Differential Equations as Enablers of Qualitative Reasoning Using Dimensional Analysis

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Dimensional analysis has been used for qualitative reasoning about simple physical systems such as the spring and for complex physical systems such as stars, heat exchangers and nuclear reactors [1, 2, 11]. The technique appears promising because its central representation is fundamental to the language of physics; nevertheless, its use has been considered problematic or even mysterious.

Most research in qualitative reasoning starts out with modeling constructs e.g. qualitative differential equations [10] and confluences [5]. These modeling constructs are then used to specify the particular problem that one wishes to solve. The dimensional approach uses as its modeling constructs, physical variables and their dimensional representations. At the surface, this approach to modeling might appear to be impoverished or knowledge-free; however, dimensional representations provide a compact and qualitative encoding of physical knowledge that has been captured by numerical laws. There are philosophical questions that have troubled physicists and others for better part of this century — is there something intrinsic and magical about dimensions? To put it another way, are there any physical laws or phenomena that do not require dimensional homogeneity? These are not the questions that have been the focus of our research. Instead we have used the technique, widely used as a numerical aid in engineering, to provide a tool for qualitative reasoning. In this paper we focus on a rich class of enablers for the technique viz. differential equations that characterize physical systems and phenomena; there will be some discussion of other enablers as well.

We also report, albeit briefly, on an exciting discovery about differential equations. Consider an abstract differential equation i.e. the symbols occurring in it have no physical dimensions; even in such a case dimensional analysis can be used to extract some information from the equation. This analysis becomes relevant and useful, if we are able to rewrite a physical differential equation in terms of dimensionless variables and parameters. Some of these points will be illustrated by example; more formal analyses and results are being prepared for submission to the mathematics community. The main reason for including this material here is that it provides mechanisms for inferring gross behavioral characteristics (e.g. oscillation) and associated variables (e.g. time-period). Another reason is that the approach provides a *weak* method for analyzing differential equations, a subject of significant interest in our community.

This paper is organized as follows: in Section 1 we present a brief review of the horizontal spring discussing some problems that arise in this context but that are quite general to dimensional analysis approach to qualitative physics. We then proceed to describe how knowledge contained in differential equations can be used to enable the dimensional analysis approach, in Section 2. Here we also present the essential insights on dimensional analysis of differential equations; some more details are included in the Appendix. In Section 3 we rework the spring example, starting with the differential equation and associated boundary conditions rather than with *a set of variables*, and use this to discuss the dimensional method as presented by us in [1]. Finally, we provide a more general discussion of the issues and concerns relating to the dimensional analysis approach.

1 Review: The Simple Horizontal Spring

Buckingham's π-theorem defines the number of dimensionless products that are possible in a given situation. If a physical system can be defined by the equation

$$f(x_1, x_2, \ldots x_n) = 0$$

then it can also described by a function F of $n - r$ dimensionless products

$$F(\pi_1, \pi_2, \ldots \pi_{n-r}) = 0$$

where r is the rank of the dimensional matrix of the variables x_i (that is a matrix with one row corresponding to every x_i and one column corresponding to every dimension, such as M, L, T, etc.; the matrix element (i, j) is the exponent of the j^{th} dimension in dimensional representation of the i^{th} variable). The πs are also referred to as *regimes*.

The analysis of the simple horizontal spring described in [1] proceeds along the following steps:

1. *Given:* the system is characterized by the three variables — the time-period T, the mass m and the spring stiffness k whose dimensional representations are $[T]$, $[M]$, and $[MT^{-2}]$ respectively.

2. *Buckingham's Theorem:* for this description $n = 3$ and $r = 2$, r be-

ing the rank of the dimensional matrix.[1] The basis must contain two variables ($r = 2$) and the system can be characterized by a single π or regime ($n - r = 1$), and hence $F(\pi_1) = 0$ will describe the system. Using $(m,\ k)$ as the basis[2], we obtain π_1 as[3]

$$\pi_1 = Tk^{1/2}m^{-1/2}.$$

3. *Reasoning from Regimes:* from the constancy of π_1, it is concluded that $\left(\frac{\partial T}{\partial m}\right)$ is positive, and that $\left(\frac{\partial T}{\partial k}\right)$ is negative. Although this was not mentioned in [1], constancy of π_1 can also be used to reason about proportional change e.g.

$$\frac{\Delta T}{T} = \frac{1}{2}\frac{\Delta m}{m} - \frac{1}{2}\frac{\Delta k}{k}.$$

   One can now answer questions of the form — *how does time-period change if m increases by 1% and k increases by 3%?*

Such analysis has usually raised the following four questions (italics below indicate the general question):

1. Why include exactly T, m *and* k? Why not include the amplitude x_{max} or the position x in the function f? *How would an AI system using dimensional reasoning determine which variables are relevant for modeling a given physical situation?*

2. Why are the mass and the spring stiffness chosen as the basis variables? Why not, for example include time-period T in the basis? *Given that the variables for describing a system are known, how would one select the basis for the problem so as to satisfy the constraints of Buckingham's π-theorem?*

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[1]The dimensional matrix is $\begin{pmatrix} 1 & 0 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}$ where the rows correspond to variables m, k and T.

[2]The basis selection heuristics are discussed in [1] and will be revisited later in the paper.

[3]The process of calculating πs consists of solving a system of linear equations to obtain the values of exponents that will render each π dimensionless.

3. Why is $\pi = Tk^{1/2}m^{-1/2}$ assumed to be constant? *Extracting qualitative information requires that the πs be assumed constant. How is this assumption justified?*

4. What is the effect of excluding the amplitude or including (say) g the acceleration due to gravity? *What is the effect on the analysis if irrelevant variables are included or relevant variables are left out?*

The last question is in fact subsumed in the first question, and we will not discuss it separately. In order to answer these questions we will use information in the underlying differential equations.

2 Differential Equations

Differential equations are a ubiquitous means for codifying knowledge in physical as well as non-physical domains. For example, in artificial intelligence, differential equations have been used in conjunction with logic to model real-world systems [13]. Two widely used representations in qualitative physics research, qualitative differential equations [10] and confluences [5], have differential equations as their underpinning. Moreover, some of the leading attempts at qualitative reasoning are based directly on innovative qualitative analysis of differential equations [12, 16]. In this section we will discuss how the knowledge encoded in ordinary differential equations can be used to justify and enable the dimensional analysis approach to qualitative reasoning.

A differential equation and the associated boundary and/or initial conditions, codify the physical knowledge underlying a device or a phenomenon, and thus provide the pertinent variables. They also allow us to organize the variables into the following classes — dependent variable, independent variable, and parameters. We use *parameter* as the collective terms for physical coefficients, and symbols that occur in the boundary and/or initial conditions. This information can be used to construct a basis selection heuristic as we shall demonstrate in the next section. Once the basis has been selected, Buckingham's procedure provides a characterization of the equation (and hence the system) in terms of dimensionless products for the dependent variable, the independent variable and for all the parameters that are not in the basis. In other words there is some function G such that

$$\pi_{dependentVar} = G(\pi_{independentVar}, \pi s\ for\ nonbasis\ parameters). \qquad (1)$$

This equation captures two kinds of information viz. the *intrinsic* behavior of the system and the role of parameters in influencing or causing this behavior. In qualitative physics research terminology, the latter has been referred to as comparative analysis[15].

The solution of the equation i.e. the form of G determines how the dependent variable behaves with respect to the independent variable or what we have termed as intrinsic behavior. Any changes in the parameters that leave the parameter πs unchanged, do not alter the intrinsic behavior of the system e.g. the system still oscillates but with a different time-period. Such reasoning is quite popular in similitude studies of engineering systems e.g [6]. In this paper we will mostly concentrate on comparative analysis; however, we believe that such analysis is crucial in providing an understanding of the how a device works. In circuit design there is a whole methodology that rests on such an approach viz. tolerance design [14].Thus the differential equation interpretation, as captured in (1), provides the clue as to which πs are held constant.[4]

In some cases a differential equation may provide information on constructing composite variables. Consider a phenomenon (e.g. natural convection) where both the density of a fluid (ρ) and the acceleration due to gravity (g) are pertinent physical variables. The differential equation might indicate that *all* occurrences of these variables are as the product ρg; now the dimensional analysis procedure can treat ρg as a composite variable, thereby reducing the number of πs by one.

Our exploration has also led to the exciting discovery that dimensional analysis is applicable to certain classes of abstract differential equations; by abstract we mean that the variables do not have any physical dimensions. Often dimensional reasoning can be used to extract aspects of qualitative behavior i.e. the form of G in Equation (1), above; we shall see an example of such use in the next section.

The essential idea turns out to be very simple:

If a differential equation is dimensionally homogeneous under a

[4]In [1] the πs correspond to the parameter πs in the differential equation interpretation.

certain dimensional assignment, then its solution is also dimensionally homogeneous under the same assignment.

The key phrase above is "under a certain dimensional assignment". For the rest of this section we ask our readers to abandon the familiar notion of dimensions (as inspired by units in the physical world) and to think of them as abstract entities. Through a simple example of a two term differential equation, we will give a flavor of the approach; some more details are provided in the Appendix.

Consider the differential equation

$$\frac{dy}{dx} = x^m y^n \tag{2}$$

where the variables have no physical interpretation. To the variables x and y we assign the dimensional representations $[X]$ and $[Y]$ respectively; from the definition of dy/dx its dimensional representation is $[YX^{-1}]$. Requiring (2) to be dimensionally homogeneous leads to

$$[X^m Y^n] = [YX^{-1}]$$

and hence the relationship

$$[Y] = [X]^{(m+1)/(1-n)}. \tag{3}$$

This relationship bewteen the dimensions $[Y]$ and $[X]$ is referred to as the *dimensional signature* of the equation and can be used to infer some aspects of the solution.

Even for the simple equation (2), dimensional signatures do not always turn out to have a simple power law form. Let us consider the following cases; the dimensional signatures are obtained by substituting specific values for m and/or n in (3):

$$[\mathrm{Y}] = \begin{cases} (m = -1,\ n \neq 1) & [\mathrm{Y}] = [\mathrm{X}]^0 \\ (m \neq -1,\ n = 1) & [\mathrm{Y}] = [\mathrm{X}]^\infty \\ (m = -1,\ n = 1) & [\mathrm{Y}] = [\mathrm{X}]^{(0/0)} \end{cases} \tag{4}$$

These cases correspond to $[X]$, $[Y]$, and both $[X]$ and $[Y]$ cancelling out when dimensional homogeneity is enforced. The intuition is that dimensional signatures can be used to organize the knowledge about differential equations; some results are summarized in the Appendix.

This theory relies in part on a rarely sold object in the mathematical hardware store, transfinite ordinals, which we use to assign dimensional representations to exponential functions; the previous approach has been to require that the arguments of all transcendental functions be dimensionless, and hence no dimensional representation has been assigned to such functions.[5]

3 The Spring using the Differential Equation

We now return to the frictionless horizontal spring, described by the following differential equation:

$$m\,\frac{d^2x}{dt^2} + kx = 0. \tag{5}$$

This equation is usually written with any two of the following three boundary conditions:

$$1:\ x(0) = x_{max},\quad 2:\ x(T) = x_{max},\quad 3:\ \dot{x}(0) = 0.$$

We choose conditions 1 and 2, and rewrite (5) as follows:[6]

$$f(m, k, t, x, x_{max}, T) = 0. \tag{6}$$

Now we apply Buckingham's π-theorem. The number of variables n is 6, and the rank of the dimensional matrix is 3.[7] As $n - r = 3$, the basis should contain three variables. We assign variables to the basis in the following heuristic order:

[5] Although this use is a mere detail, from the point of view of qualitative physics, we mention it here because we suspect it may be the first engineering application of the Cantor numbers. Information about other applications of these numbers will be much appreciated.

[6] The analysis will also hold if we choose conditions 2 and 3. However, choosing conditions 1 and 3 leads to the problem of having to discover the parameter T; this case will be discussed briefly later in this section.

[7] Now the dimensional matrix will have six rows, corresponding to the variables in (6). Since variables x and x_{max} have dimensional representation $[L]$, the dimensional matrix will have an additional column for the dimension $[L]$.

1. Assign all coefficients to the basis.
2. Assign all non-zero boundary conditions to the basis.
3. If the dimensional matrix is singular, remove variables from it starting with variables dimensionally identically to the independent variable, the dependent variable, and the parameters starting with the term dimensionally identical to the lowest-order derivative.

In the case of the spring, Step 1 adds the variables m and k to the basis, Step 2 adds x_{max} and T to the basis and Step 3 removes the variable T from the basis. Thus (6) can be reduced to

$$F(\pi_x, \pi_t, \pi_T) = 0 \tag{7}$$

where

$$\pi_x = \frac{x}{x_{max}}, \quad \pi_t = \frac{tk^{1/2}}{m^{1/2}}, \quad and \quad \pi_T = \frac{Tk^{1/2}}{m^{1/2}}.$$

The functional relation (7) can also be rewritten as[8]

$$\pi_x = G(\pi_t, \pi_T). \tag{8}$$

Now holding π_T constant, since T is a parameter, provides π_x as a function of π_t which is the essential behavior of the spring since it is a solution of the underlying differential equation. Alternately, consider the boundary condition $x = x_{max}$ when $t = T$; in terms of the πs, $\pi_x = 1$ when $\pi_t = \pi_T$. Under this condition the relation (8) reduces to[9]

$$1 = G(\pi_T, \pi_T)$$

from which we conclude that π_T is constant.[10]

[8]Such rewriting of course will not hold for all possible functions F. The intuitive justification is that we are dealing with physical variables that are in principle measurable and hence real.

[9]We now have one equation with one unknown viz. π_T. Again appealing to the physical (and measurable) origin of the variables, we are concluding that this equation must have at least one real-valued solution. For purposes of reasoning, the precise value is not important.

[10]In the general case there might be multiple parameters e.g. $\pi_x = G(\pi_t, \pi_{p_1}, \ldots, \pi_{p_m})$ rather than a single parameter. If the parameters are independent of each other, then the valid assumption is that $\pi_{p_1}, \ldots, \pi_{p_m}$ are all constant. However, if the parameters have dependencies then then a weaker assumption, $h(\pi_{p_1}, \ldots, \pi_{p_m}) = 0$ is needed.

We now return to the case where the boundary conditions specified are $x(0) = x_{max}$ and $\dot{x}(0) = 0$. Most of the previous analysis carries through but there is no parameter T and hence no π_T. The clue to discovering the missing parameter T is to somehow establish that there is oscillation. This can be accomplished in a number of ways; the traditional approach has been to draw the inference from qualitative or quantitative simulation of the equation. We will now demonstrate how the dimensional approach to analyzing the equation, discussed earlier, may be used to establish oscillation.

The original differential equation, (5), can easily be rewritten in terms of the πs as [11]

$$\frac{d^2\pi_x}{d\pi_t^2} + \pi_x = 0. \tag{9}$$

Since π_x and π_t have no physical dimensions associated with them, we can assign to them the abstract dimensions $[\Pi_x]$ and $[\Pi_t]$ respectively. Now using $[\Pi_x]$ and $[\Pi_t]$ as the basic dimensions, and requiring (9) to be dimensionally homogeneous, leads to the dimensional relation

$$[\Pi_x][\Pi_t]^{-2} = [\Pi_x].$$

Note that the dimension $[\Pi_x]$ cancels out. We have an inference procedure to conclude that the cancelling out of $[\Pi_x]$ implies a solution of the form[12]

$$\pi_x \sim e^{A\pi_t}$$

which we dub as *possible-oscillation*. Symbolic substitution of this form in (9), reveals that A is imaginary and hence the oscillation is real and has a time-period. [13]

Let us now consider the spring and mass oscillator where friction is not negligible. This introduces a damping term $\left(b\,\frac{dx}{dt}\right)$ in the original equation where the damping coefficient b has the dimensions $[MT^{-1}]$. Using

[11] There is a remarkably simple rewrite rule — variables and parameters are replaced by corresponding πs e.g. x is replaced by π_x. The basis variables e.g. m do not occur in the equation since their contribution is absorbed in the πs.

[12] More formally, the form $p \sim q$ means that the relation holds within a dimensionless constant i.e $p = Kq$ where K is dimensionless within the current interpretation.

[13] If A turned out to be real, then the behavior would be *monotonic-increase* or *monotonic-decrease* depending on the sign of A. Of course this is mathematically equivalent to saying that monotonic increase is oscillation with an imaginary time-period.

the same basis as before we now have an additional dimensionless product, $\pi_b = b/k^{1/2}m^{1/2}$. We now have two parameter πs and holding them constant amounts to having

$$g(\pi_b, \pi_T) = 0$$

which can be written as

$$\pi_T = H(\pi_b).$$

Of course, we cannot determine the sign of $\partial T/\partial b$ since we do not know the form of the function H.[14] Holding both π_T and π_b constant leads to two dimensionally homogeneous relations

$$T \sim m/b, \quad \textit{and} \quad T \sim b/k$$

depending on whether k or m is eliminated. Each of these relations respects the intra-regime partials obtained for the undamped case i.e. $\left(\frac{\partial T}{\partial m}\right)$ is positive and $\left(\frac{\partial T}{\partial k}\right)$ is negative. However, they yield different signs for $\left(\frac{\partial T}{\partial b}\right)$. The damped spring example brings out a limitation of dimensional reasoning; in such cases additional information would be needed to resolve the difficulty e.g. knowing whether π_T is directly or inversely proportional to π_b.

Consider two variants of the basic oscillator with two springs, of stiffnesses k_1 and k_2, connected either in series or in parallel. The dimensional approach cannot distinguish between these cases; it will now produce two regimes

$$\pi_T = \frac{T k_1^{1/2}}{m^{1/2}}, \quad \pi_{k_2} = \frac{k_2}{k_1}$$

using k_1 as a basis variable. The effect of changing k_1 and k_2 will be characterized by the intra-regime partial $\left(\frac{\partial T}{\partial k_1}\right)$ and the inter-regime partial $\left[\left(\frac{\partial T}{\partial k_2}\right)\right]^{k_1}$ both of which are negative. Hence both the springs exert similar effects on the time-period of oscillation.[15]

[14] In the terminology of [1], this is equivalent to saying that the inter-regime partial linking T to b is ambiguous i.e. $\left[\frac{\partial T}{\partial b}\right]^k$ is positive and $\left[\frac{\partial T}{\partial b}\right]^m$ is negative.

[15] There is a configuration where the springs can exert opposing influences on the time-period — the mass is situated between the two springs each of which is connected to a rigid support. Now while one spring is being compressed, the other one is being stretched. This case would be captured by introducing a negative sign in the regime π_{k_2} and now the intra-regime partial would still be negative but the inter-regime partial would be positive.

4 Discussion

The main objective of this paper was to demonstrate that differential equation knowledge can be useful in justifying the dimensional analysis procedure proposed in [1]; we have demonstrated this at some length in context of the spring and mass oscillator. In general, a differential equation and the associated boundary / initial conditions, codifying the physical knowledge underlying a device or a phenomenon, provide the pertinent variables and parameters. The classification of variables (dependent, independent, parameters, etc.) assists in basis selection. Moreover, the classification provides an argument as to which πs may be held constant:

- If all the πs, except those corresponding to the dependent and the independent variables, are held constant, then the Buckingham root function e.g. (6) yields an unspecified functional relationship between the dependent and the independent πs
- Any changes to the parameters and/or coefficients that respect this constancy, will yield similar behavior as predicted earlier

The spring example was chosen for its simplicity and versatility since in principle it embodies many oscillatory processes. More general cases are characterized by a system of differential e.g. stellar interior equations in astrophysics and the Lotka-Volterra equations in ecology. Both these cases have been analyzed using the same approach. One observation is that as we deal with more complex systems i.e. more parameter πs, there is greater need for analysis of the differential equations themsleves. Towards this end, our work on dimensional analysis of differential equations has proved particularly useful.

The arguments provided so far might may have led the reader to believe that differential equations are in fact the *sole* enablers of the dimensional approach. In the rest of this paper we will dispel this notion by broadening the discussion. The overall argument is that dimensional reasoning provides a framework to integrate information from many different sources (or enablers) to bring it to bear on the questions of interest. Broadly speaking, the dimensional reasoning task requires three kinds of knowledge — knowledge of pertinent variables, knowledge needed to partition these variables into basis and non-basis variables, and knowledge of what regimes may be held constant.

Knowledge of pertinent variables may come from many other sources such as expert knowledge, a library of designed components and from analogies to or conceptual perturbations of known designs, just to name a few. Similarly, the basis selection task uses information such as — is this an exogenous variable, is this an output variable for my analysis, is this a well-known constant (e.g. Newton's gravitational constant G) etc. Some more details are included in the discussion on heuristics for basis selection in [1]. Both these tasks in fact benefit from the dimensional constraints placed by Buckingham's theorem e.g. all dimensions that occur in the dimensional representations of the system variables must also occur in the dimensional representations of the basis variables. Even the task of acquiring pertinent variables can use dimensional information e.g. if a certain dimension (say $[L]$) occurs in the dimensional representation of only one variable then either this variable is irrelevant or some other relevant variable has been missed.

Holding all parameter regimes constant is a useful but initial heuristic. Often the process might result in ambiguities, as we saw in the case of damped spring example. The important part is that regimes provide a useful mechanism for focusing the expert's knowledge e.g. casting the system in the form of regimes might allow us to index into expert knowledge about the functional relationship between some subset of the regimes.

Finally, the dimensional analysis approach should not be viewed as a standalone technique; instead, as mentioned above, it provides a simple yet powerful mechanism for organizing and focusing knowledge. The system that we are building emphasizes flexibility; it has been architected to accept different kinds of knowledge that can be brought to bear on the problem.

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Appendix: A Dimensional Approach to Differential Equations

Dimensional Representation

An important task is to obtain the dimensional representations of terms in the equation in terms of the abstract primary dimensions $[Y]$ and $[X]$. The intuitive argument is that dimensional representation constitutes a vector space and both differentiation and integration are linear operators in this vector space. Hence if y and x are both members of a vector space, then so are y', y'', $\int y dx$ and so on. *Therefore they each have a dimensional representation in a system where the primary dimensions are* $[X]$ *and* $[Y]$.

The rules for generating dimensional representations of expressions are as follows:[16]

[16]Some notation: the symbol $\stackrel{\delta}{=}$ denotes that the expression on the right-hand side is the dimensional representation of the expression on the left-hand side. It is important to note that the mapping from a dimensional relationship is many-to-one; thus $\stackrel{\delta}{=}$ is not an equality relation, but shorthand notation for the mapping $\Delta(\ )$. If $x \stackrel{\delta}{=} \Omega$ (say) and $y \stackrel{\delta}{=} \Omega$, then it does not follow that $x = y$.

Dimensionless quantity	Π_i	$\stackrel{\delta}{=}$	$[1]$.
Monomials	x^n	$\stackrel{\delta}{=}$	$[X]^n$.
Multinomials	$x^\alpha y^\beta$	$\stackrel{\delta}{=}$	$[X^\alpha Y^\beta]$
General Multinomial	$\prod_{i=1}^{i=n} x_i^{\alpha_i}$	$\stackrel{\delta}{=}$	$[X_1^{\alpha_1} \dots X_i^{\alpha_i} \dots X_n^{\alpha_n}]$
Polynomials	$\sum_{i=1}^{i=n} x^i$	$\stackrel{\delta}{=}$	$[X]^n$
Differentiation	$\left(\frac{dy}{dx}\right)$	$\stackrel{\delta}{=}$	$[YX^{-1}]$
	$\left(\frac{d^2y}{dx^2}\right)$	$\stackrel{\delta}{=}$	$[YX^{-2}]$
	$\left(\frac{d^ny}{dx^n}\right)$	$\stackrel{\delta}{=}$	$[YX^{-n}]$
Integration	$\int f(x)dx$	$\stackrel{\delta}{=}$	$[\Delta(f(x))X]$

Most of these rules are conventional and obvious. They follow from the definition of any polynomial of degree n as an element in a vector space of dimension n whose basis is the set of monomials of degree n or lower. In the language of real analysis, the space of orthogonal vectors x^i describes an *orthonormal* basis.[17]. This assumption, of the polynomials of degree n or lower forming a basis (i.e the coefficients are linearly independent), is in fact made frequently, and for example, lies at the heart of the technique of *partial fractions.*[18]

We now add one rule that greatly increases the generality of the dimensional analysis. So far, expressions that are exponential are assumed to be either dimensionless or undimensioned. We assume instead the following rule:

Exponentials $e^x \stackrel{\delta}{=} [X^\omega]$

Here the exponent ω is the first transfinite ordinal. The algebra we use for these numbers is the Conway arithmetic for numbers [4, 8]. Because Conway numbers form a field, our assumption of the vector space continues to be valid.

Consider the dimensional interpretation of a polynomial, as shown above,

[17]See for example, [9, p. 143]

[18]In partial fractions, the technique is so basic that even otherwise detailed and thorough books rarely carry an explanation of the theory, e.g. [7].

viz.

$$\sum_{1}^{n} a_k x^k \stackrel{\delta}{=} [X]^n. \tag{10}$$

If we now assume that the polynomial is not restricted to be finite-dimensional, i.e. $n \leq \infty$, we see that the dimensional representation of a polynomial of infinite degree is $[X]^\infty$ which is encoded as $[X]^\omega$. Now consider such an infinite-degree polynomial in x, viz. the definition of e^x.

$$e^x = \sum_k a_k x^k, \quad a_k = (k!)^{-1}$$

where $0 \leq k \leq \infty$. Thus, when the dimensional representation of a mathematical expression is $Y = [X^\omega]$, the mathematical expression is of the form $y = e^x$. The transition is valid because the basic theorems applicable to any orthonormal basis smoothly extend to the infinite-dimensional Hilbert space of polynomials x^k.

Reasoning from Dimensional Signatures

The dimensional signature provides important clues to the form of the solution. For each of the four classes mentioned above, we summarize the rules of inference, and provide some notes on coverage. The notation $y \sim x^m$ that y is a polynomial of degree n in x; otherwise $y \sim f(x)$, where f is a transcendental function, means that $y = Kf(x)$ and K is dimensionless in the $([X], [Y])$ system of dimensions.

Class	Dimensional Signature	Inference Rules
Yields [Y] = [X]^m Relationship	$[Y]=[X]^m$ $m \neq 0$	$y \sim x^m$ *or* $y \sim (x_0^p + x^p)^{m/p}$
[Y] Cancels Out	$[Y]^0=[X]^m$ $m \neq 0$	$f(y',y,x)=0 \quad \Rightarrow \quad y \sim e^{Ax^{m+1}}$ $f(y^{(k)},\cdots,y'',y',y)=0 \quad \Rightarrow \quad y \sim e^{Ax}$
[X] Cancels Out	$[Y]^m=[X]^0$ $m \neq 0$	$f(y',y,x)=0 \quad \Rightarrow \quad y^A \sim \ln x$ $f(y^{(k)},\cdots,y'',y',x)=0 \quad \Rightarrow \quad y \sim \ln x$
Both [X] and [Y] Cancel Out	$[Y]^0=[X]^0$	$y \sim x^A$ *or* $y \sim (x_0^p + x^p)^{A/p}$

- For the class *Yields* $[Y]=[X]^m$ *Relationship* the equation be reduced to the lowest possible order prior to computing the dimensional signature. For example the equation $y''=(y')^2/x$ must be reduced to a first order equation $z'=z^2/x$ where $z=y'$, prior to computing the dimensional signature.

- For the classes $[X]$ *Cancels Out* and $[Y]$ *Cancels Out* equations of the form

$$f(y^{(k)},\cdots,y'',y',y,x)=0$$

for example: $y''=xy$ or $y''=y^3/x^2$ are not covered.

- The rules of inference contain parameters p and A which can be crucial to determining the form of the solution and hence its qualitative characteristics. The inference procedure consists of symbolic substitution of the form in the actual equation in order to determine the value of the parameters. Often just the sign of the value or its type (real or imaginary) will suffice for purposes of qualitative behavior.