Qualitative modelling of continuous-variable systems by means of nondeterministic automata*

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Abstract

The paper considers the problem of qualitative modelling of discrete-time continuous-variable dynamical systems, for which only a quantised measurement $[\mathbf{x}(k)]$ of the state $\mathbf{x}(k)$ is available. The qualitative model has to describe the qualitative trajectory $\mathbf{x}(1), \mathbf{x}(2), \ldots$ for given qualitative initial state $\mathbf{x}(0)$ and qualitative input sequence. First, it is shown that the qualitative trajectory of the system is ambiguous. Hence, the qualitative model has to be nondeterministic. Second, it is shown that nondeterministic automata provide reasonable qualitative models of the continuous-variable system. The relation between the automaton and the given system makes obvious which knowledge about the system has to be available if the qualitative model should be set up. Third, it is proposed to use stochastic automata which provide means for weighting each state concerning its appearance on the qualitative trajectory of the continuous-variable system. On this basis, the set of spurious solutions, which exist for any qualitative model, can be reduced. The appropriateness of the model becomes obvious by designing a qualitative controller. The results are illustrated by the problem of stabilising an 'inverted pendulum'.

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1 Introduction

Qualitative modelling of dynamical systems has been dealt with in two separate lines of research. Artificial intelligence approaches have the primary aim of modelling human understanding and reasoning about physical systems (for a survey cf [13]). Typically, rough characterisations $[\mathrm{x}(k)]$ are used as qualitative description of the system state x at time step k. For example, for deKleer and Brown [1] $[\mathrm{x}(k)]$ denotes the sign of $\mathrm{x}(k)$, whereas for Kuipers [4] $[\mathrm{x}(k)]$ refers to a large but quantitatively unspecified interval.

On the other hand, qualitative modelling and analysis has been a topic of active research in systems and control theory for many years. Graph-theoretic analysis of dynamical systems [12], uncertain systems modelling and robust control [5], or the qualitative analysis of interconnected systems [10], [9] has been fields where qualitative rather than quantitative results are the main motivation of research. These investigations try to follow the way an control expert goes. Experienced control engineers are able to solve their control tasks even if many details of the system dynamics are not known or deliberately neglected, because the knowledge about the principal behavioural patterns such as the existence of oscillations, saturation effects, or limit cycles or about the current output of the process in terms of subsets of the state space rather than accurate quantitative values are sufficient for many control purposes.

Although motivated by different aims and developed independently, both lines of research have considerable similarities. In [6] it has been shown that the relation between quantitative and qualitative models can be described by an abstraction operator. On this basis, results from both the artificial intelligence approach and systems theory can be presented in a unified framework. The methods developed in these fields differ, virtually, in the abstraction operator.

One of the main problems in qualitative modelling is the conservatism of the results. Even for simple examples such as the mass—spring system, qualitative models yield a large set of trajectories. Although it can be proved that this set includes the qualitative description of the real system trajectory, this set does include also many behavioural forms that no physically real dynamical system can perform (*spurious solutions*). The main reason for this is that the qualitative model is based on too less information about the real system, because the quantity spaces used are too coarse.

To circumvent this situation is the motivation of this paper. A new form of qualitative models is proposed, which is capable of including more information about the system. This new kind of qualitative models has the form of nondeterministic or stochastic automata.

In more detail, it is assumed that the qualitative value $[\mathrm{x}(k)]$ of the system state $\mathbf{x}(k)$ is received by means of a directionwise quantiser (Section 2). As shown in [7], for a qualitatively given initial state $[\mathbf{x}(0)]$ the qualitative system trajectory is ambigous. This result is reviewed in Section 3 and extended to systems with inputs on autonomous systems. As a consequence of the ambiguities of the system performance, nondeterministic and stochastic automata are proposed as reasonable forms of qualitative models (Section 4). These kind of models can be used to analyse the qualitative behaviour of the system and, moreover, to design a qualitative controller. The control problem will be solved in Section 5. The example of the pole balancing problem explained in Section 6 demonstrates that the conservatism of the qualitative model is so low that the resulting qualitative controller does stabilise the real unstable system.

This paper follows the view on qualitative modelling and control which has been proposed in [2] and [7]. It extends the automata–theoretic model that has been described in [7] for autonomous systems to systems with inputs. Another generalisation of [7] concerns the quantisation, which has been assumed to be equidistant in [7] but has a more general form here. In addition to that, by solving a typical regulator problem with the help of this new kind of qualitative model, it will be shown that the model includes enough information about the system behaviour not only for analysis but also for design purposes.

Notations. $\{a, b, c, \dots\}$ denotes a set of unordered elements $a, b, c, \dots$, whereas $(a, b, c, \dots)$ is the description of an n-tupel with ordered elements. Hence, if $x_1, x_2, \dots, x_n$ are scalars and $\mathbf{u}_1, \dots, \mathbf{u}_m$ vectors, $(x_1, x_2, \dots, x_n)$ denotes an n-row vector and $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ a matrix with m columns, respectively. In general, vectors are denoted by boldface lower case letters, eg $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, matrices by boldface upper–case letters, eg $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and scalars by lower–case italics, eg x_i, z_j, g_{ij}. $\mathbf{R}$ and $\mathbf{Z}$ are the sets of real or integer numbers.

2 Continuous-variable systems with quantised state measurements

Consider the linear discrete-time continuous-variable system

$$\mathbf{x}(k+1) = \mathbf{A}\ \mathbf{x}(k) + \mathbf{B}\ \mathbf{u}(k), \quad \mathbf{x}(0) = \mathbf{x}_o \tag{1}$$

where

$$\mathbf{x} = (x_1,\ x_2, \dots, x_n)' \quad \text{and} \quad \mathbf{u} = (u_1,\ u_2, \dots, u_m)'$$

denote the vectors of the systems state or input variables, respectively. The prime is the symbol for vector transposition. $\mathbf{A}$ and $\mathbf{B}$ are matrices of appropriate dimensions with constant elements. For every given input sequence

$$\mathrm{U} = (\mathbf{u}(0),\ \mathbf{u}(1), \ldots, \mathbf{u}(T-1))$$

with fixed observation horizon T the system (1) has the trajectory

$$\underline{\mathrm{X}}(\mathbf{x}(0),\ \mathbf{U}) = (\mathbf{x}(0),\ \mathbf{x}(1), \ldots, \mathbf{x}(T)) \tag{2}$$

where

$$\mathbf{x}(k) = \mathbf{A}^k\ \mathbf{x}(0) + \sum_{l=0}^{k-1}\ \mathbf{A}^{k-l-1}\ \mathbf{B}\ \mathbf{u}(l) \tag{3}$$

holds.

Now, it is assumed that the state $\mathbf{x}(k)$ cannot be measured quantitatively but is quantised by a directionwise quantiser that maps the state variables into a set of intervals. The intervals are bounded by given values

$$g_{ij} \quad (i = 1, 2, \ldots, n; \quad j = f_i^-, f_i^- + 1, \ldots, f_i^+)$$

and defined independently for all components x_i of the state vector $\mathbf{x}$. Hence, the state variables x_i that belong to the same set

$$Q_{x_i}(z_i) = \{x_i \mid g_{i,z_i} \leq x_i < g_{i,z_i+1}\} \tag{4}$$

are qualitatively equivalent and represented by the same qualitative value $[x_i(k)] = z_i$. The sets $Q_{x_i}(z_i)$ are defined for

$$z_i \in \{f_i^-, f_i + 1, \ldots, f_i^+ - 1\}.$$

The quantised state vector $[\mathbf{x}(k)]$ is given by

$$[\mathbf{x}(k)] = ([x_1(k)],\ [x_2(k)], \ldots, [x_n(k)])' \tag{5}$$

where

$$[x_i(k)] = z_i \quad \text{holds if and only if} \quad x_i(k) \in Q_{x_i}(z_i).$$

The set $Q_{x_i}(z_i)$ defined in eqn (4) can be written as

$$Q_{x_i}(z_i) = \{x_i \mid [x_i] = z_i\}.$$

For all

$$\mathbf{z} \in \mathbf{Z}_x = \{(z_1, z_2, \ldots, z_n)' \mid z_i \in \{f_i^-, f_i + 1, \ldots, f_i^+ - 1\}\} \subseteq \mathbf{Z}^n \tag{6}$$

$Q_x(\mathbf{z})$ can be defined by

$$Q_x(\mathbf{z}) = Q_{x_1}(z_1) \otimes Q_{x_2}(z_2) \otimes \ldots \otimes Q_{x_n}(z_n) = \{\mathbf{x} \mid [\mathbf{x}] = \mathbf{z}\} \tag{7}$$

with $\otimes$ denoting the Cartesian product. All these sets together cover the subspace $\mathbf{R}_x \subseteq \mathbf{R}^n$

$$\mathbf{R}_x = \cup_{\mathbf{z} \,\varepsilon\, \mathbf{Z}_x} Q_x(\mathbf{z}). \tag{8}$$

For

$$g_{i,f_i^-} = -\infty \quad \text{and} \quad g_{i,f_i^+} = \infty \tag{9}$$

the whole state space $\mathbf{R}^n$ is partitioned into the sets $Q_x(\mathbf{z})$:

$$\mathbf{R} = \cup_{\mathbf{z} \,\varepsilon\, \mathbf{Z}_x} Q_x(\mathbf{z}). \tag{10}$$

In the case of equidistant intervals with interval length q_{x_i}

$$g_{ij} = (j - \frac{1}{2})q_{x_i}, \quad j \,\varepsilon\, \mathbf{Z} \tag{11}$$

holds. Then

$$\mathbf{Z}_x = \mathbf{Z}^n \tag{12}$$

and

$$\mathbf{R}^n = \cup_{\mathbf{z} \,\varepsilon\, \mathbf{Z}^n} \quad Q_x(\mathbf{z})$$

follow.

It is further assumed that the inputs u_i can assume one of a set of given values u_i^j

$$u_i(k) \,\varepsilon\, \mathbf{Z}_{u_i} = \{u_i^j \mid j = g_i^-, g_i^- + 1, \ldots, g_i^+\}. \tag{13}$$

Therefore, the element j of $\mathbf{Z}_{u_i}$ is the qualitative value of $u_i(k)$, ie

$$[\mathbf{u}(k)] = ([u_1(k)],\ [u_2(k)], \ldots, [u_m(k)])' \tag{14}$$

where

$$[u_i(k)] = j \quad \text{holds if and only if} \quad u_i(k) = u_i^j.$$

Hence, $[\mathbf{u}(k)]$ belongs to the set $\mathbf{Z}_u$:

$$\mathbf{z} \,\varepsilon\, \mathbf{Z}_u = \{(z_1, z_2, \ldots, z_m)' \mid z_i \,\varepsilon\, \{g_i^-, g_i^- + 1, \ldots, g_i^+\}\} \subseteq \mathbf{Z}^n. \tag{15}$$

The qualitative input sequence $[\mathbf{U}]$ is represented by

$$[\mathbf{U}] = ([\mathbf{u}(0)], [\mathbf{u}(1)], \ldots, [\mathbf{u}(T-1)]).$$

The qualitative trajectory of the system (1) is given by

$$[\mathbf{X}(\mathbf{x}(0),\ \mathbf{U})] = ([\mathbf{x}(0)],\ [\mathbf{x}(1)], \ldots, [\mathbf{x}(T)]). \tag{16}$$

Obviously, for a given initial state $\mathbf{x}(0)$ the system (1) has a unique qualitative trajectory $[\mathbf{X}]$.

3 Nondeterminism of the qualitative behaviour

For the qualitative model, only the qualitative values $[\mathbf{x}(k)]$ and $[\mathbf{u}(k)]$ are relevant. The initial state $\mathbf{x}(0)$ is only known to belong to the set $Q_x(\mathbf{z}(0))$ for some given $\mathbf{z}(0)$:

$$\mathbf{x}(0) \,\varepsilon\, Q_x(\mathbf{z}(0)) \tag{17}$$

The system input is described by some qualitative input sequence

$$\mathbf{V} = (\mathbf{v}(0), \mathbf{v}(1), \ldots, \mathbf{v}(T-1))$$

where

$$[\mathbf{u}(k)] = \mathbf{v}(k) \quad (k = 0, 1, 2, \ldots, T-1) \tag{18}$$

holds. Therefore, the system (1) can perform any trajectory that starts from some $\mathbf{x}(0)$ given in eqn (17) under the control sequence described by $\mathbf{V}$ and eqn (18). These quantitative trajectories form the set

$$\tilde{\mathbf{X}}(\mathbf{z}(0), \mathbf{V}) = \{\mathbf{X}(\mathbf{x}(0), \mathbf{U}) \mid \mathbf{x}(0) \,\varepsilon\, Q_x(\mathbf{z}(0)),\ [\mathbf{U}] = \mathbf{V}\}. \tag{19}$$

The model has to generate the qualitative trajectories that result from the set $\tilde{\mathbf{X}}$ and form the set $[\tilde{\mathbf{X}}]$:

$$[\tilde{\mathbf{X}}(\mathbf{z}(0), \mathbf{V})] = \{[\mathbf{X}] \mid \mathbf{X} \,\varepsilon\, \tilde{\mathbf{X}}\}. \tag{20}$$

It has been shown in [7] for autonomous systems (eqn (1) with $\mathbf{u}(k) = \mathbf{0}$) and equidistant quantisation (cf eqn (11)) that the set $[\tilde{\mathbf{X}}]$ is, in general, not a singleton but has more than one element. In order to extend this result to the class of systems (1) considered here, the sets

$$\begin{aligned}
M_x(0) &= Q_x(\mathbf{z}(0)) & (21)\\
M_x(k) &= \{\mathbf{A}^k\, \mathbf{x}(0) + \sum_{l=0}^{k-1} \mathbf{A}^{k-l-1}\, \mathbf{B}\, \mathbf{u}(l) \quad | \quad \mathbf{x}(0) \,\varepsilon\, Q_x(\mathbf{z}(0)), \\
&\quad\ [\mathbf{u}(l)] = \mathbf{v}(l) \quad (l = 0, 1, \ldots, k-1)\} \\
&= \{\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad | \quad \mathbf{x} \,\varepsilon M_x(k-1),\ [\mathbf{u}] = \mathbf{v}(k)\}. & (22)
\end{aligned}$$

are defined and qualitatively described by

$$[M_x(k)] = \{[\mathbf{x}] \mid \mathbf{x} \,\varepsilon\, M_x(k)\}. \tag{23}$$

Obviously, the system (1) has a unique qualitative trajectory if and only if

$$M_x(k) \subseteq Q_x(\mathbf{z}) \quad \text{for some} \quad \mathbf{z} \,\varepsilon\, \mathbf{Z}_x \tag{24}$$

holds for all $k = 0, 1, \ldots, T$.

Theorem 1 *The system (1) has for an arbitrarily given qualitative initial state* $\mathbf{z}(0)$ *and for an arbitrarily given input sequence* $\mathbf{V}$ *a unique qualitative trajectory if and only if eqn (24) holds for all* $k = 0, 1, \ldots, T$.

The following theorem shows that the conditions of Theorem 1 are, in general, not satisfied.

Theorem 2 [7] *Consider the system (1) for* $\mathbf{u}(k) = 0$ $(k = 0, 1, \ldots, T-1)$ *and equidistant quantisation as described by eqn (11). Assume that* $\det A \neq 0$ *holds and define*

$$\bar{\mathbf{A}} = \mathrm{diag}\frac{1}{q_{x_i}}\ \mathbf{A}\ \mathrm{diag} q_{x_i}.$$

The system (1) has for all qualitative initial states $\mathbf{z}(0) \in \mathbf{Z}_x$ *a unique qualitative trajectory* $[\tilde{\mathbf{X}}(\mathbf{z}(0),\ \mathbf{0})]$ *if and only if*

$$\bar{\mathbf{A}} = \mathrm{diag}\frac{1}{2n_i + 1}\ \mathbf{P}$$

holds where $\mathbf{P}$ *denotes a permutation matrix and* $n_i \in \mathbf{Z}$ *holds for* $i = 1, 2, \ldots, n$.

Obviously, an autonomous system (eqn (1) for $\mathbf{u} = \mathbf{0}$) has to satisfy a rather restrictive condition on the matrix $\mathbf{A}$. For systems with nonvanishing inputs, additional conditions have to be met. Hence, almost all systems (1) have a set of qualitative trajectories rather than a unique qualitative trajectory. This fact has severe consequences for qualitative modelling since it shows that any qualitative model has be nondeterministic. That is, for given qualitative initial state and qualitative input sequence the model has to generate a set of trajectories rather than a unique output sequence.

4 Qualitative modelling by means of nondeterministic or stochastic automata

4.1 Modelling by nondeterministic automata

The nondeterminism of the qualitative trajectory of the system (1) suggests to use a nondeterministic or stochastic automaton as qualitative model. First, the nondeterministic automaton $N(\mathbf{Z}_x, \mathbf{Z}_u, H, \mathbf{z}(0), \mathbf{V})$ is considered, where $\mathbf{Z}_x$ denotes the set of states, $\mathbf{Z}_u$ the set of inputs. $H : \mathbf{Z}_x \times \mathbf{Z}_u \to 2^{\mathbf{Z}_x}$ the transition relation, $\mathbf{z}(0)$ the initial state and $\mathbf{V}$ the input sequence. $2^{\mathbf{Z}_x}$ is the power set of $\mathbf{Z}_x$. $\mathbf{Z}_x$ and $\mathbf{Z}_u$ are the sets defined in eqns (6) or (15), respectively.

The transition function H has to be found so that the automaton generates the set $\tilde{\mathbf{X}}$ of qualitative trajectories of the system (1). In order to do this, the sets

$$\begin{aligned} M_z(0) &= \{\mathbf{z}(0)\} && (25) \\ M_z(k+1) &= \{H(\tilde{\mathbf{z}}, \mathbf{v}(k)) \mid \tilde{\mathbf{z}} \in M_z(k), \quad (k = 0, 1, 2, \ldots, T-1)\} && (26) \end{aligned}$$

are introduced. Then the set $\tilde{\mathbf{Z}}(\mathbf{z}(0), \mathbf{V})$ of all trajectories of the nondeterministic automaton is given by

$$\tilde{\mathbf{Z}}(\mathbf{z}(0), \mathbf{V}) = M_z(0) \otimes M_z(1) \otimes \ldots \otimes M_z(T). \tag{27}$$

Obviously, the automaton N is a nice qualitative model of the system (1) if for arbitrary

$$\mathbf{z}(0) \quad \text{and} \quad \mathbf{V}$$

the relation

$$\tilde{\mathbf{Z}}(\mathbf{z}(0), \mathbf{V}) \supseteq [\tilde{\mathbf{X}}(\mathbf{z}(0), \mathbf{V})] \tag{28}$$

holds. Then, the automaton generates all qualitative trajectories of the system (1). As an extension of a result of [7], which concerns autonomous systems, the following theorem can be proved analogously to [7].

Theorem 3 *For arbitrarily given initial state* $\mathbf{x}(0)$ *and input sequence* $\mathbf{U}$, *for the nondeterministic automaton* $N(\mathbf{Z}_x, \mathbf{Z}_u, H, [\mathbf{x}(0)], [\mathbf{U}])$ *the relations*

$$M_z(k) \supseteq [M_x(k)] \tag{29}$$

and (28) hold if and only if the transition function H *satisfies the relation*

$$H(\mathbf{z}, \mathbf{v}) \supseteq \{[\mathbf{x}] \mid \mathbf{x} \in D_o(\mathbf{z}, \mathbf{v})\} \tag{30}$$

with

$$D_o(\mathbf{z}, \mathbf{v}) = \{\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \mid \mathbf{x} \in Q_x(\mathbf{z}),\ [\mathbf{u}] = \mathbf{v}\}. \tag{31}$$

If the $\supseteq$ -sign in eqn (30) is replaced by '=', the function H generates the smallest sets $H(\mathbf{z}, \mathbf{v})$. Then the smallest sets $M_z(k)$ and $\tilde{\mathbf{Z}}(\mathbf{z}(0), \mathbf{V})$ are obtained. The corresponding sets and the automaton N is marked by an asterisk: $H^*, N^*(\mathbf{Z}_x, \mathbf{Z}_u, H^*, [\mathbf{x}(0)], [\mathbf{U}]), M_z^*(k)$ and $\tilde{\mathbf{Z}}^*(\mathbf{z}(0), \mathbf{V})$.

The nondeterministic automaton N^* generates the smallest set of spurious solutions. This set is given by

$$T(\mathbf{z}(0), \mathbf{V}) = \tilde{\mathbf{Z}}^*(\mathbf{z}(0), \mathbf{V}) \setminus [\tilde{\mathbf{X}}(\mathbf{z}(0), \mathbf{V})]. \tag{32}$$

The reason why, in general, the transition function H^* cannot be chosen so that the set T is empty is based on the Markov property that the nondeterministic automaton possesses. Accordingly, the set $M_z(k)$ depends merely on the set $M_z(k-1)$ but not on the automaton states at time instances earlier than $k-1$ (cf eqn (26)). On the other hand, the set $[M_x(k)]$ cannot be uniquely determined from the set $[M_x(k-1)]$. Hence, the automaton can, in general, not generate exactly the same qualitative trajectories as the system (1).

Theorem 3 has an interesting impact on knowledge acquisition. The theorem describes the relation between the qualitative model N and the real system (1). Accordingly, a qualitative model can be found if it is known which set $D_o(\mathrm{z}, \mathrm{v})$ of states will follow at time $k+1$ the state z at time k and the qualitative input v. D_o can be determined by means of eqn (31) if the model (1) is known. On the other hand, this set can also be found from experience with the system behaviour. Eqn (30) shows that a reasonable qualitative model can be determined even if instead of D_o a superset $\tilde{D}_o$ of D_o is available. An operator that has experience with the qualitative behaviour of a given system does know all possible directions where the system can 'go'. Hence, he can describe the set $\tilde{D}_o$ and, thus, find a qualitative model that, according to eqn (28), does generate all qualitative trajectories of the system.

The more the operator knows the system, the better will be the set $\tilde{D}_o$, i.e. the smaller will be the difference between this set and D_o. Eventually, if the operator knows his system precisely, he is able to determine the best possible qualitative model N^*, which is the model with the least set of spurious solutions that can be found for the system under consideration.

This remark points to the fact that for dynamical systems the process of knowledge acquisition can be based on both the model (1) of the system or experience of an operator.

4.2 Modelling by stochastic automata

The qualitative description of the system (1) can be improved if the states $\mathrm{z} \in M_z(k)$ of the automaton are weighted concerning the probability with which that the system (1) does really assume these qualitative states z at time k. The weight is the probability $p_z(\mathrm{z}, k)$ with which the state $\mathrm{z} \in M_z(k)$ belongs to $[M_x(k)]$.

This weighting is based on the following consideration of the system (1) for qualitatively given inital state and input sequence. The system (1) is started from different initial states $x(0)$ that satisfy the relation (17), and the trajectories are observed. Since $\mathrm{x}(0)$ is only known to belong to the set $Q_x(\mathrm{z}(0))$ it is reasonable to assume that the initial states $\mathrm{x}(0)$ used in these experiments are uniformly distributed over $Q_x(\mathrm{z}(0))$. Then,

$$p_z(\mathrm{z}, k) = \mathrm{Prob}\{[\mathrm{x}(k)] = \mathrm{z}\}$$

is the probability of the state x to have the qualitative value z.

This is the reason for using the stochastic automaton $S(\mathbf{Z}_x, \mathbf{Z}_u, P, \mathrm{z}(0), \mathrm{V})$ as qualitative model where

$$P : \mathbf{Z}_x \times \mathbf{Z}_x \times \mathbf{Z}_u \to \mathbf{R}$$

represents the transition probability function. $P(\mathrm{z}, \tilde{\mathrm{z}}, \mathrm{v})$ is the probability that the automaton that has at time k the state z and gets the input v goes to the state $\tilde{\mathrm{z}}$ at time $k+1$.

In analogy to the nondeterministic automaton, the following sets are introduced:

$$\begin{aligned} H_S(\mathrm{z}, \mathrm{v}) &= \{\tilde{\mathrm{z}} \mid P(\mathrm{z}, \tilde{\mathrm{z}}, \mathrm{v}) \neq 0\} && (33) \\ M_{Sz}(0) &= \{\mathrm{z}(0)\} \\ M_{Sz}(k+1) &= \{H_S(\mathrm{z}, \mathrm{v}(k)) \mid \mathrm{z} \in M_{Sz}(k)\}, \quad (k = 0, 1, \ldots, T-1). && (34) \end{aligned}$$

The performance of the stochastic automaton is described by the probability $t(\mathrm{z}, k)$ with which the automaton is at time k in state z. The following relations hold for the automaton that has the initial state $\mathrm{z}(0) = \mathrm{z}_0$:

$$\begin{aligned} t(\mathrm{z}_0, 0) &= 1 \quad \text{and} \quad t(\mathrm{z}, 0) = 0 \quad \text{for all} \quad \mathrm{z} \neq \mathrm{z}_0 \\ t(\mathrm{z}, k+1) &= \sum_{\tilde{\mathrm{z}} \in \mathbf{Z}_x} P(\mathrm{z}, \tilde{\mathrm{z}}, \mathrm{v})\, t(\tilde{\mathrm{z}}, k) \quad (k = 0, 1, \ldots, T-1). && (35) \end{aligned}$$

Obviously,

$$M_{Sz}(k) = \{\mathrm{z} \mid t(\mathrm{z}, k) \neq 0\}$$

holds. The set of trajectories of the stochastic automaton is given, analogously to eqn (27), by

$$\tilde{\mathrm{Z}}_S(\mathrm{z}(0), \mathrm{V}) = M_{Sz}(0) \otimes M_{Sz}(1) \otimes \ldots \otimes M_{Sz}(T). \tag{36}$$

The following theorem is an extension of the results on autonomous systems given in [7] to the broader class of systems (1) considered here:

Theorem 4 *For arbitrarily given initial state* $\mathrm{x}(0)$ *and input sequence* U, *for the stochastic automaton* $S(\mathbf{Z}_x, \mathbf{Z}_u, P, [\mathrm{x}(0)], [\mathrm{U}])$ *the relations*

$$M_{Sz}(k) \supseteq [M_x(k)] \tag{37}$$

and

$$\tilde{\mathrm{Z}}_S(\mathrm{z}(0), \mathrm{V}) \supseteq [\tilde{\mathrm{X}}(\mathrm{z}(0), \mathrm{V})] \tag{38}$$

hold if and only if the transition probability function P satisfies the relation

$$P(\tilde{\mathrm{z}}, \mathrm{z}, \mathrm{v}) = \frac{\int_{D(\tilde{\mathrm{z}},\mathrm{z},\mathrm{v})} dx}{\int_{D_o(\mathrm{z},\mathrm{v})} dx} \tag{39}$$

with

$$D(\tilde{\mathrm{z}}, \mathrm{z}, \mathrm{v}) = D_o(\mathrm{z}, \mathrm{v}) \cap Q_x(\tilde{\mathrm{z}}) \tag{40}$$

and D_o as in eqn (31).

For this stochastic automaton, $t(\mathrm{z}, k)$ is an approximation of $p_z(\mathrm{z}, k)$.

The best nondeterministic automaton N^* and the stochastic automaton described in the theorem above have the same sets of trajectories because the following relations can be proved:

$$\begin{aligned} H^*(\mathrm{z}, \mathrm{v}) &= H_S(\mathrm{z}, \mathrm{v}) \quad \text{for all} \quad \mathrm{z}, \mathrm{v} \\ \tilde{\mathbf{Z}}^*(\mathrm{z}(0), \mathbf{V}) &= \tilde{\mathbf{Z}}_S(\mathrm{z}(0), \mathbf{V}) \quad \text{for all} \quad \mathrm{z}(0), \mathbf{V}. \end{aligned}$$

However, the stochastic automaton yields a better characterisation of the qualitative performance of the system (1) since it generates together with each set $M_{Sz}(k)$ a weighting function $t(\mathbf{z}, k)$ that describes the probability of the state $\mathbf{z} \varepsilon M_{Sz}(k)$ to be really assumed by the system (1).

The additional characterisation of the states of the qualitative model by the probability $t(\mathbf{z}, k)$ makes it possible to reduce the set $\tilde{\mathbf{Z}}_S(\mathbf{z}(0), \mathbf{V})$. If $t(\mathbf{z}, k)$ has a low value the state $\mathbf{z}$ can be assumed to belong not to the qualitative trajectory of the system (1) but to spurious solutions. Therefore, such states can be deleted. For a given threshold s only those states $\mathbf{z}$ for which

$$t(\mathbf{z}, k) > s$$

holds are used for determining the set $\tilde{\mathbf{Z}}_S(\mathbf{z}(0), \mathbf{V})$ of qualitative trajectories according to eqn (36).

A reasonable representation of the stochastic automaton is obtained if the probabilities $t(\mathrm{z}, k)$ of all states $\mathrm{z} \; \varepsilon \; \mathbf{Z}_x$ are written in the vector

$$\mathrm{t}(k) = (t(\mathrm{z}_1, k),\ t(\mathrm{z}_2, k), \ldots t(\mathrm{z}_q, k))'$$

where q is the number of elements of $\mathbf{Z}_x$. Then the performance of the stochastic automaton is represented by

$$\mathrm{t}(k+1) = \mathrm{P}(\mathrm{v}(k))\, \mathrm{t}(k), \quad (k = 0, 1, \ldots, T-1) \tag{41}$$

where the matrix $\mathrm{P}(\mathrm{v}(k))$ represents the probability function P for given input $\mathrm{v}(k)$.

5 Qualitative control

The qualitative model proposed in the preceding sections can be used in qualitative analysis and simulation. An example is given in [7] where the free motion of an oscillator has been qualitatively described by a nondeterministic and a stochastic automaton. Numerical examples have demonstrated that the automata represent reasonable qualitative models because the set of spurious solutions is very small.

The aim of this section is to go a step further. A method will be proposed for designing a qualitative feedback controller that stabilises an unstable system (1) although the state variables cannot be measured quantitatively. The possibility to use the qualitative model for control purposes shows that the qualitative model is really a powerful means for dealing with dynamical systems.

The following is a summary of a method for qualitative controller design which has been proposed in [8]. The main idea of this method is the following. It is assumed, without loss of generality, that the equilibrium state of the system (1) is given by $\mathbf{x} = \mathbf{0}$ and $\mathbf{z} = [\mathbf{x}] = \mathbf{0}$. Therefore, the aim of stabilising the system is to find a qualitative controller

$$[\mathbf{u}(k)] = \mathbf{f}([\mathbf{x}(k)]) \tag{42}$$

that moves the system into the equilibrium state. Since only the qualitative state $[\mathbf{x}]$ is available, the system cannot be asymptotically stabilised as it would be possible with some quantitative controller

$$\mathbf{u}(k) = \mathbf{f}(\mathbf{x}(k))$$

[11]. Therefore, the control aim is to hold the system in the surroundings of the equilibrium state. The control law $\mathbf{f}$ has to be chosen so that the probability $p_z(\mathbf{0}, k)$ of the stochastic automaton in connection with the control sequence that results from eqn (42) is maximised.

The control law can be determined by means of the qualitative model if the controller (42) is applied to the model in the form

$$\mathbf{v}(k) = \mathbf{f}(\mathbf{z}(k)) \tag{43}$$

where the same control law $\mathbf{f}$ is used as in eqn (42). Then, $\mathbf{V} = \mathbf{V}_f = (\mathbf{f}(\mathbf{z}(0)), \mathbf{f}(\mathbf{z}(1)), \ldots, \mathbf{f}(\mathbf{z}(T-1))$ holds and the qualitative model of the closed loop system is given by $S_f(\mathbf{Z}_x, \mathbf{Z}_u, \mathbf{P}_f, [\mathbf{X}(0)], \mathbf{0})$ with

$$\mathbf{P}_f = (p_{f_{ij}}) \ , \quad p_{f_{ij}} = p_{ij}(\mathbf{f}(\mathbf{z}_j))$$

where $p_{ij}(\mathbf{v}_k)$ is the ij–element of the matrix $\mathbf{P}(\mathbf{v}_k)$, cf. eqn (41). Then the aim is to maximise the probability $t(\mathbf{0}, k)$, which can be determined by means of the model S_f. The ultimate aim is to obtain a stable closed loop system S_f. That is, for any given initial state $\mathbf{z}(0)$ the model should eventually reach the equilibrium state $\mathbf{z} = \mathbf{0}$, i.e.

$$t(0, k) \longrightarrow 1 \quad \text{as} \quad k \to \infty\,. \tag{44}$$

However, the qualitative model S has spurious solutions and so has the closed loop model S_f. Hence, the control aim (44) may not be satisfied even if the original aim

$$p_z(0, k) \longrightarrow 1 \quad \text{as} \quad k \to \infty$$

is reached. Therefore, the following design method is used in which the aim (44) is replaced by maximizing $t(0, k)$ at time $k = 0, 1, 2, \ldots$

For $k = 1$ the task is

$$\max\ t(0, 1) = \max\ (\mathbf{P}(\mathbf{f}(\mathbf{z}_i))_0\ \mathbf{e}_i \tag{45}$$

where $\mathbf{e}_i$ denotes a q-vector with vanishing elements besides a '1' in the i-th row. The symbol $(\ )_0$ denotes the row of the given matrix that belongs to the equilibrium state $\mathbf{z} = \mathbf{0}$. By means of eqn (45) the function $\mathbf{f}$ can be chosen for all states $\mathbf{z}$ for which there is a nonvanishing entry in the row $(\mathbf{P}(\mathbf{f}(\mathbf{z}_i))_0$. The resulting partial control law is denoted by $\mathbf{f}^1$. It is defined for those states of the model S, from which the equilibrium state $\mathbf{0}$ can be reached in one time step. The control law $\mathbf{f}^1$ ensures that the model reaches the equilibrium state with maximum probability.

For other states, $\mathbf{f}$ can be chosen if $k = 2$ is considered. Then the task is given by

$$\max\ t(0, 2) = \max\ (\mathbf{P}^2(\mathbf{f}(\mathbf{z}_i))_0\ \mathbf{e}_i. \tag{46}$$

The solution yields the function $\mathbf{f}$ for all states from which the equilibrium state can be reached in no less than two time steps. The control law ensures for these states that the model reaches its equilibrium state with maximum probability.

If necessary, further time steps have to be considered in order to get the complete control law $\mathbf{f}$. This method for qualitative control will be demonstrated by an example in the following section.

6 Example: The pole balancing problem

The following application study demonstrates the methods of qualitative modelling and control that have been proposed in this paper. The problem is to stabilise an 'inverted pendulum' (Fig. 1) by pushing the vehicle reasonably to the left or to the right. This problem is a real benchmark for qualitative modelling because this stabilisation problem is really difficult. Experiences with this experiment show that the existence of a solution to this control problem depends strongly on the quality of the sensor signals. If the angle and the angle velocity can be measured precisely, a systems theoretic approach to the control problem is reasonable. Accordingly, a discrete-time model (1) is set up and the feedback controller

$$\mathbf{u}(k) = \mathbf{K}\mathbf{x}(k)$$

can be designed by well known methods [11].

However, this way of solution does not take into account the deterioration of the closed-loop system performance in case of bad sensor information. It is typical for this experiment that the angle and the angle velocity cannot be measured precisely but only with severe measurement errors. Then the following qualitative approach is reasonable where the quality of the sensor data can be explicitly taken into consideration by using appropriate quantisations of the angle and the angle velocity.

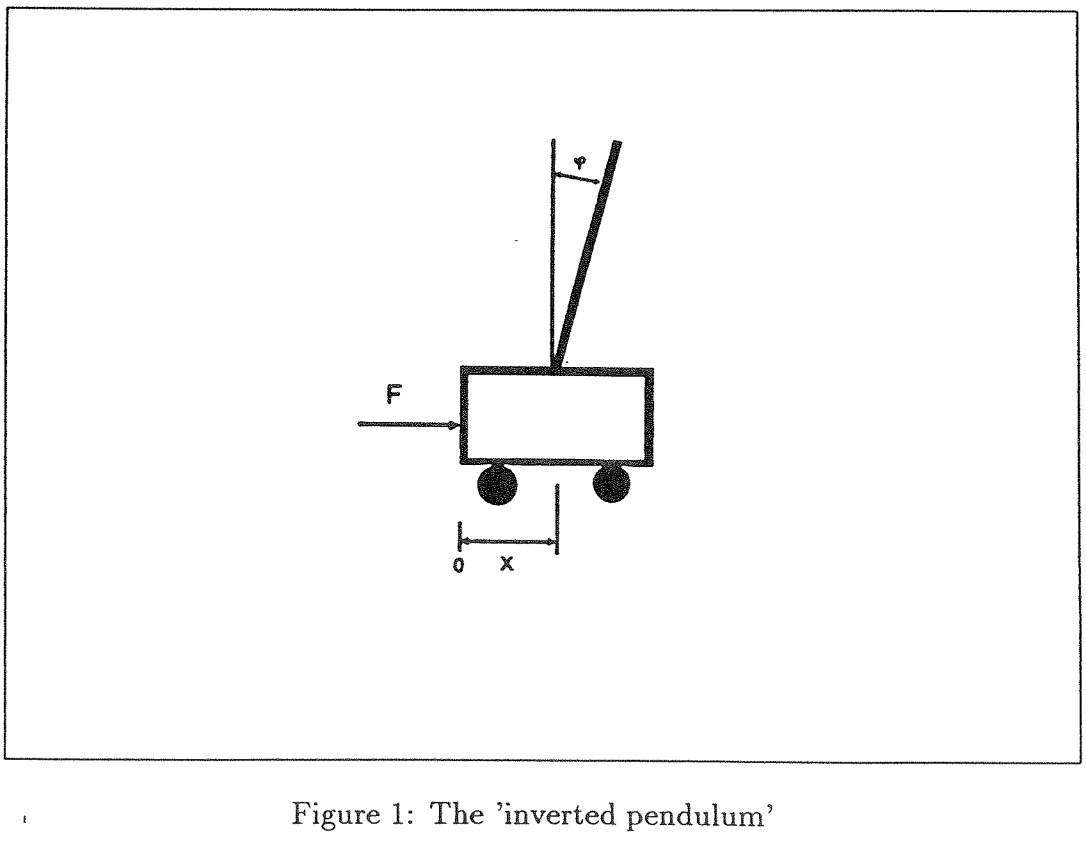

Figure 1: The 'inverted pendulum'

For the inverted pendulum shown in Fig. 1 the following model can be set up, cf [3]:

$$(m + m_c)x - ml\dot{\varphi}^2 \sin\varphi + ml\varphi\cos\varphi = 0$$

$$ml^2\ddot{\varphi} - mgl\sin\varphi + mlx\cos\varphi = 0.$$

In these equations the following signals and parameters have been used:

$F(k)$ - force on the vehicle (input signal)
$x(k)$ - position of the vehicle
$\varphi(k)$ - angle
m - mass of the pole
m_c - mass of the vehicle
l - length of the pole.

After linearisation of the equations around the equilibrium point $\varphi = 0$ and with $x, \dot{x}, \varphi$ and $\dot{\varphi}$ as entries of the state vector x and F as unique input $u(k)$, the following model is obtained:

$$\dot{\mathrm{x}}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{m_c} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(m+m_c)g}{m_c l} & 0 \end{pmatrix} \mathrm{x}(t) + \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \\ -\frac{1}{m_c l} \end{pmatrix} u(t). \tag{47}$$

With the parameter values

$$m_c = 1kg, \quad m = 0.1kg, \quad l = 0.5m, \quad g = 9.81\frac{m}{s^2}$$

and for the sampling time of $0.02s$ the following model of the form (1) results:

$$\mathrm{x}(k+1) = \begin{pmatrix} 1 & 0.02 & -0.002 & 0 \\ 0 & 1 & -0.0196 & -0.002 \\ 0 & 0 & 1.0043 & 0.02 \\ 0 & 0 & 0.4318 & 1.0043 \end{pmatrix} \mathrm{x}(k) + \begin{pmatrix} 0 \\ 0.2 \\ 0 \\ -0.4 \end{pmatrix} u(k). \tag{48}$$

A reasonable quantisation can be derived from the following arguments, which reflect practical experiences with the system. A simulation study shows that the pole can no longer be stabilised if the following bounds are exceeded

$$|x_3| > 0.21, \quad |x_4| > 0.87. \tag{49}$$

Due to these bounds and the measurement insensitivity of 0.0175 (1°) for φ and 0.0175 (1^o) per sampling period for $\dot{\varphi}$ the bounds g_{ij} are fixed:

$$g_{3,-1} = -0.210, \quad g_{3,0} = -0.0175, \quad g_{3,1} = 0.0175, \quad g_{3,2} = 0.210$$

$$g_{4,-1} = -0.870, \quad g_{4,0} = -0.0175, \quad g_{4,1} = 0.0175, \quad g_{4,2} = 0.870.$$

All states that satisfy at least one of the inequalities given in (49) are outside the 'working space' of the system. They belong to the same qualitative state z_{10}. The other states are consecutively numbered as follows:

$$\mathrm{z}_1 = (-1,-1)', \quad \mathrm{z}_2 = (-1,0)', \quad \mathrm{z}_3 = (-1,1)', \ \ldots, \ \mathrm{z}_9 = (1,1)'.$$

The input signal $u(k)$ has three qualitative levels:

$$\begin{aligned} u(k) = 10 &\iff v(k) = 1 \\ u(k) = 0 &\iff v(k) = 0 \\ u(k) = -10 &\iff v(k) = -1, \end{aligned}$$

ie the force can be chosen to be zero or maximum in both directions.

Under these assumptions, a stochastic automaton S can be found as described in Theorem 4, which is represented by eqn (41) with

$$\mathbf{P}(0) = 0.01 \begin{pmatrix} 94 & 81 & 2 & 23 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 19 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 91 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 75 & 5 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 90 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 5 & 75 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 91 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 19 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 23 & 2 & 81 & 94 & 0 \\ 6 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 6 & 100 \end{pmatrix},$$

$$\mathbf{P}(-1) = 0.01 \begin{pmatrix} 49 & 0 & 0 & 15 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 39 & 100 & 60 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 34 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 44 & 100 & 75 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 45 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 100 & 48 & 0 \\ 9 & 0 & 39 & 0 & 0 & 17 & 0 & 0 & 52 & 100 \end{pmatrix},$$

$$\mathbf{P}(1) = 0.01 \begin{pmatrix} 48 & 100 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 55 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 45 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 46 & 100 & 44 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 34 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 60 & 100 & 39 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 49 & 0 \\ 52 & 0 & 0 & 46 & 0 & 0 & 39 & 0 & 9 & 100 \end{pmatrix}.$$

The control aim is to stabilise the system in the state $\mathbf{z}_5 = (0,0)'$. The stabilising controller can be found by means of the procedure proposed in Section 5. The control law is given by the following table:

z(k) =	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9
u(k) =	-1	0	0	0	0	0	0	0	1

Note that for the implementation of this qualitative controller only a rought measurement of the angle and the angle velocity is necessary. The controller needs only information about the qualitative value of both signals. A further simplification is possible because the resulting control law is rather simple. The controller has only to known whether both the angle and the angle velocity have the same sign.

The simulation result given in Fig. 2 shows that this controller actually stabilises the inverted pendulum. The closed-loop system eventually reaches a cyclic trajectory, which can be compared with a limit cycle of a nonlinear system. In fact, the controller brings about a severe nonlinearity of the closed-loop system. Due to the roughness of the measurement, the equilibrium point cannot be approached asymptotically, but the system obviously remains in the near surrounding of this point.

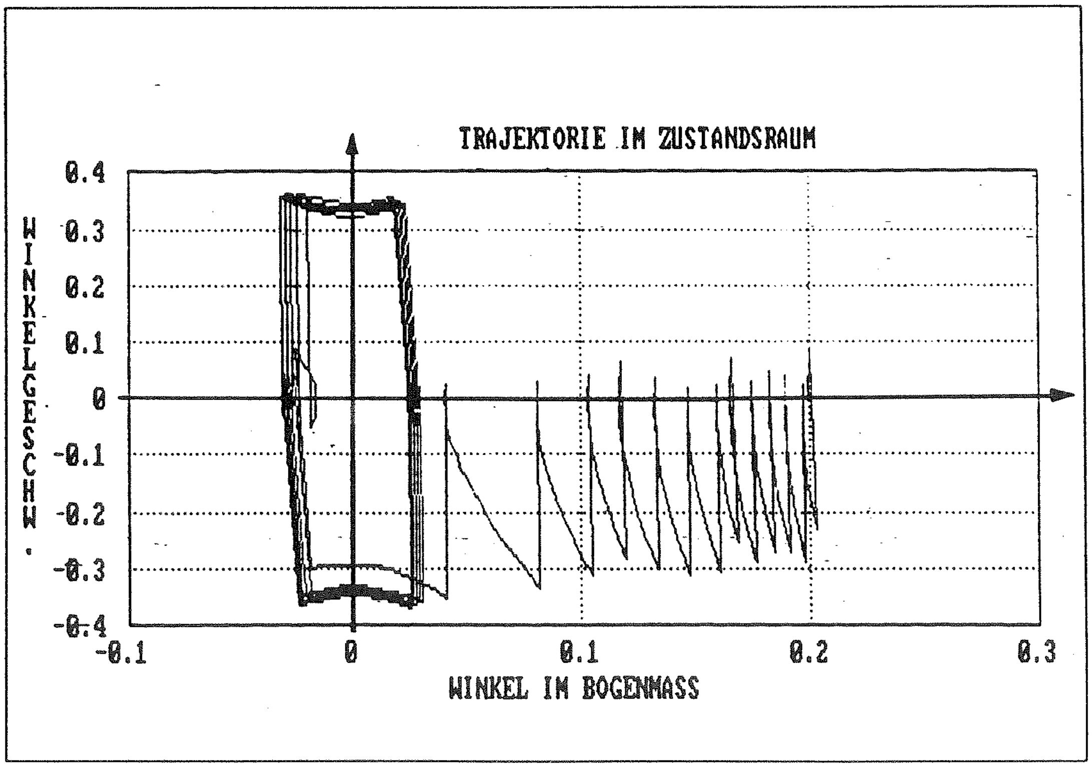

Figure 2: Trajectory of the 'inverted pendulum' with qualitative controller

7 Conclusions

The paper has proposed a new form of qualitative models, which is appropriate for systems whose state variables can be measured through a quantiser. As it became obvious in Section 3 that the qualitative trajectory of the system is, in general, ambiguous, nonlinear or stochastic automata provide a reasonable framework for qualitative modelling.

This new form of qualitative models makes it possible to use well known results on discrete event systems, which have been elaborated with emphasis to 'real' discrete event systems such as manufacturing systems or computer nets. These tools can be applied now to continuous-variable systems if these systems are considered on the qualitative level of abstraction rather than by means of quantitative models like (1).

As an interesting byproduct, the paper bridges the gap between the fields of continuous-variable and discrete event systems. Until now, both classes of systems have been investigated separately. The paper demonstrates that both forms of considerations can be applied simultaneously. Theorems 3 and 4 describe how automata have to be chosen if they should describe a continuous-variable system (1) qualitatively. With these results in mind, it should be investigated whether analysis and control methods that have been developed for 'real' discrete event systems are really reasonable for qualitative analysis and control of continuous-variable system or whether it is necessary to elaborate new methods that take into account the continuous nature of the system under consideration. This is the aim of current research whose results will be described in forthcoming papers.

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