

Discrete and Continuous Time in Physical Systems

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Abstract

We investigate the relationship between discrete and continuous time in dynamic physical systems. Employing the common-sense picture behind derivatives, differential equations are translated into discrete-time analogues. The properties of qualitative simulation in discrete time are discussed. It is furthermore shown that discrete time can be embedded in continuous time in a natural way without the need to introduce the notion of infinitesimals. This provides a new perspective on the extended prediction problem. In addition, it is demonstrated that qualitative differential equations (QDEs) are abstractions not only of ordinary differential equations (ODEs), but also of an important class of integro-differential equations (IDEs). This extends qualitative simulation to the realm of IDEs.

1 Introduction

There are many ways to model time. A broad distinction can be made between the point-based and period-based paradigms [1]. Much of the work in AI is based upon the period approach [2]. In reasoning about physical systems, however, it is common to work within the point paradigm. As evidenced by the pervasive use of differential equations, physics itself conventionally uses a continuous time, whereby time is represented by the structure of the reals. Differential equations are also the starting point of much computational work, including numerical simulation and qualitative reasoning [3; 4]. But since the computer is a discrete machine, it is necessary in AI and other branches of computer science such as simulation to take a mixed view and to also employ discrete models of time, *e.g.*, in terms of the structure of the integers. There are sometimes also strictly physical reasons to consider discrete time, for example when the dynamics of the system is naturally modeled in terms of instantaneous events such as collisions between particles. In such cases it sometimes even happens (for example in nuclear-reaction theory) that both discrete and continuous physical theories exist for the same phenomenon.

Hence, there is already an interesting variety within the point ontology of time. This is one of the first papers that explores the *interrelationship* of different point representations of time in dynamic physical systems. As far as I know, neither in mathematical physics nor in AI much work has been done on this topic. In dynamical systems theory, a central distinction is made between continuous dynamics (differential equations) and discrete dynamics (iterated maps, difference equations), but usually only a single, separate representation of time is considered. For example, the KAM system of Yip [5] restricts itself to discrete maps, while the POINCARÉ system of Sacks [6] analyzes differential dynamics only. A partial exception is [7], studying how discrete processes can be aggregated into a larger continuous process. The present paper however investigates the connection between different temporal views on one and the same process. We develop a formal mathematical theory of the relationship of discrete and continuous dynamics, and new aspects of quantitative as well as qualitative reasoning are brought forward.

First, in Sec. 2 we describe how differential equation systems may be translated into common-sense analogues in discrete time. Next, in Sec. 3 we show how these analogues can be related to the original differential equations in an alternative way that avoids the notion of derivatives and infinitesimals. We then discuss some ensuing consequences for the extended prediction problem [8; 9]. Section 4 generalizes qualitative simulation to discrete-time systems. Finally, in Sec. 5 we prove that qualitative differential equations [4] are abstractions not only of ordinary differential equations but also of infinite delay equations. This provides a method to qualitatively simulate a new class of physical systems.

2 Doing Away with Derivatives

We consider here continuous-time processes that are modeled by differential equation systems of the form:

$$\frac{d}{dt}x_i(t) = f_i(x_1(t), \dots, x_j(t), \dots, x_N(t)), \quad i = 1, \dots, N. \quad (1)$$

Here, we employ the usual continuous time parameter t , represented by the real numbers. The x_i are real variables. In the present paper we restrict ourselves to functions f_i that are *linear* in their arguments x_j . As is well known, higher-order differential equations can always be rewritten in the form (1). There is no restriction on the system order or dimension N .

For the discrete-time processes to be considered we will employ a discrete time parameter S (for 'step') that is represented by the integers. Clearly, if one wants to obtain a discrete system analogous to Eq. (1) one has to get rid of the derivatives. This can be done in many ways, but the most straightforward one is to look at the formal definition of the differential quotient:

$$\frac{d}{dt}x(t) \stackrel{\text{def}}{=} \lim_{\tau \rightarrow 0} \frac{x(t + \tau) - x(t)}{\tau}$$

and to drop the limiting process, thus imposing that τ takes on a *finite, non-zero* value. We can now replace the real continuous-time variable $x(t)$ by a real discrete-time variable $X(S)$, assuming that the continuous time point t links to the discrete time point S and $t + \tau$ to the next discrete instant $S + 1$. For convenience, we introduce the notation

$$\Delta X(S) \stackrel{\text{def}}{=} X(S + 1) - X(S).$$

On this basis, the differential equation system (1) becomes translated into the following discrete-time system:

$$\frac{\Delta}{\tau} X_i(S) = f_i(X_1(S), \dots, X_j(S), \dots, X_N(S)), \quad i = 1, \dots, N. \quad (2)$$

We will call this discrete-time system the *common-sense analogue* of the continuous system (1). Other discretizations are possible and are standardly used in numerical simulation, but Eq. (2) is the simplest one and is closest to intuition. In actual fact, it matches the way in which Leibniz himself introduced the differential quotient in his original work [10], namely, as a *finite divided difference* — albeit with special calculus rules. Obviously, the connection between the above discrete and continuous dynamic systems may be simply established by taking the limit $\tau \rightarrow 0$. This leads us into the problem how to make predictions over extended periods of time in a finite number of steps [8]. As we will see in the next section, however, there are other ways to establish this link that avoid this limiting process.

3 A Discrete Perspective on the Extended Prediction Problem

As borne out by out by Eq. (1), differential dynamics consists of constraints on the simultaneous t values of physical quantities. Shoham [8] states that this instantaneous flavor of the rules of physics is the reason that predictions about extended periods are hard to make. Namely, a differential equation allows prediction only for a very short (in fact: infinitesimally small) interval forwards, and this has to be iterated many (in fact: an infinite number of) times. This he calls the extended prediction problem. In his comment [9], Rayner calls this way of interpreting derivatives a “cardinal sin”. Now, let us commit this cardinal sin¹, but still show that efficient extended prediction is possible on this basis. To this end, we will develop a novel perspective on differential dynamics.

3.1 Probabilistic embedding of discrete dynamics

As pointed out, we will consider the time lapse τ in Eq. (2) not as an infinitesimal quantity (leading to a proper differential equation) but as a finite, non-zero one. It is then easy to rewrite the common-sense discrete-time analogue (2) of the differential system (1) into a form that is suitable for extended prediction:

$$X_i(S + 1) = X_i(S) + \tau f_i(X_1(S), \dots, X_j(S), \dots, X_N(S)), \quad i = 1, \dots, N. \quad (3)$$

This is called an iterated map in dynamical systems theory. The situation at the next discrete time point is found by performing one iteration. This can be done very efficiently, since for each iterate only one function evaluation (see the right-hand side) is necessary. Accordingly, extended prediction for the discrete-time system is simple.

The everyday approach to extended prediction for the analogous continuous-time system now is to simply use this discrete method (or similar, more sophisticated ones) also for the differential dynamics, thereby making τ sufficiently small and striking a careful balance between the required accuracy and additional computational costs incurred by calculating more points. This seems to me an adequate approach (and here Rayner rightfully refers to the common-sense view of working scientists) because formal estimates can be given for the local and global errors made in using discrete methods (*i.e.*, finite τ) for integrating differential problems.

There is however an interesting alternative. First we note that in the conventional approach the discrete time points S are uniformly embedded in continuous time, with

¹To stress his criticism, Rayner more than once refers to the common-sense views of mathematicians and physicists. I am certain, however, that committing this cardinal sin is a daily practice among working physicists. Leibniz is my other witness. Formally, the evolution operators f_i related to differentiation are called the generators of the infinitesimal time translation!

fixed spacing τ . Thus, the assumption is that $X(S) = x(t = S\tau)$, in other words, the discrete time points are obtained by uniform sampling from the continuous time line. But one can introduce other, nonuniform and probabilistic, ways of sampling by introducing the probability $\mathcal{P}(t, S)$ that at continuous time point t precisely S discrete steps have occurred². Then, if we know the discrete-time variable $X(S)$ from extended prediction in discrete time, we can obtain the corresponding continuous-time variable $x(t)$ from

$$x(t) = \sum_{S=0}^{\infty} \mathcal{P}(t, S) X(S). \quad (4)$$

One can imagine many different possibilities for the way in which the discrete points S are randomly distributed in continuous time, and later on in Sec. 5 we will construct a whole class of them. In this section we will be content with a single form of probabilistic embedding. Let us suppose that the discrete steps take place on the continuous time axis *completely independent* from each other. It is known that occurrence of uncorrelated events is associated with a Poisson distribution:

$$\mathcal{P}_P(t, S) = \frac{1}{S!} \left(\frac{t}{\tau}\right)^S \exp\left(-\frac{t}{\tau}\right). \quad (5)$$

Here, the finite parameter τ represents the *average* time lapse (spacing) between two subsequent discrete time points.

3.2 Extended prediction of differential dynamics

Now, the connection between discrete and differential dynamics is provided by the following theorem.

Proposition 1 *Let $X(S)$ be the solution of the discrete system (2), and let $x(t)$ be the corresponding continuous-time variable satisfying Eqs. (4) and (5). Then $x(t)$ is the solution of the differential equation system (1) for all times t .*

Proof. By direct verification. Substitute Eq. (5) into (4) and differentiate both sides with respect to t . Rearranging the terms with respect to S and inserting Eq. (2) shows that $x(t)$ satisfies Eq. (1). \square

Phrased in more popular terms, we may say that *differential equations are equal to discrete-time equations plus probabilistic Poisson sampling*.³ Thus, we can deal with differential equations in a way that explicitly avoids the notion of derivatives or infinitesimal time steps, as shown by the right-hand side of this pseudo-equation. This yields an alternative solution to the extended prediction problem. A straightforward algorithm can be given for this:

²This idea stems from the theory of stochastic processes, see [11; 12].

³Incidentally, this idea can be used to show that the graph-theoretic approach to computational and physical causality proposed by [13] generalizes to discrete-time systems.

Step 1. Compute the solution $X(S)$ of the discrete-time system (2) through the iterated map algorithm (3). If the continuous-time interval of interest is $(0, T)$, the needed total number of discrete steps is on the order of $M = T/\tau$.

Step 2a. If one is interested in only one or a few future time points t , it is easiest to calculate Eqs. (4) and (5) directly, cutting off the summation at the point where additional terms contribute less than a preset negligible fraction.

Step 2b. If one is interested in a prediction covering the whole interval, one can generate the needed continuous time points by the following sequential Poisson sampling algorithm: draw a random number sequence $p_i \in (0, 1)$ using a standard pseudo-random number generator; then calculate the spacing to the next continuous time point as $\Delta t_i = -\tau \ln(p_i)$.⁴ If so desired, accuracy can be increased by repeating this procedure and averaging the results using so-called stratified sampling.

The computational complexity of each of these steps is $\mathcal{O}(M)$. Several variations or improvements are possible. It is interesting to note that this stochastic Poisson sampling also occurs in computer graphics as a technique to improve the quality of images [14; 15]. Here, other algorithms can be found (such as the dart-throwing algorithm) that are applicable to more dimensions but are somewhat less efficient in one dimension. In sum, efficient extended prediction based on differential equations is possible through probabilistic embeddings of discrete common-sense analogues in continuous time, without invoking differentials or infinitesimals.

Illustration 1: the radioactive decay law. A nice and simple example to illustrate the above is radioactive decay. Suppose we have a block of radioactive material and a Geiger device counting the decay events. Discrete time S is naturally provided by subsequent (groups of) clicks of the Geiger device and $X(S)$ is the fraction of nuclei that has not yet decayed. An obvious model is to postulate that in a given finite time span τ a constant fraction decays, so $\frac{\Delta}{\tau} X(S) = -\frac{\alpha}{\tau} X(S)$ ($0 < \alpha < 1$), and that the nuclear decay events occur independently, so that Eq. (5) applies. By symbolically solving Eqs. (3) and (4) we immediately see without using differential calculus (as standard text books do) that the fraction $x(t)$ in continuous time decreases in an exponential fashion. The algorithm discussed above gives a realistic animation of the Geiger counting process, in strictly discrete and finite terms.

4 Qualitative simulation in discrete time

The previous section has focussed on quantitative reasoning about discrete dynamics in its relation to continuous dynamics. An interesting issue is how this extends to qualitative

⁴That this indeed yields a Poisson distribution can be mathematically proved, cf. Sec. 5.

reasoning. We will follow here the mathematically transparent exposition of Kuipers [4] regarding qualitative simulation of ordinary differential equations (ODEs). Does this qualitative simulation machinery generalize to discrete-time systems of the finite difference equation (FDE) type (2), and if so, how?

Already upon a first inspection it is clear that one has to tackle several problems in the discrete case. First, the abstraction of time is necessarily different: Kuipers abstracts continuous time as an alternating sequence of distinguished time points and open intervals. But open intervals do not exist in discrete time. More importantly, a discrete-time function may transition between qualitative regions without assuming the landmark value itself at all. Hence, the very notion of distinguished time points becomes problematic. Second, several proofs in [4] depend on trend functions being ‘reasonable’ (essentially, continuously differentiable and behaving not too pathologically). For example, the intermediate value theorem is used. However, the notions of continuity and differentiability are lost in discrete time. The latter problem was already pointed out in [7]. On the other hand, it is evident that quantitative simulations with respect to Eqs. (1) and (2) are close if τ is not too large. Therefore, one would expect that a similar situation applies to qualitative simulation.

Upon translation from continuous to discrete time, the definition of the qualitative value *qval* of a trend function $X: \mathbf{Z} \rightarrow \mathbf{R}$ remains intact. The same holds for the qualitative constraint predicates ADD, MULT, MINUS and M^\pm . Due to the lack of continuous differentiability, redefinitions are needed for the DERIV predicate and for the qualitative change *qdir*. These can be furnished however by taking into account the common-sense ideas underlying derivatives discussed in Sec. 2:

Definition 2 *The qualitative change value of a discrete-time trend function $X(S)$ is given by $qdir = inc, std, dec$, if $\Delta X(S) > 0, = 0, < 0$.*

Definition 3 *The qualitative constraint predicate $DDIFF(Y, X)$ holds iff $Y(S) = \Delta X(S)$ for all $S \in [a, b] \subset \mathbf{Z}$. $DDIFF$ is the discrete-time analogue of $DERIV$.*

On this basis, any FDE (2) can be abstracted to qualitative constraint sets, just as in the case of ODEs. We now propose to abstract the discrete time axis as an alternating sequence of (closed) intervals relating to qualitatively steady state regions and of distinguished time points, whereby if necessary the latter are simply *added* if the trend function does not assume the landmark value itself. It is then straightforward to check that the QSIM I- and P-rules for possible transitions still apply. Thus, we have established the following proposition.

Proposition 4 *Qualitative simulations of the ODE (1) and of its common-sense FDE analogue (2) in discrete time yield the same results, with the possible exception of the distinguished time points.*

Hence, the QSIM machinery *does* extend to discrete time, but we have to give a special position to the distinguished time points. This idea can be made more precise,

by generalizing Kuipers' notion of a 'reasonable' function such that the distinguished time points of the ODE (1) are separated out *without* spoiling the reasonable nature of the function to be integrated. This can be achieved by using the so-called *Lebesgue integration theory*, which constitutes a generalization over standard Riemann integration (which is also used in qualitative simulation). The basic idea is that an integrable (or reasonable) function may be changed at isolated points without changing the value of the integral, because isolated points have 'no length' (formally: form a set of measure zero on the real line). One says that a property P holds 'almost everywhere' if the points where P does not hold form a set of measure zero. We will not go into detail here but refer the reader to standard text books such as [16].

By virtue of the Lebesgue theory the following proposition (a formally more precise version of the above proposition) is valid.

Proposition 5 (i) *The distinguished time points form a set of measure zero.*
(ii) *Let C be the class of functions that are equal to the solution $x(t)$ of the ODE (1) almost everywhere, namely, with the possible exception of the distinguished time points. Then qualitative simulation of the FDE (2) yields all behaviors that are compatible with the function class C.*

In sum, qualitative simulation in discrete time is '*almost equivalent*' to qualitative simulation in continuous time: the former yields the same results but is slightly more crude, in the sense that it cannot distinguish between 'everywhere' and 'almost everywhere'. This completes our discussion of discrete qualitative simulation.

Illustration 2: decay and oscillation. The previous example of radioactive decay is very naturally modeled qualitatively by the axioms $DDIFF(Y,X)$ and $M_0^-(Y,X)$. Then the discrete version of QSIM, like the continuous one, correctly predicts that the radioactive fraction always keeps decreasing and becomes zero at infinity. Another example is the harmonic oscillator. A 'stroboscopic' view on the undamped spring gives the quantitative model $\Delta^2 X(S) + \omega^2 \tau^2 X(S) = 0$, and the qualitative constraints $DDIFF(Y,X)$, $DDIFF(Z,Y)$ and $M_0^-(Z,X)$. Qualitative simulation produces the same oscillatory behaviors as in the continuous case (including the ambiguities concerning damping, conservativeness or runaway), while numerical simulation of the FDE shows oscillatory motion whereby the landmark value zero might be missed in discrete time, depending on the actual values of ω and τ . This shows the 'almost equivalence' of discrete and continuous qualitative simulation.

5 Are QDEs abstractions of ODEs only?

This section will lead us to a rather remarkable theorem about qualitative abstraction and simulation of system types that have not previously been considered in AI. We have already

shown that differential and discrete dynamics are almost equivalent from the viewpoint of qualitative simulation. We have also seen that these forms of dynamics are quantitatively equivalent if (and only if) they are linked by probabilistic Poisson sampling. So, a natural question is: what happens if we use other stochastic sampling methods that do not yield a Poisson distribution? Although this requires some mathematical sophistication, the results are rewarding.

5.1 A construction method for probabilistic embeddings

In order to tackle this problem, we will first give one method to construct different probabilistic embeddings \mathcal{P} to be used in Eq. (4) [11; 12]. As in the Poisson case, we suppose that the discrete steps S occur at random instants in continuous time. For each step we assume that the time lapsed since the occurrence of the preceding step is determined by a generic probability density $\varphi(t)$ satisfying

$$\forall t \varphi(t) \geq 0; \int_0^{\infty} \varphi(t) dt = 1. \quad (6)$$

There are no other restrictions on φ . As a result we obtain a very wide class of embeddings \mathcal{P} , each suitable function φ yielding an instance. The probability $\mathcal{P}(t, S)$ can be inductively constructed from φ as follows. For $S = 0$ we take $\mathcal{P}(t, S)$ to be the probability that no step has occurred at continuous time t . This is formally expressed by:

$$\mathcal{P}(t, 0) \stackrel{\text{def}}{=} \chi(t) = 1 - \int_0^t \varphi(t') dt'. \quad (7)$$

Next, $\mathcal{P}(t, S = 1)$ is the probability that precisely one step has occurred at time t . This is formally given by the convolution expression

$$\mathcal{P}(t, 1) = \text{Conv}[\varphi, \chi] \stackrel{\text{def}}{=} \int_0^t \chi(t - t') \varphi(t') dt'. \quad (8)$$

This procedure is continued to yield a recursive expression for \mathcal{P} for all S :

$$\mathcal{P}(t, S + 1) = \text{Conv}[\varphi, \mathcal{P}(t, S)]. \quad (9)$$

One obtains the Poisson distribution (5) via this procedure by taking for φ an exponential distribution: $\varphi_P(t) = \frac{1}{\tau} \exp(-\frac{t}{\tau})$. We mention in passing that this provides the rationale for the sequential Poisson sampling algorithm discussed in Sec. 3.

5.2 Quantitative and qualitative reasoning implications

After these preliminaries we are able to derive both a quantitative and a qualitative theorem, the latter being the central result of this section.

Proposition 6 *Let $X(S)$ be the solution of the discrete system (2), and let $x(t)$ be the corresponding continuous-time variable satisfying Eqs. (4) and (7)–(9). Then for all times t , $x(t)$ satisfies an integro-differential equation (IDE) system of the form:*

$$\frac{d}{dt}x_i(t) = \int_0^t dt' \mathcal{M}(t-t') f_i(x_1(t'), \dots, x_j(t'), \dots, x_N(t')), \quad i = 1, \dots, N. \quad (10)$$

Proof. The proof is most elegantly set up in matrix notation. Take the Laplace transform of Eq. (4). Then, the summation can be carried out symbolically using the Faltung theorem, upon insertion of Eqs. (7)–(9) and of Eq. (2) in its iterated map formulation (3). Comparison with the solution for $x_i(t)$ obtained by Laplace transformation of Eq. (10) now establishes the proposition, in addition yielding the relation between \mathcal{M} and φ .⁵ \square

Unlike an ODE, the IDE (10) is not a constraint on simultaneous values of quantities. One also needs all past state values in order to evaluate the time evolution of the system. Thus, IDEs (also called infinite delay equations) have a memory \mathcal{M} , whereas ODEs are memory-less (that is, \mathcal{M} is a Dirac delta function). IDEs naturally occur in nonequilibrium statistical mechanics and quantum mechanics, and have applications for example in nuclear and chemical physics. The above proposition says that IDEs can be seen as discrete equations plus probabilistic sampling. Hence, also for IDEs extended prediction is possible without using the notion of infinitesimals, along the same lines as discussed in Sec. 3.

The view that IDEs and ODEs all constitute discrete-time equations of the type (2) plus some form of probabilistic embedding in continuous time, has also interesting consequences for qualitative reasoning. Qualitative simulation abstracts the continuous time axis such that it does not know anything about the *duration* of the qualitatively steady intervals. Only the *fact that* they occur and their *ordering* may be deduced by qualitative simulation. Consequently, it is not visible for qualitative simulators such as QSIM what the precise locations of the distinguished time points or their relative spacings are on the continuous time axis. By implication, it is immaterial to qualitative simulation how the discrete steps S are embedded in continuous time; we may locally ‘stretch’ or ‘shrink’ the time axis without changing the qualitative simulation results. The different ways of probabilistic embedding discussed above only differ in the way they execute this positioning of discrete events in continuous time. In other words, qualitative simulation cannot distinguish between different shapes of the memory function \mathcal{M} . This leads us to the following conclusion.

Proposition 7 *Qualitative simulation cannot distinguish between ODEs (1) and IDEs (10). They fall into the same equivalence class of abstraction to qualitative differential equations.*

This has two consequences. First, not one (the ODE) but many (the IDEs) differential equations are abstracted to the same qualitative differential equation. Therefore, qualita-

⁵The relationship between the memory kernel \mathcal{M} and the event spacing distribution φ is given by $\mathcal{M}^* = u/[1/\varphi^* - 1]$ or, equivalently, $\varphi^* = [u/\mathcal{M}^* + 1]^{-1}$. Here, u is the complex frequency (the variable conjugated to time) and the asterisk denotes the Laplace transform.

tive simulation is much more coarse than previously thought. Second, it has become clear how to qualitatively simulate a new class of dynamic systems, *viz.* the IDEs, namely, via replacing them by the corresponding ODE.

Illustration 3: nuclear reactions. If in illustration 1 about radioactive decay we would postulate *correlated* decay events, that is, a non-Poisson distribution \mathcal{P} or a non-exponential φ , we would obtain an IDE (10) in continuous time. It is not difficult (*e.g.*, for φ being a Dirac delta function) to verify that the qualitative behavior as discussed in illustration 2 remains the same (we add that this also applies to the example of the spring). Correlated events in this case are physically not really expected. However, in other nuclear domains (*cf.* [12] and references therein) there are recent claims that correlated events actually give a better fit to experiment, for example in the field of heavy-ion collisions. Here, $x_i(t)$ is read as the probability that a heavy ion has mass i , S represents the subsequent exchange events of protons and neutrons between the colliding heavy ions, and the memory kernel \mathcal{M} may be taken to be exponential or Gaussian.

6 Conclusion

The dynamics of physical systems may be expressed in terms of both discrete time and continuous time. Theoretical physics typically prefers real continuous time, as exemplified by its use of differential equations. On the other hand, modeling in discrete time (in the structure of the integers) is of great computational interest. This work has studied the question how these two conceptions of time are related in reasoning about dynamical systems.

The major results of this paper are:

- Efficient extended prediction of differential equation systems is possible, since differential equations have been proved to be equivalent to discrete dynamics plus probabilistic sampling of the continuous time line. This new approach completely avoids the notion of infinitesimal time steps and differentials.
- The machinery of qualitative simulation of differential dynamics has been shown to carry over to discrete-time systems with minor adjustments. With the possible exception of sets of measure zero on the time axis, qualitative simulation yields the same results for differential equations and their discrete-time common-sense analogues.
- Qualitative differential equations have been proved to be abstractions not only of ordinary differential equations, but also of integro-differential equations that contain a memory function and that naturally occur in statistical and quantum mechanics. In other words, the equivalence class of qualitative abstraction is much larger than

previously thought. At the same time this result shows how to carry out qualitative simulation of integro-differential equations.

These results are illustrated by examples from recent literature on nuclear physics. Thus, our approach of introducing probabilistic embeddings of discrete point events in continuous time proves very fruitful in developing a formal theory of the quantization of time in dynamic physical systems.

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