Reasoning about Constant Coefficient Dynamic Systems

Juan Flores Art Farley
Computer and Information Science Department
University of Oregon
Eugene, Oregon, 97403
phone (503) 346-4416 fax (503) 683-5373
email juan@cs.uoregon.edu

Abstract: One of the main objectives of qualitative physics is to derive the behavior of a system from the description of its components and their interrelationships. Observation of the problems that qualitative simulation techniques present, motivated us to study physical systems from a different perspective, their analytical solutions. In this paper we present a framework to reason about dynamic systems, and its implementation, called NumGen. This framework, based on the properties of the analytic solutions, produces descriptions of the behavior of systems that fall into this category. The set of possible behaviors, can be connected in a graph representing how changes in the parameters of the system correspond to changes in its response. abstract

1 Introduction

Quantitative Physics models a system in terms of Ordinary Differential Equations (ODEs). It discards causality for the sake of conciseness. On the other hand, explanations found in textbooks and classrooms describe the interactions between the components of the system in terms of cause and effect, paying little or no attention to the precise values of parameters.

There have been several successful efforts in trying to derive qualitative behavior from structure. De Kleer and Brown [DB-90] model physical systems in a representation that captured causality and allowed reasoning about change. They developed the concepts of confluence (Qualitative Differential Equation or QDE), qualitative state, a qualitative version of calculus, and a reasoning framework (envisioning), based on confluences. Williams proposed a technique that performs large signal analysis of a mechanism described by confluences [W-90]. Forbus describes physical situations as objects and processes that affect them [F-90]. Kuipers presents an alternative to the idea of qualitative simulation and implements it in a system called QSIM. QSIM's representation is based on constraint equations, derivable directly from the differential equations that govern the mechanism [K-90].

Kuipers' work has been a reference to establish comparisons with many works in the field. Nevertheless, it has several drawbacks. The local nature of the analysis performed in QSIM gives place to the production of spurious behaviors. Ambiguity arises due to the presence of negative feedback loops, as a result of the lack of additive inverse of the sign algebra. Also, as pointed out by Kuipers et al [K-91], the derivatives of certain variables are only constrained by continuity (an intrinsic property of the model), leading to an effect known as chattering. Ambiguity and chattering are translated into an intractable branching factor in the behavior tree.

This leads to the following problems when using QSIM. If you limit the number of nodes to be produced, the tree is so wide, that you will not get very far in the simulation, missing all the important details. On the other hand, if you do not, the number of nodes produced increases rapidly and with them the number of spurious behaviors. The difference between those behaviors is sometimes so subtle that it is hard for the user to perceive it. The user is unable to distinguish between a real and a spurious behavior. QSIM's graphic capabilities are very limited; if we know that responses to dynamic systems are either sinusoidal or exponential, why draw them as straight lines?

These observations motivated us to search for a better solution. In this paper we present a framework for reasoning about dynamic systems represented as a linear ordinary differential equation with constant coefficients. Given an nth order system, we want to generate descriptions of all possible qualitative responses. Then for each response, we want to generate an instance of a quantitative nth order equation that exhibits that qualitative behavior.

A question may arise: Why are we trying to solve this problem apparently limited in scope? Many important problems in engineering, and physics, can be expressed in mathematical terms by means of a linear ordinary differential equation with constant coefficients. On the other hand, linear differential equations that do not have constant coefficients, and non-linear equations can be reasoned about if we apply linear transformations and decompose them into piece wise linear approximations of the original equation.

The rest of this paper is organized as follows. In section two, the mathematical background about linear ordinary differential equations with constant coefficients is presented. We follow the approach of exploiting available mathematical knowledge prior to transforming the problem to a qualitative form [S-92].

In section three, we present a Qualitative Analysis of the responses of systems governed by this kind of differential equations. Section four gives the bases to generate numerical examples and plot the response. Section five mentions a couple of questions that are being investigated. The last section proposes future work, compares NumGen with other systems, and presents the conclusions.

2 Linear Ordinary Differential Equations with Constant Coefficients

As mentioned before, linear ODEs with constant coefficients are the most studied kind of differential equations; they have complete analytical solutions. Also, there is a good number of problems that can be described by this kind of equation, and more complicated cases can be reduced to one or several of these equations.

In this section, the theory of solution of linear ODEs with constant coefficients, and a qualitative interpretation is presented. The facts presented in this section are the basis for the framework developed and presented in the next section.

Consider the homogeneous nth order ODE given in equation 1.

$$a_0\frac{d^nx(t)}{dt^n}+a_1\frac{d^{n-1}x(t)}{dt^{n-1}}+\cdots+a_{n-1}\frac{dx(t)}{dt}+a_nx(t)=0 \quad (1)$$

where $a_0,\ldots,a_n$ are real constants.

The solution represents the behavior of the system in response to the forcing function and initial conditions $X(0), X'(0), ..., X^{(n)}(0)$.

It is quite natural to think of an exponential function as a candidate solution to that equation. Substituting $x = e^{rt}$ in equation 1 and factoring e^{rt} yields equation 2.

$$e^{rt}(a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n) = e^{rt}Z(r) \quad (2)$$

$x(t)$ is a solution of equation 2, for those values of r that satisfy the characteristic equation. i.e. the roots of polynomial $Z(r)$. The general solution of equation 1 is of the form:

$$x(t) = c_1e^{r_1t} + \cdots + c_ne^{r_nt} \quad (3)$$

We can see that the natural response of an nth order system, is the sum of n exponential terms. One for each root of the characteristic equation of the ODE. If it has positive roots, the system is unstable, otherwise, it is stable. If the roots of the characteristic equation of the ODE are all real, the system's response is non-cyclic. If the characteristic equation has complex roots, they come in conjugate pairs, in which case, the general solution is still of the form of equation 3, only that each pair of complex roots $(r \pm i\omega)$ becomes an exponential sinusoidal function. This property is known as Euler's identity.

$$C_1e^{(r+i\omega)t} + C_2e^{(r-i\omega)t} = e^{rt}(A_1\cos\omega t + A_2\sin\omega t) \quad (4)$$

So, if we restrict the kind of systems we are to analyze to those that can be expressed by an nth order ordinary differential equation with constant coefficients, we know the kind of responses we are to get. We can express the behavior of a system in terms of the exponential and sinusoidal components in the response. We define

$$E_n(t) = \sum_{1\leq i\leq n} a_ie^{r_it} \quad (5)$$

as a summation with at most n exponential terms, and

$$ES_n(t) = \sum_{1\leq i\leq n} a_ie^{r_it}\sin\omega_it \quad (6)$$

as a summation of exponentially decreasing sinusoidal functions. Noting that we are not interested in giving analytical solutions to the differential equation, but a qualitative description of its behaviors. That is, the kind of qualitative shapes that responses to equation 1 can have.

Theorem 1 *Given a system of order n, the response can be expressed as in equation 7.*

$$X(t) = E_{n_1}(t) + ES_{n_2}(t) \quad (7)$$

where $n_1+2n_2 = n$. This result is evident from equation 3, equation 4, and the definitions of equation 5 and equation 6. This general form of the responses to this kind of dynamic systems will be called High Level Qualitative Description (HLQD).

We see that if the second term of equation 7 does not exist, the response will be acyclic. Otherwise, it is a sinusoidal wave, where $E_{n_1}(t)$ represents its axis or attractor, and $ES_{m_2}(t)$ its sinusoidal components.

Note that of we include a forcing function, the system's response would be decomposed into Natural (the solution to the homogeneous equation) and Forced responses. If we restrict the forcing functions to be of the form $e^{\alpha t}\sin\beta t$ (i.e. constant, exponential, or sinusoidal), the forced response always has the same qualitative form as the forcing function [BD-65]. This would preserve the qualitative form of the response, and only add one more exponential or sinusoidal term to the response.

3 Qualitative Analysis of the Response

The HLQD of the response represents a family of functions. If we want to be more precise and give a better characterization of the responses, we need to study how $E_{n_1}(t)$ and $ES_{n_2(t)}$ behave.

A study of exponential functions has taken us to the following conjecture. A function $f(t) = E_{n_1}(t)$ has at most n_1 places where $f'(t) = 0$. What this says is that the number of peaks (maxima and minima, including the zero when $t \rightarrow \infty$) cannot be greater than $n - 1$. That is, the qualitative shape of $E_{n_1}(t)$ depends on the order of magnitude relations between its coefficients and exponents. For example, figures 1, 2, and 3 show all possible qualitative shapes of $E_3(t)$ (their mirror images are also possibilities).

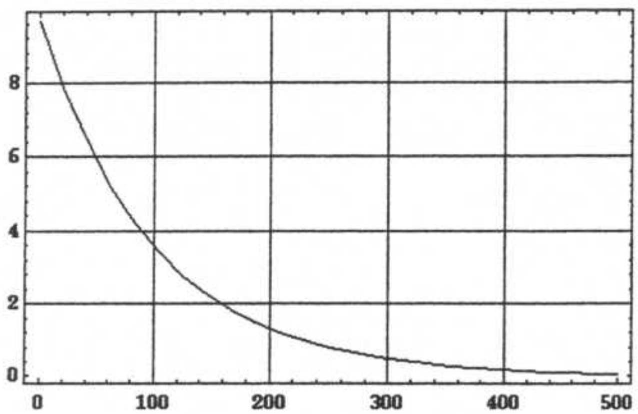

Figure 1: $E_3(t)$ reducing to $E_1(t)$

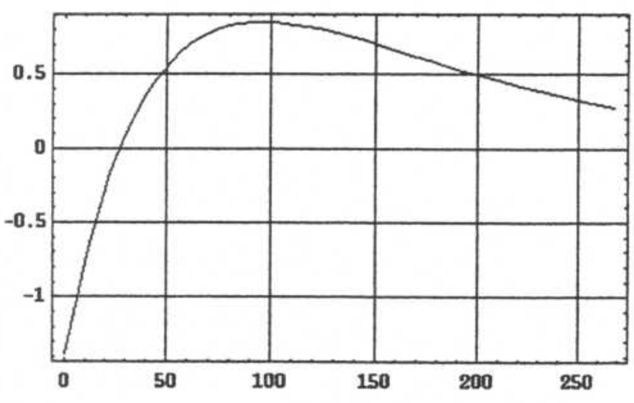

Figure 2: $E_3(t)$ reducing to $E_2(t)$

As we can see, the possible different behaviors of $E_n(t)$ are the behaviors of all $E_i(t)$ (for $0 < i < n$), (with i maxima/minima).

$ES_{n_2}(t)$ is a summation of n_2 exponentially decreasing sinusoidal components. We can also derive reduction rules for this kind of functions. For instance, if the decay rate of one component is too high, or its amplitude is too small, this component becomes negligible, and the result is a reduction in order of the

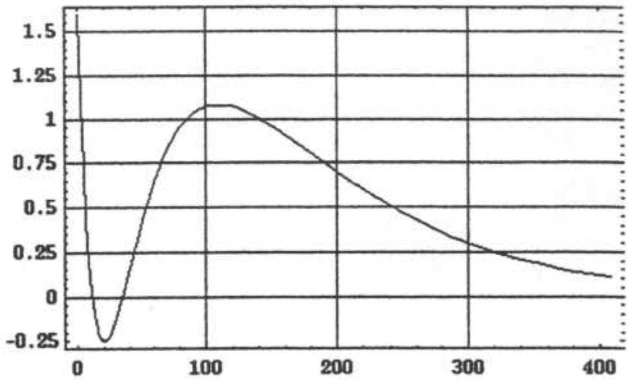

Figure 3: $E_3(t)$ presenting the maximum number of peaks

function. Assuming all the decay rates and amplitudes are comparable, the important parameters to determine the qualitative shape of the response are the frequencies.

If we order the n_2 frequencies of $ES_{n_2}(t)$, we can form a vector of order of magnitude relations among its components. These will determine the different qualitative shapes that the function can have. For instance, for $ES_2(t)$, the possible order of magnitude relations between the two frequencies are $\omega_1 < \omega_2$, $\omega_1 \sim< \omega_2$, and $\omega_1 = \omega_2$. The first case can be seen as the faster sine mounted on a slower one. The second one, where the two frequencies are very close but not equal, creates a beating response. The third case, where both frequencies are equal, reduces to a single sine response, i.e. $ES_1(t)$. Figures 4, 5, and 6, show the three cases.

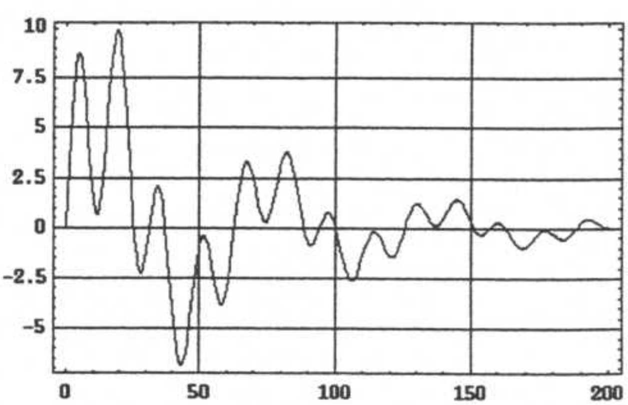

Figure 4: $ES_2(t)$ presenting two mounted sines

In general, the frequency vector for $ES_{n_2}(t)$ can be expressed as in equation 8:

$$\omega_1 \ \Theta_{i_1} \ \omega_2 \ \Theta_{i_2} \ \ldots \ \omega_{n-1} \ \Theta_{i_{n-1}} \ \omega_n \qquad (8)$$

where Θ_{i_j} can be any of $=, \sim<$, or $<$. These operators have the same meaning as the operators $=, \sim<$,

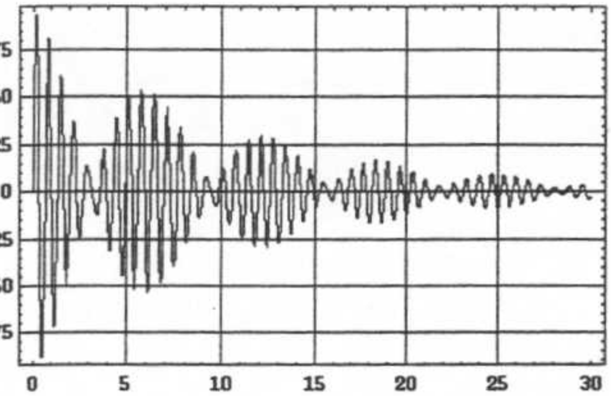

Figure 5: $ES_2(t)$ presenting two beating sines

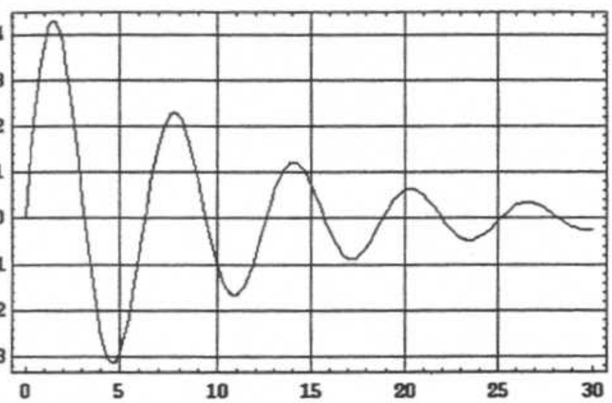

Figure 6: $ES_2(t)$ reducing to one sine

and $<< \cdots - <$, mentioned in [M 87], for a parameter $e = 0.1$.

The mounting and beating relations that the different sine components may have, can be explicitly stated, to give a more intuitive meaning to the frequency vector. In that way, $n \geq 2$ frequencies related by the operator $\sim<$, could be represented as nB, and n frequencies related by the operator $<$ can be represented as nM. A sequence of n frequencies related by the equality operator, $=$, collapse into a single element (i.e. the summation of n sines of the same frequency can be expressed as a single sine). For instance, the vector $\omega_1 < \omega_2 \sim< \omega_3 < \omega_4$ can be represented as $1M\ 2B\ 1M$. This means the sine with the fastest frequency ω_4 is mounted on the next two, that are beating, and the result is mounted on a single sine. As we see, the notation is read from right to left. Figure 7 shows a graphical example of a function that presents those characteristics.

The total response of a dynamic system can be seen as a pair, whose first element is a number that represents the number of exponential terms in the response, followed by the frequency vector, expressed in the above MB notation.

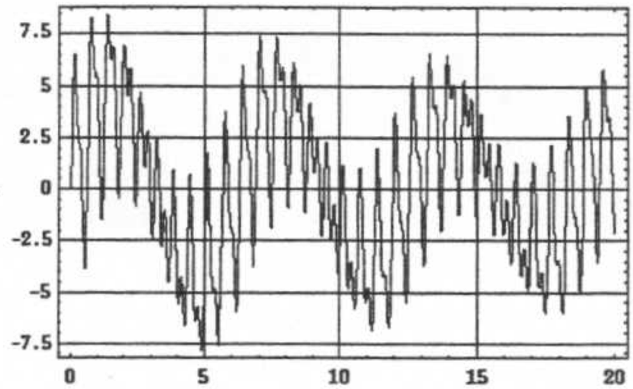

Figure 7: A function that presents the $(1M\ 2B\ 1M)$ characteristic

4 Numeric Example Generation

In the previous sections, we have characterized the possible responses to a time invariant dynamic system. This description may not be adequate for users; they would rather have a graphical description of the expected responses. One idea is to give a qualitative plot of the response. The other idea, and the one presented here, is to generate an example of a numeric function that exhibits the desired qualitative behavior and plot it. Since the response has exponential and sinusoidal components, $E_n(t)$ and $ES_n(t)$, we will treat them separately.

For any given qualitative behavior, the conditions for the parameters of $ES_n(t)$ are clearly stated. Therefore we can produce a numeric vector of frequencies that keeps the same relations of order of magnitude. We start with a unit frequency (the slowest) and produce the rest of the vector according to the following rules:

$$\begin{array}{c} \omega_1 \sim< \omega_2 \Rightarrow \omega_1/\omega_2 = 1.1 \\ \omega_1 < \omega_2 \Rightarrow \omega_1/\omega_2 = 4 \end{array} \qquad (9)$$

That is, two frequencies need to be very close together to beat, and, by observation, a factor of 1.1 makes two frequencies beat. If two sines are to be mounted, it would be convenient to have at least four complete periods of the fast mounted on the slow one, so that the user can see the variations more clearly. This also preserves the order of magnitude relations proposed in [M 87] for a parameter $e = 0.1$. The amplitudes and decay rates of the sines are generated randomly, in ranges from 1 to 5 and -0.1 to -0.2, respectively. These values were chosen so that the waves do not vanish before the interesting features of the sines appear.

The necessary order of magnitude conditions that guarantee that an exponential function $E_{n_1}(t)$ will show the desired characteristics have not been so precisely defined. Nonetheless, some relations among them have been established well enough to generate the desired functions in an iterative way.

If we consider the terms of $E_n(t) = \sum_{1 \leq i \leq n} a_i e^{r_i t}$ ordered from slowest to fastest rate of decay:

1. The sign of the amplitudes must alternate (i.e. $[a_i] \neq [a_{i+1}]$).
2. The absolute value of magnitude a_i, must be greater than the absolute value of the summation of the previous magnitudes (i.e. $\mid a_i \mid \geq \sum_{1 \leq j < i} a_j$).
3. The effect of the ith term must be negligible at the time of the first zero of the function of the i_1st terms. See figure 8.

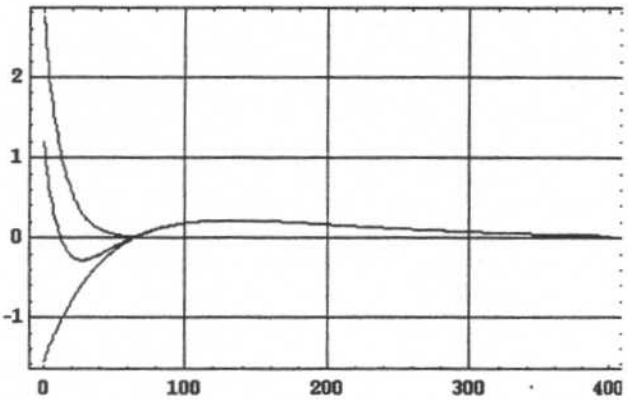

Figure 8: Inclusion of the 1st term to $E_3(t)$

The first two conditions are easy to satisfy, since they are clearly stated; but the third one is not. To ensure the produced function has the desired qualitative properties, we will not only enforce the number of maxima/minima, but will make the function alternate around zero. The procedure starts with a decay rate of $-.1$, and a randomly generated amplitude. We double the decay rate, generate a random amplitude that preserves conditions (1) and (2), and check if the function presents a new zero between $t = 0$ and the time of the last zero. If it is not the case, the decay rate is doubled again and again, until a zero is produced.

Finally, we put the two components together. For example, if the desired response is $((E1)(2B))$, the system generates the function shown in 10, which is shown in figure 9.

$$\frac{20}{e^{.05t}} + \frac{7.635 \sin t}{e^{.01525t}} + \frac{6.205 \sin 1.1t}{e^{.01555t}} \qquad (10)$$

So, we can show the user all qualitatively different responses to the system at different levels: the HLQD, the MB notation, and a plot of the numerical example.

We can go a little further than that. If we have a numerical example of the response of the system, we

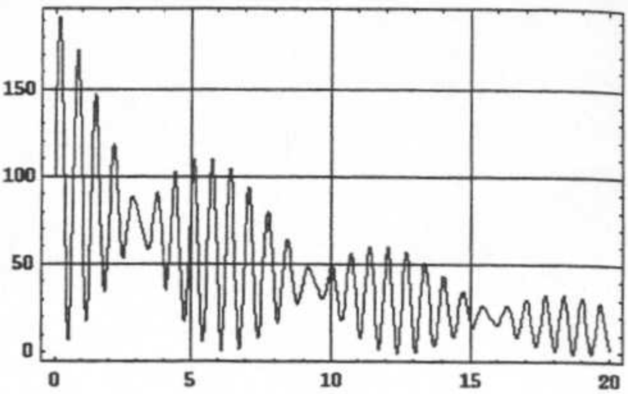

Figure 9: Plot of example of $((E1)\ (2B))$

$(r + 0.05)$
$(r + 0.01525 + I)(r + 0.01525 - I)$
$(r + 0.01555 + 1.1I)(r + 0.01555 - 1.1\mathrm{I})$

(a) The factored characteristic polynomial

$0.0605262X^{(0)} + 1.21392X^{(1)} +$
$0.178591X^{(2)} + 2.2145X^{(3)} +$
$0.1116X^{(4)} + 0.1116X^{(5)} + X^{(6)} = 0$

(b) The differential Equation

Figure 10: Generation of a differential equation for a numerical example

can generate the differential equation that produced that response. The decay rates and frequencies of the response are the components of the roots of the characteristic equation. With the roots, we form the factored polynomial, expand it and have the characteristic equation, whose coefficients are the coefficients of the differential equation.

For the example above, the factored characteristic equation and the resulting differential equations are shown in figure 10.

The generated function, equation 3, is the actual solution to the differential equation. Thus, to know the initial conditions, we just obtain its derivatives and substitute for $t = 0$. The result for this example is shown in figure 11

We have shown how to derive all possible qualitative different responses for a time invariant system of order n and how to generate a numerical example of

$X^{(0)}(0) = 20,$	$X^{(1)}(0) = 13.4605,$
$X^{(2)}(0) = 0.395141,$	$X^{(3)}(0) = -15.8861,$
$X^{(4)}(0) = 0.97935,$	$X^{(5)}(0) = 17.5905$

Figure 11: Generation of the initial conditions

a system that exhibits that behavior.

This framework has been implemented in Allegro Common Lisp, interfacing with Mathematica [Wo-88] to do some mathematical manipulation and plotting. The implementation (called NumGen), has been tested on a number of examples.

5 Current Research

In this section, we present the topics we are currently investigating at this time, as well as several interesting questions that still need to be solved. The first one deals with differential analysis, the second with the recognition of qualitative features and system design.

In the previous sections, based on the properties of the analytic solutions, we showed how to produce descriptions of behavior of time invariant systems. The set of all possible behaviors can be connected in a graph to represent how changes in the parameters of the system correspond to changes in the response. Differential Analysis answers these questions such as: If the system is generating qualitative behavior QB_i, what kind of behavior is obtained if a_i increases (i.e. $\partial a_i = +$)?. Note that the changes in the response can be either qualitative (i.e. the new response has a different shape), or quantitative (i.e. the new response has the same shape, but different amplitude, decay rate, or frequency).

In a behavior transition graph, the nodes will represent the different behaviors and the arcs the possible transitions among them. The frequency vector indicates what are the possible changes a response can have. For example, for the vector $\omega_1 < \omega_2 < \omega_3 < \omega_4$ (corresponding to 4M), if a given frequency changes, let us say ω_2 decreases, the only possibility is that ω_1 and ω_2 get close enough to beat, that is $\omega_1 \sim< \omega_2 < \omega_3 < \omega_4$ (corresponding to $2B\ 2M$). Further reduction in ω_2 leads to equality between ω_1 and ω_2, reducing this term to $3M$. Thus the arcs of the transition graph are labeled by the kind of changes that produce them, and since those changes are reflexive, we only indicate the parameter whose change produces the transition. Figure 12 shows the total envisionment graph for $ES_4(t)$, where frequencies are altered. Note that big changes in frequencies are accounted for by continuous small changes.

A similar analysis can be produced for the exponential part of the responses. All possible changes in all the parameters generate the neighboring nodes to the one in question.

An important issue arises here. The frequencies and decay rates of the different components correspond to the location of the roots of the characteristic equation in the complex plane. We do not want to express the transitions in terms of parameters that we are not allowed to affect directly. It would be desirable to tag the arcs with conditions on the coefficients a_i of equation 1. The question is how to map changes in the

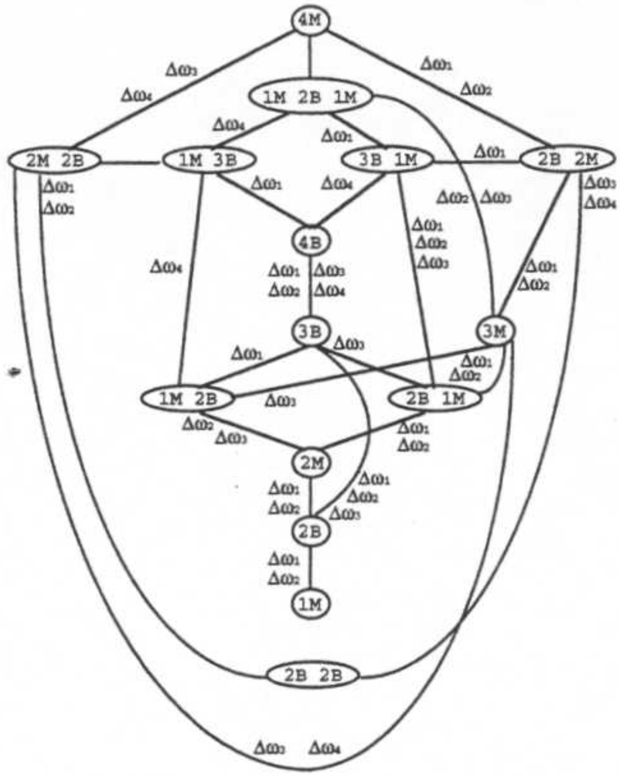

Figure 12: Behavior transition graph for $ES_4(t)$

coefficients to changes in the parameters of the solutions? In this discussion, we will talk about changes in the coefficients, understanding that changes in the initial conditions can be accounted for in the same way.

We are considering qualitative behavior QB_i, and want to tag the arcs of the graph with changes in the coefficients of the differential equation. To accomplish that, we can generate a numeric example of a differential equation that exhibits qualitative behavior QB_i, then modify each coefficient by a certain amount and observe the result. When we modify a coefficient, a slightly different differential equation is obtained, which may exhibit a different qualitative behavior. To obtain this behavior we solve it (i.e. factor the characteristic equation to find the parameters of the new solution). The form of the solution can be analyzed and its qualitative shape determined (i.e. how many exponential terms does it have? how many sinusoidal terms? are they mounted, or do they beat?, what are their amplitudes?, etc.) and compared with the original response.

If both responses have the same qualitative shape, we tag the arc from QB_i to QB_i with confluences, indicating the changes in ai and the resulting changes in any parameter. For instance, if a_n increase in a_3 resulted in a decrease in the frequency ω_4, this would be expressed as $\partial a_3 + \partial\omega_4 = 0$. We understand this confluence is unidirectional, that is, a_3 can affect ω_4, but not vice verse. If the change on a_i takes us to a different behavior QB_j, we tag the corresponding arc from QB_i to QB_j with the change in the coefficient. This transition is reflexive, that is, if $\partial a_i = +$ transforms QB_i into QB_j, the opposite change $\partial a_i = -$, would transform QB_j into QB_i.

Unfortunately, a simple change in a coefficient a_i can produce changes in several parameters of the response (i.e. more than one frequency, or amplitude). More work is needed on this problem.

Another area we are investigating is the development of an algorithm to determine qualitative characteristics from a quantitative representation of the response of an unknown system (possibly generated by physical measurements). In obtaining the qualitative shape of the response, we need to verify what orders of magnitude are present among the different parameters, and convert those relations into the MB notation. With the qualitative properties of a numerical observation, we know what kind of components are present in the response. If we know the components of the response, we already illustrated how to design a quantitative differential equation that exhibits a similar behavior.

An interesting question is how to design a physical system that could be modeled by a certain differential equation. That would complete the design phase. An important detail here is that the relation from physical systems to differential equations is many-to-one. So we would need to find a way to design components and their interactions to produce the desired model, or have a library of physical systems of different orders. In any case, it would be interesting to determine order of magnitude relations on the system parameters that would produce the desired characteristic in the response.

The first part allows us to reason about how changes in the system impact the response. The second part, to reason in two directions: producing all possible behaviors of an nth order system, or to design a system that exhibits a given behavior.

6 Conclusions

One of the main objectives of Qualitative Physics is to derive the behavior of a system from its structure. We propose a framework to reason about dynamic systems, and discuss its implementation, NumGen. NumGen, given an nth order system, generates descriptions of all possible qualitative responses. Then for each kind of response, generates a quantitative nth order equation that exhibits that behavior. NumGen expresses the response at two levels of abstraction. At the high level of abstraction (HLQD) the response is expressed in terms of qualitative descriptions of exponential and sinusoidal functions. At the low level (the MB notation), a function is described by a the characteristics its components may present.

This framework can be compared with previous works in the field of qualitative physics.

QSIM's system representation [K-90] is more general; in NumGen we trade this generality for more specific descriptions of the response. This trade off is reflected in the replacement of monotonic relations by proportionalities. By using global knowledge about the overall shape of the response, instead of the local transition table given in QSIM, we eliminate the production of spurious behaviors. Also, the numeric example generator allows us to yield a more accurate graphical description of the response.

Compared our work to that developed by Schaefer [Sc-91], we see he solves a wider range of problems. But his representation is less precise; it only includes an abstraction of how the frequency and amplitude of the response change with time. NumGen represents the response at both high and low levels.

Sacks work [S-85] is probably the one that yields the most complete form of response. He derives an analytical expression for the response (for systems solvable by the Laplace method) and then describes it in an interval based representation. An important feature of NumGen is that it does not have to do any algebraic manipulation on the equations, and still yields an acceptable form of description. In solving the differential analysis problem, we do use some algebraic manipulation (or numeric simulation).

There are still some problems to be addressed:

- For second order systems, the relation between system parameters and differential equation coefficients is clear and intuitive. For higher order systems, the differential equation coefficients are functions of the various physical system parameters. At this point, to perform differential analysis, we allow only one coefficient to vary at a time. If we allow a single system parameter to vary, this could cause many variations on the values of the coefficients of the differential equation. We need to explore how to account for multiple changes in the coefficients at the same time.

- The algorithm we have to extract qualitative features from a numeric representation of a response needs to be fully developed and tested.

- To provide a framework to express specific knowledge about different kinds of systems, such as non-constant coefficient or non-linear systems.

It can be seen, when comparing this framework and the solutions it is able to produce, with other works, that by using mathematical knowledge and restricting the domain of application, more precise answers can be given, less spurious behaviors generated, and the branching factor on the search for solutions reduced.

7 Acknowledgments

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