\}), ((r_1, r_2), +), \{\}), ((r_2, 0), \{+\}), ((r_2, r_0), -), \{\}), ((r_0, 0), \{\})\}.$$

The algorithm traverses all landmarks in r 's trajectory. When it is analyzing landmark $r = 0$, it finds a negative perturbation. Since the system is at a stable fixed point, the system will be attracted back to the fixed point at 0, and remain in the same BDS. When r is in qualitative state $(0, +)$, the BDS ends. Since there is another BDS starting at the same point, the algorithm does not detect any fall-off event. When the algorithm reaches landmark $r = r_2$, it finds another perturbation, a positive one this time. Since at that point the system was in an unstable fixed point, the perturbation drives the system away from it. The next fixed point in the direction of the perturbation is the one that belongs to the BDS $\langle (r_1, \infty), ((x_2, x_1), +), s \rangle$. The system has moved from an unstable fixed point at 0, to a stable fixed point somewhere in the interval (x_2, x_1) . At that point, r starts decreasing. When r reaches r_1 , the BDS ends; this time there is no BDS at the end of it, so the algorithm has detected a fall-off event. The closest attracting fixed point at that landmark is the one that belongs to the BDS $\langle (-\infty, 0), (0, 0), s \rangle$. The system has moved back to the stable fixed point at 0. The system continues at 0 with no change until the end of the simulation.

In the results derived from the simulation, we find two catastrophic events: an increase in magnitude at r_3 due to an external perturbation and a restoration to its original position at 0, due to a bifurcation point. The fact that the system is taking a trajectory when r increases and returning on a different one, when the parameter r decreases, indicates the system is an irreversible one, exhibiting a hysteretical behavior. Figure 10 shows a qualitative plot of the results of the simulation. Note that, for clarity, we have drawn the return to 0 slightly off place, so that it does not meet with the previous trajectory. Also note that we are totally disregarding the transient behavior of the system when it moves from fixed point to fixed point.

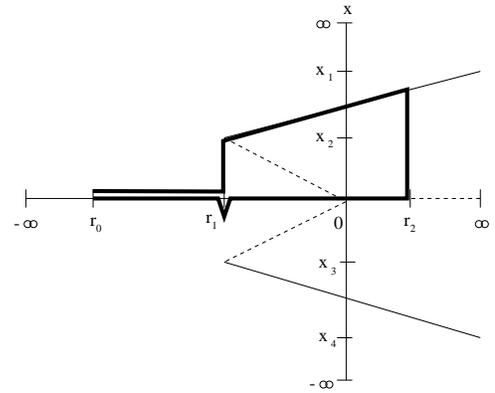


Figure 10: Results of the simulation in graphical form

QRBD was capable of producing all events found in

the simulation, but does not intend to produce the natural language explanations provided in the previous paragraphs. Those explanations are interpretations, provided by the authors of this paper, of the actual simulation outputs. The actual output of the simulation is presented in Figure 11.

QStates of r and x , and events present during simulation:
 $x = 0$.
 $r = (r_0, +)$.
 $r = (r_1, +)$, - pert., stable FP, $x = 0$.
 $r = (0, +)$.
 $r = (r_2, 0)$, + pert., unstable FP, $x = (x_2, x_1)$.
 $r = (0, -)$.
 $r = (r_1, -)$, Fall off, $x = 0$.
 $r = (r_0, 0)$.

Figure 11: Output of the simulation

Future Work

There are several directions to continue the work reported here. Among others, we have the following. The current algorithms and implementation only work with first-order systems; to broaden the applicability of the system, we need to deal with higher-order systems. There are some complications that need to be solved to go to higher-order systems; the first one is that bifurcation diagrams are no longer unidimensional, they are multidimensional. Since bifurcation diagrams represent all phase diagrams for a given range of r , they have one dimension more than the system itself. In that case, what were lines in a bifurcation diagram for first-order systems, become hypersurfaces for the general case, fixed points could become fixed regions, and so on. Also, starting in second order systems, new elements of a phase diagram appear (mainly saddles and limit cycles). Saddles are semi-stable fixed points, so they do not really introduce any novelties to the representation. A limit cycle is a closed path on a phase portrait for a second-order system; under changes of parameter r , the limit cycle shrinks and finally collapses to a fixed point. That would produce a funnel like element on a 3D bifurcation diagram.

Another direction to work on is on allowing more than one parameter to vary. For the general case (an n -th order system) with k changing parameters, the bifurcation diagram lies in a $n + k$ -th dimensional space. Again, representation needs to be enhanced to accommodate all possible elements in more general diagrams.

The last two directions of work deal with applications. The authors are seeking interaction with people from different areas, looking for potential applications, specially in the area of electrical engineering, where a number of people are working on bifurcation analysis for electrical power systems (for an example, see (Garcia-Kasusky, Fuerte-Esquivel, & Torres-Lucio 2003)).

Now, if we are to apply this framework to an engineering domain, we definitely need to produce not

only qualitative, but semiquantitative, or quantitative results. Otherwise, our results are of limited interest to engineers. For instance, if we know the system will undergo a catastrophic change, an engineer would like to know the order of magnitude of the change in the state variables.

Conclusions

This paper presents the representation and algorithms necessary to perform simulation of dynamic systems exhibiting bifurcations, on the presence of perturbations. Those simulations take place at a qualitative level, producing accounts of the different phenomena taking place in the dynamic system.

The system, QRBD, produces qualitative descriptions and qualitative plots of the results of the simulation. Those results include accounts of catastrophic events and hysteresis. Those two phenomena put together are of large importance to areas like electrical systems, biology, electronics, etc.

We propose several extensions to the work, and are committed to working on them.

The system QRBD has been implemented entirely on Mathematica (Wolfram 2003). The Mathematica notebooks and programs can be downloaded from (<http://lsc.fie.umich.mx/~juan/qrbd>).

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