Comparative Analysis and Qualitative Integral Representations*

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1 Introduction

The goal of comparative analysis [Forbus, 1984; Weld, 1987, 1988] is to determine how a perturbation to one aspect of a system affects the behavior of other aspects of the system, particularly when the system is incompletely known and described by a qualitative differential equation (QDE) model.

In terms of the QSIM representation for qualitative structure and behavior [Kuipers, 1986], a predicted behavior is a sequence of qualitatively distinct sets of values for the variables in the QDE. The behavior implies a set of relationships among the *landmark values* of the variables. The goal of comparative analysis is to analyze these relations to determine the direction of effect of a perturbation to the value associated with a landmark p on the value associated with the landmark q:

$$C(p,q) = \operatorname{sign}\left(\frac{\partial p}{\partial q}\right).$$

1.1 Our Approach

Our approach here is "Qualitative Physics" (or "Qualitative Mathematics") rather than "Naive Physics." A qualitative differential equation expresses a

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state of incomplete knowledge about the structure of a system that leaves traditional numerical or analytical methods inapplicable to the problem, due to incomplete knowledge of monotonic functional relations or the landmark values in quantity spaces.

However, once a QDE expresses a state of incomplete knowledge, we allow arbitrary (computable) algebraic or analytic reasoning to extract the strongest possible conclusion from the available knowledge. As discussed by Struss [1988] and Kuipers [1988], the set of qualitative conclusions that can be drawn from a given set of constraints are changed by the application of truth-preserving algebraic manipulations such as the associative and distributive laws.

This is certainly a familiar phenomenon in physics, where much mathematical work consists of reformulating the equations describing a situation so that a desired conclusion can be drawn unambiguously. Within the qualitative reasoning community, this approach of applying sophisticated mathematics to extract useful conclusions from incomplete knowledge is shared by (among others): Dormoy [1988], Kuipers [1986], Kuipers & Chiu [1987], Lee & Kuipers [1988], Raiman [1986], Sacks [1987, 1988], Struss [1988], Weld [1987, 1988], Williams [1988].

Our approach appears to handle all the comparative analysis examples discussed in the qualitative reasoning literature. It is a superset of Weld's [1987] rigorous method of comparative analysis, which in turn is a superset of Forbus' [1984] DQ analysis. The relationship with Weld's [1988] heuristic method of exaggeration is not clear. Dimensional analysis has also been used for qualitative reasoning [Kokar, 1986] and comparative analysis [Bhaskar and Nigam, 1988]. However, that method relies on a multiplicative form for universal laws, and has problems distinguishing among multiple variables with the same dimensions and qualitatively different roles in the behavior of the mechanism.

1.2 Overview

In this paper, we reduce the problem of comparative analysis to one of algebraic manipulation and simplification. There are several steps to this reduction:

1. Qualitative *integral* equations are required because they make explicit relations among time-points and time-intervals that are implicit in the equivalent qualitative *differential* equations.

2. Given the goal of determining $C(p,q)$, a search process must find an expression $R(p,q;r,s,t,\ldots)$ that follows from the QDE, where all landmarks other than p and q are constant.

    - Where this is impossible, the goal can be broken into a conjunct of simpler subgoals:

    $$C(p,q) = C(p,r) * C(r,q).$$

3. Given an expression $R(p,q;r,s,t,\ldots)$, take the partial derivative with respect to q and solve for $\partial p/\partial q$. There are several approaches to this differentiation, depending on the properties of the expression R.

4. The resulting expression for the partial derivative $\partial p/\partial q$ can appear awesomely complex, but frequently turns out to be relatively easy to simplify since only its sign is of interest, and not its actual value. This simplification step also requires search of a space of truth-preserving transformations

5. The result is
$$C(p,q) = \mathrm{sign}\left(\frac{\partial p}{\partial q}\right).$$

2 Qualitative Integral Equations

Integral equations express the same information as differential equations, but make explicit more of the objects that appear in the behavior. A qualitative differential equation might include the derivative constraint,

$$rate(t) = \frac{d}{dt}amount(t),$$

which the qualitative simulator can use to predict behaviors at time-points t_0, t_1, etc. A corresponding integral equation includes explicit reference to time-points and landmark values of variables:

$$amount(t_1) = amount(t_0) + \int_{t_0}^{t_1} rate(t)\, dt.$$

In fact, this is just a restatement of the Fundamental Theorem of the Calculus. Since we are doing qualitative reasoning, and our knowledge about the function $rate(t)$ may include partially specified monotonic functions, we cannot use the familiar inferential methods of the calculus to evaluate an integral. On the other hand, we can use the definite integral as a descriptive term, and many of the familiar theorems about definite integrals will be useful axioms for our qualitative reasoning method. See the section on **Simplification and Evaluation.**

Note that although the above example involves integration over t, we will have occasion to integrate with respect to other variables in the QDE.

The definite integral provides a new type of constraint, relating time-points and landmark values explicitly with the variables for rate and amount. Thus, quantity spaces (including time) are extended from purely ordinal spaces to ones where the length of intervals can be considered.

3 Deriving the Qualitative Integral Equations

From the available constraints, derive the integral equation $p = I(q)$.

- For a QDE with following factorizable form:

$$\frac{\partial p}{\partial q} = f(p)g(q),$$

  the corresponding QIR is given by:

$$\int_{p_1}^{p_2} \frac{dp}{f(p)} = \int_{q_1}^{q_2} g(q)dq$$

- For a conservative system, where QDEs are:

$$\frac{dx}{dt} = y, \quad and \quad \frac{dy}{dt} = F(x),$$

  following relation must be satisfied (see Sec. 2.2):

$$y_2^2 - y_1^2 = 2\int_{x_1}^{x_2} F(x)dx$$

  This implies that for fixed initial values: x_1 and y_1, $y_2(= y)$ is a function of $x_2(= x)$ only. So the QDE: $y = dx/dt$, leads to following QIR:

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{y}$$

- We also use the chain rule, to break a problem into tractable subcases:

$$\frac{\partial p}{\partial q} = \frac{\partial p}{\partial r}\frac{\partial r}{\partial q}$$

So far we have restricted our considerations to the general forms given above.

4 Taking the Partial Derivative

Suppose we have found a way to express p as an integral expression involving q,

$$p = I(q) = \int_a^b G(x,q)dx. \tag{1}$$

There are several ways of exploiting this integral equation to derive an expression for $\partial p/\partial q$.

4.1 The Easier Case

If things go well — i.e. we can integrate $\partial G(x,q)/\partial q$ — we can use the identity:

$$\frac{\partial p}{\partial q} = \frac{\partial b}{\partial q}G(b,q) - \frac{\partial a}{\partial q}G(a,q) + \int_a^b \frac{\partial G(x,q)}{\partial q}dx. \tag{2}$$

4.2 The Not-So-Easy Case

If we are not so fortunate, we can still reason with the more complex equation

$$\frac{\partial p}{\partial q} = \lim_{dq\to 0} \frac{I(q+dq) - I(q)}{dq}. \tag{3}$$

5 Simplification and Evaluation

The partial derivative expressions look awesomely complex, especially when one appreciates that $G(x, q)$ and $I(q)$ may themselves be substantial expressions. However, it turns out that the sign of $\partial p/\partial q$ can (at least sometimes) be evaluated with surprising ease, by applying a set of qualitative integral simplification rules.

5.1 Definitions

Notation:

Suppose $I = \int_a^b F(x)dx$, where a and b are finite.

- $\{F\}$ = "global sign" of $F(x)$, where consistent over (a, b).
- $[I] = sign\ I = \{F\}$

Qualitative integral rules

- If $F(x)$ has a definite sign for $x \in (a, b)$ (except for isolated zeros) then $[I] = \{F\} = -,\ 0,\ or\ +$.
- If $F(x)$ has both $+$ and $-$ signs for $x \in (a, b)$, then $[I] = \{F\} = nil$.

5.2 Qualitative Simplification Rules

1. **Removal of positive multiplicatives**

   For $g > 0$, except for isolated zeros:

   $$[I] = \{fg\} = \{f\}.$$

2. **Elimination of common denominators**

   For f_1 and f_2 arbitrary functions and $\{g_1, g_2\} > 0$,

   $$[I] = \{\frac{f_1}{g_1} - \frac{f_2}{g_2}\} = \{f_1g_2 - f_2g_1\}$$

3. **Linearization of differences**

   For $f, g > 0$ and monotonic function h,

   $$[I] = \{h(f) - h(g)\} = \{f - g\}$$

   A useful specific case, when $a > 0$, is:

   $$[I] = \{f^a - g^a\} = \{f - g\}.$$

5.3 Ordinary Algebra With Integrals

1. **Partial derivative rules:**

   - Differential definition rule
   - Integral derivative rule (for regular case)
   - Chain rule

2. **Change limits of integrals:**

   - Displacement rule:

   $$\int_a^b f(x)dx = \int_0^{b-a} f(x+a)dx$$

   - Scaling rule:

   $$\int_0^c f(x)dx = \int_0^1 \frac{f(cx)}{c}dx$$

3. **Combining two integrals:**

   $$\int_a^b f dx + \int_a^b g dx = \int_a^b (f+g)dx$$

   To combine two integrals with different limits, use rule 2 to convert to a common range, e.g. 0 to 1.

4. **Combining ranges:** If $t_1 < t_2 < t_3$, then

   $$\int_{t_1}^{t_2} f(t)\,dt + \int_{t_2}^{t_3} f(t)\,dt = \int_{t_1}^{t_3} f(t)\,dt.$$

6 Examples

Many of our test examples (in this paper, all except the first) have been taken from a family of related problems of motion in one dimension.

$$v = \frac{dy}{dt} \qquad a = \frac{dv}{dt} \qquad a = -k\ f(y)$$

- **ODEs:** f has an explicit algebraic form.
  - $f = 1$. Constant gravity.
  - $f = r^{-2}$. Decreasing gravity.
  - $f = y$. Simple spring.
  - $f = \theta$. Pendulum: small amplitude approximation.
  - $f = \sin\theta$. Pendulum: arbitrary amplitude.
  - $f = s^a$. Rolling on a concave surface, $a > 0$.
- **QDEs:** f is partially specified.
  - $f(y) = M^+(y)$, increasing monotonic function.
  - $f(y) = M^-(y)$, decreasing monotonic function.

6.1 Problem 1.

"If water pours into the tank fast, it will take less time to fill".

A. From QDE-to QIR:

- Denote water level: y, filling rate: $cf(t)$, where c controls the overall rate. The QDE is:

$$\frac{dy}{dt} = cf(t).$$

- QIR: From $t_1,\ y_1$ to $t_2,\ y_2$,

$$y_2 - y_1 = c\int_{t_1}^{t_2} f(t)dt$$

B. Define correlation problem: Fill the tank from time $t_1 = 0$ at arbitrary height y_1, to time $t_2 = T$ at the top h.

- To show: $\partial T/\partial c < 0$.

- Present QIR:

$$h - y_1 = c \int_0^T f(t)dt$$

C. Evaluate correlation:

- Take partial derivative of present QIR with respective to c, keeping t_1, y_1, h fixed.

$$0 = \frac{\partial c}{\partial c} \int_0^T f(t)dt + c\frac{\partial T}{\partial c} f(T)$$

- Evaluate signs:

$$0 = (+) \ + \ (+)\frac{\partial T}{\partial c}(+), \ or \ \frac{\partial T}{\partial c} < 0$$

6.2 Problem 2.

For the rising ball, where gravity decreases with height y, show $\partial h/\partial v_i > 0$.

A. From QDEs to QIRs:

- QEDs:

$$a = -gf(y), \ a = dv/dt, \ \ v = dy/dt, \ \ with \ \ g > 0, \ \ f' < 0.$$

  Notice a depends on y only, the system is conservative.

- QIRs:

$$v = \frac{dy}{dt} \ \ gives \ \ t_2 - t_1 = \int_{y_1}^{y_2} \frac{dy}{v} \quad (I)$$

  where dependence of v on y can be obtained through:

$$v_2^2 - v_1^2 = -2g \int_{y_1}^{y_2} f(y)dy \quad (II)$$

B. Define correlation problem:

- Present QIRs: From $t_1 = 0$, $y_1 = 0$, $v_1 = v_i$ to $t_2 = t_{rise}$, $y_2 = h$, $v_2 = 0$.

- QIR-I': (I) gives

$$t_{rise} = \int_0^h \frac{dy}{v}$$

  * Evaluate v based on II:
  * From $y_1 = y,\ v_1 = v\ to\ y_2 = h,\ v_2 = 0$, obtain

$$v^2 = -2g \int_y^h f(y)dy,\ or\ \ v = v(y,\ h;\ g)$$

  * or $F_{I'} = F_{I'}(v,\ y,\ h;\ g)$, with

$$t_{rise} = \int_0^h \frac{dy}{v(y,\ h;\ g)} \quad (I')$$

- QIR-II': From $y_1 = 0$, $v_1 = v_i$ to $y_2 = h$, $v_2 = 0$,

$$v_i^2 = 2g \int_0^h f(y)dy \qquad (II')$$

  where $F_{II'} = F_{II'}(v_i,\ h;\ g)$.

- Define correlation problem:
  - To determine: $\partial h/\partial v_i$.
  - QIR: Select $F_{II'}(v_i,\ h;\ g)$, i.e.

$$v_i^2 = 2g \int_0^h f(y)dy.$$

C. Evaluate correlation:

- Take partial derivative of (II') with respect to v_i,

$$2v_i = 2g \frac{\partial h}{\partial v_i} f(h)$$

- Evaluate signs:

$$(+) = (+)\frac{\partial h}{\partial v_i}(+),\quad or\quad \frac{\partial h}{\partial v_i} > 0.$$

6.3 Problem 3:

For the rising ball, where gravity decreases with height y, to find $\partial t_{rise}/\partial v_i$.

A. From QDEs to QIRs: All equations here are identical to those of problem 2.

B. Define correlation problem:

- To determine $\partial t_rise/\partial v_i > 0$.
- Apply chain rule: $\partial t_{rise}/\partial v_i = (\partial t_{rise}/\partial h)(\partial h/\partial v_i)$
- $\partial h/\partial v_i$ is already known.
- Proceed to determine $\partial t_{rise}/\partial h$).

C. Evaluate correlation:

- QIR: Choose $F_{I'}(t_{rise},\ h;\ g)$, i.e.

$$t_{rise} = \int_0^h \frac{dy}{v(y,\ h,\ g)}$$

- Taking partial of $F_{II'}$ with respect to h, gives

$$\frac{\partial t_{rise}}{\partial h} = \frac{1}{v(h,h,g)} + ... = \frac{1}{0} + ...,$$

  problematic!

- Alternative strategy: To evaluate

$$\frac{\partial t_{rise}}{\partial h} = \lim_{dh \to 0} \frac{t_{rise}(h + dh) - t_{rise(h)}}{dh}$$

  - Through the set of qualitative integral simplification rules given in Sec 4, it can be shown:

$$sign\left(\frac{\partial t_{rise}}{\partial h}\right) = sign\left(\frac{f(y)}{y} - f'(y)\right),\ for\ 0 < y < h$$

  - Since $f' \leq 0$,

$$sign\frac{\partial t_{rise}}{\partial v_i} = sign\left(\frac{\partial t_{rise}}{\partial h} \cdot \frac{\partial t_{rise}}{\partial h}\right) = (+)(+) = +.$$

6.4 Problem 4:

Consider a spring system with restoring acceleration $a = -cf(y)$, where $f' > 0$, $c > 0$. Find the effect on the period due to perturbation on the oscillation amplitude.

- It can be shown that the sign of correlation here is the same as that of $\partial t_{rise}/\partial h$. The latter problem is essentially solved in problem 3. It gives:

$$sign\left(\frac{\partial t_{rise}}{\partial h}\right) = sign((+) - (+)) = nil$$

- Disambiguate the sign: It turns out that more detail information on f disambiguate the signs here:
  - $sign\,(\partial t_{rise}/\partial h) > 0,\ requires\ f'' < 0.$
  - $sign\,(\partial t_{rise}/\partial h) = 0,\ requires\ f'' = 0.$
  - $sign\,(\partial t_{rise}/\partial h) < 0,\ requires\ f'' > 0.$

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