Dualistic Algebra for Qualitative Analysis

François Guerrin

Institut National de la Recherche Agronomique (INRA) - Biometrics and Artificial Intelligence Unit
BP 27 — 31326 Castanet-Tolosan Cedex (F)
phone +33 61 28 52 88 fax +33 61 28 53 35
email guerrin@toulouse.inra.fr

Abstract: A central point in qualitative models is that of calculation. The problem is to represent, compare, and combine, approximate or symbolic values with the major constraint of consistency with some numerical counterpart. Aware of the fact that many qualitative characterizations are in the form of antagonistic concepts (such as low vs. high, weak vs. strong, bad vs. good, etc.) more or less explicitly separated by a reference (we call the *norm*), we exploit this idea to build a *dualistic algebra* based on a linear order and an homomorphism with the set of integers Z, which accounts for the ranks of qualitative values (viewed as labels expressing magnitudes) ordered with respect to the norm. We define operations on those values, globally consistent with classical arithmetic (+, –, ×, /, power, root) as well as fitting common sense in reasoning.

1 Introduction

The major goal of qualitative reasoning (QR) is to provide engineers with computational theories to allow them to draw inferences about systems, using information of various origins and different natures. Due to the complexity of systems, qualitative models should be able to involve variables having possibly different domains such as real numbers, numerical intervals or even purely symbolic elements. Measurements (numerical values) as well as basic observations (symbolic values) are often translated into more meaningful concepts within the framework of the system under study: 0.476 is *far above normal*, 150 is *very low*, or small is *beautiful*. Concepts used to characterize quantities are often antagonistic: low/high, good/bad, above/below, false/true. This dualism implies the existence of a reference representing normality within the context of reasoning. It is either explicitly stated or implicit. Actually it corresponds to the idea of being located between both antagonistic concepts which yields an order w.r.t. the norm. It may be necessary to go beyond a binary opposition and characterize a quantity at different granularity, i.e. distinguishing among *good* and *bad* concepts more subtle characterizations such as *very good* or *very very bad*. Another big point is compositionality, i.e. how to combine *very high* with *bad*, *below* with *average* and even *blue* with *beautiful* and *cold*! These needs of comparison and combination of heterogeneous things lead us to find some set to play the role of a "common currency". Because qualitative values are intrinsically discreet entities, we chose the integers Z. Since we do not base principally our formalism on a partition of the real line (as opposed to most of QR approaches), we will refer to *quality spaces* (in the sense of Hayes, 1985) to denote the set of possible values of a given qualitative variable.

This paper addresses three main aspects: (i) how to build a quality space (QS) with a dualistic structure from a set of labels (section 2); (ii) how to compare values of (single or distinct) variables expressed on different QS at varying granularity (section 3); (iii) how to combine values through operations, consistent with both conventional arithmetic and common sense expectations (section 4). Then, we provide two examples of qualitative analysis (section 5). Finally we discuss (too briefly) this so called *dualistic algebra* (*DuAl*) formalism in the light of quantity spaces theory (section 6).

2 Quality Spaces

2.1 Dualistic Variable

Def. 2.1: A dualistic variable is a *surjective* function x: $D_v \rightarrow QS(x)$, with domain the codomain D_v of any variable (e.g. measurement, observation, or even another dualistic variable) and codomain a *quality space* QS(x), that is a finite set of ordered labels, i.e. qualitative values, as defined below.

Let L(x) be a finite non-empty set of labels of any kind λ_α, ..., λ_ω such that each of them is assumed to be a possible value for a variable x, and all of them cover the whole codomain of x. Let us assume they can be totally ordered (by any kind of objective or even arbitrary means) according to a precedence relation '≤'. Thus, $(L(x), \leq)$ as a linear order, is reflexive, anti-symmetric, transitive, and obeys the dichotomy law ($\forall \lambda_\iota, \lambda_\varphi \in L(x)$, either $\lambda_\iota \leq \lambda_\varphi$ or $\lambda_\varphi \leq \lambda_\iota$).

2.2 Description Space

Def. 2.2: Denoted $QS^*(x)$, it is the partition of $L(x)$ resulting from the family of distinct equivalence classes for the *equality* relation (from anti-symmetry of ≤):

$$[\lambda_\iota]_= = \{\lambda_{\iota'} \in L(x), \lambda_\iota \leq \lambda_{\iota'} \text{ and } \lambda_{\iota'} \leq \lambda_\iota\},$$

so $QS^*(x) = \{[\lambda_\iota]_=\}$.

We will denote $q_i, q_j, \ldots$ the $[\lambda_\iota]_=, [\lambda_\varphi]_=, \ldots$ respectively. Since $QS^*(x)$ is a partition of $L(x)$:

$$\forall q_i \in QS^*(x), \quad \bigcup q_i = L(x),$$

and $\forall q_i, q_j \in QS^*(x), \; q_i \cap q_j = \varnothing$.

2.3 Order Relation (∠)

Def. 2.3: $\forall q_i, q_j \in QS^*(x)$,

$$q_i \angle q_j \text{ iff } \forall \lambda_\iota \in q_i, \forall \lambda_\varphi \in q_j, \lambda_\iota \leq \lambda_\varphi.$$

$(QS^*(x), \angle)$ is also a linear order (reflexive, anti-symmetric, transitive, and obeying the dichotomy law).

2.4 Norm

Def. 2.4: Assuming there exists at least a special label $\lambda_\mu \in L(x)$, used as a reference for classifying the others, then $[\lambda_\mu]_=$ is called the *norm*, and is denoted q_m.

Writing $QS^{*-}(x) = \{q_i : q_i \angle q_m\}$

$QS^{*+}(x) = \{q_j : q_m \angle q_j\}$,

then: $QS^*(x) = QS^{*-}(x) \cup QS^{*+}(x)$

and $q_m = QS^-(x) \cap QS^+(x)$.

2.5 Quality Space

Def. 2.5: A description space $QS^*(x)$ is a *quality space*, denoted $QS(x)$, iff:

$$|QS^{*-}(x)| = |QS^{*+}(x)|$$

(where |.| denotes the cardinality of sets). Thus, any QS has at least one element, q_m, an odd cardinality, and is symmetric with respect to the norm: $q_i \angle q_m \angle q_j$ for all $q_i \in QS^{*-}(x)$ and $q_j \in QS^{*+}(x)$. This is in keeping with Kuipers' (1986) discretization of a continuous variable's codomain as landmark values and intervals between them. Characterizing a variable according to n landmarks ($n \geq 1$), leads to consider $2n + 1$ possible qualitative values, i.e. an odd number. In our QS, if n is odd, the norm is a landmark; if n is even, the norm is an interval between two landmarks.

P.2.5.1: $(QS(x), \angle)$, linear order, has least and greatest elements, q_{min} and q_{max} respectively: $\forall q_i \in QS_n(x), q_{min} \angle q_i \angle q_{max}$.
For sake of clarity, we may write extensively a QS as: $\{q_{min}, \ldots, q_i, \ldots, q_{max}\}$, where the middle element is the norm q_m (e.g. in {small, medium, tall}, medium is the norm).

Example 2.5.1: QS for a measurable variable. Let {0, 100} be the landmarks for the *temperature of water*. 0°C, freezing point, and 100°C, boiling point. An appropriate description space, covering the whole variable's range, may be chosen as: QS*(*water_temp*) = {(–273, 0), 0, (0, 100), 100, (100, +∞)}, where –273°C is the absolute zero. Its elements could also be denoted as labels corresponding to water states, e.g.: QS'*(*water_temp*) = {ice, freezing, liquid, boiling, vapor}. If one takes the liquid state ("liquid" or (0, 100)) as norms, then both QS* and QS'* are quality spaces for *water_temp*. They may be merged also within a single quality space, using equivalence between temperatures and states: QS(*water_temp*) = {{(–273, 0), ice}, {0, freezing}, {(0, 100), liquid}, {100, boiling}, {(100, +∞), vapor}}, where {(0, 100), liquid} is taken as the norm.

Example 2.5.2: QS for a binary variable.
• L(*bits*) = {1, no, true, zero, yes, one, 0, false)
• QS*(*bits*) = {{0, no, zero, false}, {1, yes, one, true}} is not a QS unless we add a norm to it, arbitrarily or not, depending on the granularity at which the phenomenon is observed (see how Williams, 1984, represents a digital signal as a 5 valued set); e.g. let us take the symbol ?. Assuming 0 ∠ ? ∠ 1, we write: QS(*bits*) = {{0, no, zero, false}, {?}, {1, yes, one, true}).

Example 2.5.3: QS for a non-measurable variable.
It may seem obvious that values of any measurable variable can be ordered. Let us now consider *color* with possible values: yellow, green and red; how linearly order these labels ? For this, we must define more precisely what do we mean by *color*. We give hereafter several QS, depending on the definition given to *color* as:
• my favorite color for cars: {yellow, green, red}
• increasing wavelength ranges in the light spec-

trum: {green, yellow, red}
• traffic-light spots enumerated top-down: {red, yellow, green}
• denoting the freshness of maple-tree leaves: {yellow, red, green}
• my increasing preference to paint my dining-room : {red, green, yellow}
• the lexical order of their names: {green, red, yellow}.
In every case the middle element has the meaning of being intermediate between the lowest and the highest preference. Of course, one must consider each characterization as alternative choices: *color* based on different QS are distinct variables, since the QS is what makes the variable's semantics, not the converse.

2.6 Distance

Def. 2.6: The distance between two QS elements is a metric function with integers codomain d: $QS(x) \times QS(x) \rightarrow Z$, defined for any pair ($q_i \angle q_j$) by:
• $d(q_i, q_j) = 0$, iff $q_i = q_j$
• $d(q_i, q_j) = +1$, iff: $\neg\exists\ q_k$ s.t. $q_i \angle q_k \angle q_j$
• $d(q_i, q_j) = d(q_i, q_{j'}) + d(q_{j'}, q_j)$, such that $d(q_{j'}, q_j) = 1$
• $d(q_i, q_j) = -d(q_j, q_i)$

P. 2.6.1: Composition of distances
If: $q_i \angle q_k$ ($q_i, q_k \in QS(x)$), then:
$\forall q_j \in QS(x)\ d(q_i, q_k) = d(q_i, q_j) + d(q_j, q_k)$

P. 2.6.2: Order
$\forall q_i, q_j \in QS(x)$, $q_i \angle q_j$, iff: $d(q_i, q_j) \geq 0$.

P. 2.6.3: QS cardinality
$|QS(x)| = d(q_{min}, q_{max}) + 1$

P. 2.6.4: The norm is the middle element of a QS: $d(q_{min}, q_m) = d(q_m, q_{max})$.

2.7 Ranking Function of QS Elements

Def. 2.7: The ranking function of a quality space, $\mathcal{R}$: $QS(x) \rightarrow Z$ is defined by:
$\forall q_i \in QS(x)$, $\mathcal{R}(q_i) = d(q_m, q_i)$
for Z the integers and q_m the norm of $QS(x)$.
Note that w.r.t. the definition:
$\mathcal{R}(q_m) = d(q_m, q_m) = 0$.
In the following we will denote a quality space as $QS_n(x)$ with $n = \mathcal{R}(q_{max}) = -\mathcal{R}(q_{min})$, and will call the image of $\mathcal{R}$ (also denoted $\mathcal{R}_n$) the *rank space* associated to $QS_n(x)$.

P.2.7.1: $\mathcal{R}$ is order-preserving, i.e.
$\forall\ q_i, q_j \in QS_n(x)$, $q_i \angle q_j \Rightarrow \mathcal{R}(q_i) \leq \mathcal{R}(q_j)$.

P.2.7.2: $\mathcal{R}$ is injective.

2.8 Interpretation Function

Def. 2.8: Let Z denote the integers and $QS_n(x)$ be a quality space; we define an *interpretation* as a dualistic variable I: $Z \rightarrow QS_n(x)$, by: $\forall\ z \in Z$

$$I(z) = \begin{cases} q_{max} \text{ if } z > n \\ q_i \text{ such that: } \mathcal{R}(q_i) = z, \text{ if } -n \leq z \leq n \\ q_{min} \text{ if } z < -n \end{cases}$$

(I may also be denoted I_n). By Def. 2.7, since $\mathcal{R}(q_m) = 0$, then: $I(0) = q_m$.

P.2.8.1: I is order-preserving, i.e.
$\forall\ z_i, z_j \in Z$, $z_i \leq z_j \Rightarrow I(z_i) \angle I(z_j)$

P.2.8.2: I is surjective.
Note that in general I is not the inverse of $\mathcal{R}$, since they are not bijective.

P.2.8.3: From the surjectivity of I it follows that : $|QS_n(x)| \leq |Z|$ and so: $\forall\ z \in Z$,
• if $z < \mathcal{R}(q_{min})$, then $z < \mathcal{R}(I(z))$
• if $\mathcal{R}(q_{min}) \leq z \leq \mathcal{R}(q_{max})$, then $z = \mathcal{R}(I(z))$
• if $z > \mathcal{R}(q_{max})$, then $z > \mathcal{R}(I(z))$

P.2.8.4: However, $I|_{RS_n(x)}$ is bijective; since $RS_n(x)$ and $QS_n(x)$ are finite sets they have the same cardinality. Therefore, with domain restricted to $RS_n(x)$: $I = \mathcal{R}^{-1}$ and $I^{-1} = \mathcal{R}$. As a consequence, the composition ($\circ$) of associate ranking and interpretation functions yields the identity:
$\forall\ q_i \in QS_n(x)$, $I \circ \mathcal{R}(q_i) = q_i$, and
$\forall\ z_i \in RS_n(x)$, $\mathcal{R} \circ I(z_i) = z_i$.

3 Comparison of Quality Spaces

3.1 Extension of a QS

Def. 3.1: Extending a quality space of a variable x, can be achieved recursively by adding new least and greatest elements (refining the extremities):
• $QS_0(x) = q_m$ with q_m the norm,
• $QS_{n+1}(x) = \{q_j\} \cup QS_n(x) \cup \{q_i\}$
such that: $\mathcal{R}(q_i) = -\mathcal{R}(q_j) = n + 1$.

P.3.1.1: Hierarchical inclusion.
Given $QS_n(x)$ and $QS_{n'}(y)$, and their rank spaces $RS_n(x)$ and $RS_{n'}(y)$, the lower-dimensional rank space is included within the larger one:

$$n \leq n' \Rightarrow RS_n(x) \subseteq RS_{n'}(y).$$

P.3.1.2: Given $QS_n(x)$ and $QS_{n'}(x)$, relative to the same variable x, such that $RS_n(x) \subseteq RS_{n'}(x)$, their difference is the interpretation of the difference of their rank spaces: $n \leq n' \Rightarrow$
$QS_{n'}(x) - QS_n(x) = \{I(z): z \in RS_{n'}(x) - RS_n(x)\}$
for I: $RS_{n'}(x) \rightarrow QS_{n'}(x)$, an interpretation.

3.2 Equivalence of Distinct QS Elements

Def. 3.2: Two elements of any quality space, $q_i \in QS_n(x)$ and $q_j \in QS_{n'}(y)$, are equivalent at level k (denoted $\approx_k$) if both have the same interpretation in a common quality space $QS_k(z)$:

$$q_i \approx_k q_j \text{ iff } I_k(\mathcal{R}_n(q_i)) = I_k(\mathcal{R}_{n'}(q_j)).$$

The equivalence class of any $q_i \in QS_n(x)$ at level k on any $QS_{n'}(y)$ is:

$$[q_i]_{\approx k/QSn'(y)} = \{q_j \in QS_{n'}(y) : q_i \approx_k q_j\}.$$

Figure 1 illustrates the equivalence links between different QS elements.

P.3.2: Properties of the equivalence relation
- $\approx_k$ is reflexive, symmetric, and transitive;
- $\forall\, q_i \in QS_n(x)$ and $\forall\, q_j \in QS_{n'}(y)$,

— if $\mathcal{R}_n(q_i)$ and $\mathcal{R}_{n'}(q_j)$ are the same sign, then:
$\mathcal{R}_n(q_i) \neq \mathcal{R}_{n'}(q_j) \Rightarrow q_i \approx_k q_j \Leftrightarrow k \leq min(n, n')$
$\mathcal{R}_n(q_i) = \mathcal{R}_{n'}(q_j) \Rightarrow q_i \approx_k q_j$ for all k
i.e. two elements both located on the same side w.r.t. the norm are equivalent only at more abstract levels if their ranks differ, at any level if they are the same rank.
— if $\mathcal{R}_n(q_i)$ and $\mathcal{R}_{n'}(q_j)$ are opposite signs, then: $q_i \approx_k q_j$ for $k = 0$ only, i.e. two elements located on opposite sides w.r.t. the norm can only be equivalent at the most abstract level QS_0 (see Fig. 1).

3.3 Mapping QS Elements to other QS

Def. 3.3: For any variable x, changing its value $q_i \in QS_n(x)$ into its equivalent value(s) q_j within another quality space, $QS_{n'}(x)$ (n' possibly different from n), is achieved by a mapping c: $QS_n(x) \rightarrow QS_{n'}(x)$, defined by:

$$\forall\, q_i \in QS_n(x),\ q_i \mapsto c(q_i) = [q_i]_{\approx n'/QSn'(x)}$$

If $n \leq n'$, c is called a *refinement*; if $n \geq n'$ it is called an *abstraction* (see Fig. 1).

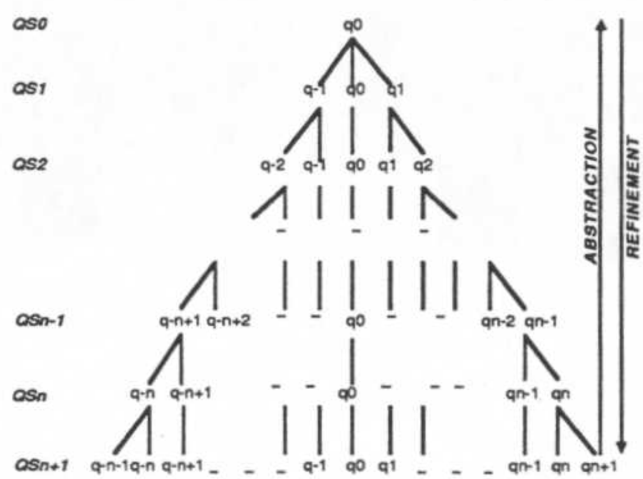

Figure 1. Equivalence ($\approx_k$) links between QS_k elements ($k = 0, \ldots, n + 1$); elements are indexed by their ranks, i.e. denoted as $q_{\mathcal{R}(q)}$.

4 Operations on QS

4.1 General Definitions and Properties

Def. 4.1.1: Internal operation.
Let $*_k$ be a k-place operation on Z ($k = 1, 2$), a k-place operation on any $QS_n(x)$ is a function $©_k$ defined by:
- $©_1$: $QS_n(x) \rightarrow QS_n(x)$, $\forall\, q_i \in QS_n(x)$,
  $©_1\, q_i = I_n(*_1\, \mathcal{R}_n(q_i))$
- $©_2$: $[QS_n(x)]^2 \rightarrow QS_n(x)$, $\forall\, q_i, q_j \in [QS_n(x)]^2$,
  $q_i\, ©_2\, q_j = I_n(\mathcal{R}_n(q_i) *_2 \mathcal{R}_n(q_j))$

P.4.1.1: Closure.
If Z is closed under $*_k$, so is $QS_n(x)$ under $©_k$.

Def. 4.1.2: Generalizing operations on any QS. Let any $QS_n(x)$, $QS_{n'}(y)$, $QS_{n''}(z)$, and any $q_i \in QS_n(x)$, $q'_j \in QS_{n'}(y)$; external operations are transformed into internal operations by changing their operands into their equivalents onto $QS_{n''}(z)$ using a function c as defined by D.3.3, that is (for ::= the rewriting sign):
- $©_1$: $QS_n(x) \rightarrow QS_{n''}(z)$, is changed into a function $QS_{n''}(z) \rightarrow QS_{n''}(z)$, by:
  $©_1\, q_i$ ::= $©_1\, c(q_i)$
- $©_2$: $QS_n(x) \times QS_{n'}(y) \rightarrow QS_{n''}(z)$ is changed into a function $[QS_{n''}(z)]^2 \rightarrow QS_{n''}(z)$ by:
  $(q_i\, ©_2\, q'_j)$ ::= $(c(q_i)\, ©_2\, c(q'_j))$

with: $c(q_i) = [q_i]_{\approx n''/QSn''(z)}$
and $c(q'_j) = [q'_j]_{\approx n''/QSn''(z)}$.

If: $n < n''$ and $n' < n''$ (transformations by c are refinements), $©_1$ yields $(n'' - n + 1)$ solutions and $©_2$ $(n'' - n + 1)(n'' - n' + 1)$; otherwise if: $n'' < n$ and $n'' < n'$ (transformations by c are abstractions) they yield only one solution.

P.4.1.2: Morphism.

An interpretation I is a homomorphism from $\langle Z, \leq, *_k, 0\rangle$ into $\langle QS_n(x), \angle, ©_k, q_m\rangle$, since I is a function (by Def. 2.8), and it is structure preserving w.r.t. order (by P.2.8.1), operations (by Def. 4.1.1) and special element: $I(0) = q_m$ (see Def. 2.8). $I|_{RS_n(x)}$, is bijective (by P.2.8.4) and so, it is an isomorphism from $\langle RS_n(x), \leq, *_k, 0\rangle$ into $\langle QS_n(x), \angle, ©_k, q_m\rangle$.

4.2 Inverse

Def. 4.2.1: The inverse of an element is the element of opposite rank:

$$\forall\, q_i \in QS_n(x),\quad inv(q_i) = I(-\mathcal{R}(q_i))$$

P.4.2: Properties of *inv*.

$\forall\, q_i, q_j \in QS_n(x)$ and q_m the norm:

- q_i and $inv(q_i)$ are symmetrical w.r.t. q_m:
  $d(inv(q_i), q_m) + d(q_i, q_m) = 0$
  If $\mathcal{R}(q_i) \geq 0$, then: $inv(q_i) \angle q_m \angle q_i$;
  if $\mathcal{R}(q_i) \leq 0$, then: $q_i \angle q_m \angle inv(q_i)$.
- fixed point: $inv(q_m) = q_m$
- involution: $inv(inv(q_i)) = q_i$
- order: $q_i \angle q_j \Leftrightarrow inv(q_j) \angle inv(q_i)$.

4.3 Addition and Subtraction

Def.4.3.1: Addition ($\oplus$) of two elements yields the *max* of both:

$$\forall\, q_i, q_j \in QS_n(x),\ q_i \oplus q_j = max(q_i, q_j)$$

i.e. q_j, if $q_i \angle q_j$, q_i otherwise.

P.4.3.1: Properties of $\oplus$

- $\oplus$ is commutative, associative, q_{min} is the identity element; then, the structures $\langle QS_n(x), \oplus\rangle$ are abelian monoids;
- $\oplus$ is idempotent, q_{max} is absorbent.

Since the identity element of addition $\oplus$ is the smallest element of a QS (i.e., q_{min}), there exists no additive inverse for all the elements of a QS (an inverse only exists for q_{min}). However we may need some kind of counterpart of subtraction in the reals; it is thus defined below.

Def.4.3.2: We define subtraction (Θ) of two elements as the first of both:

$$\forall\, q_i, q_j \in QS_n(x),\quad q_i \Theta q_j = q_i$$

Although roughly defined, this operation has some nice properties:

P.4.3.2: Complementarity of Θ with $\oplus$

- $\forall\, q_i \in QS_n(x),\ q_i \Theta q_i = q_i$
- $\forall\, q_i, q_j, q_k \in QS_n(x)$,
  $q_i \oplus q_j = q_k \Leftrightarrow q_i = q_k \Theta q_j = q_k$ iff: $q_j \angle q_i$
  (this restriction must be kept in mind when transforming equations).

$$(q_i \oplus q_j) \Theta q_k = q_i \oplus (q_j \Theta q_k) = q_i \oplus q_j$$

4.4 Multiplication and Division

D.4.4.1: The multiplication ($\otimes$) of two elements is the interpretation of the arithmetic sum of their ranks:

$$\forall\, q_i, q_j \in QS_n(x),\quad q_i \otimes q_j = I(\mathcal{R}(q_i) + \mathcal{R}(q_j))$$

P.4.4.1: Properties of $\otimes$

- $\otimes$ is commutative, associative, q_m is the identity element but it has no absorbent element.
- *inv* yields the multiplicative inverse:
  $\forall\, q_i,\ q_i \otimes inv(q_i) = q_m$

These properties give $\langle QS_n(x), \otimes\rangle$ a structure of abelian group. Therefore the right and left cancellation laws hold: the equation $q_i \otimes q = q_j$ has a unique solution $q = inv(q_i) \otimes q_j$.

- *inv* is distributive over $\otimes$:
  $inv(q_i \otimes q_j) = inv(q_i) \otimes inv(q_j)$
- $\otimes$ is distributive over $\oplus$ and Θ:
  $q_i \otimes (q_j \oplus q_k) = (q_i \otimes q_j) \oplus (q_i \otimes q_k)$
  $q_i \otimes (q_j \Theta q_k) = (q_i \otimes q_j) \Theta (q_i \otimes q_k)$
  $= (q_i \otimes q_j)$

D.4.4.2: The division ($\varnothing$) of two elements is the interpretation of the difference of their ranks:

$$\forall\, q_i, q_j \in QS_n(x),\quad q_i \varnothing q_j = I(\mathcal{R}(q_i) - \mathcal{R}(q_j))$$

P.4.4.2: Properties of $\varnothing$

- Complementarity with $\otimes$:
  $\forall\, q_i, q_j,\quad q_i \varnothing q_j = q_k$ iff: $q_i = q_j \otimes q_k$
- $\forall\, q_i, q_j,\quad q_i \varnothing q_j = q_i \otimes inv(q_j)$ and so:

$q_i \varnothing inv(q_j) = q_i \otimes q_j$, $q_m \varnothing q_i = inv(q_i)$,

$inv(q_i \varnothing q_j) = q_j \varnothing q_i$, $q_i \varnothing q_i = q_m$.

4.5 Power and Root

D.4.5.1: An element power ($[.]\cdot$) another is the interpretation of the product of both ranks:

$$\forall\ q_i, q_j \in QS_n(x),\ [q_i]^{q_j} = I(\mathcal{R}(q_i)\cdot\mathcal{R}(q_j))$$

P.4.5.1: Properties of $[.]\cdot$

- The power is commutative and associative.
- $I(1)$ (element of rank 1) is its identity element:

$$\forall\ q_i,\ [q_i]^{I(1)} = [I(1)]^{q_i} = q_i$$

Since the existence of an inverse for all elements is lacking $\langle QS_n(x), [.]\cdot\rangle$ are not groups but abelian monoids.

- Combination with *inv*:

$$[q_i]^{inv(q_j)} = inv([q_i]^{q_j}) = q_m \varnothing [q_i]^{q_j}$$

- The norm is absorbent: $\forall\ q_i,\ [q_i]^{q_m} = q_m$
- Compatibility with $\otimes$:

$$[q_i]^{q_j} = q_i \otimes \ldots \otimes q_i,\quad \mathcal{R}(q_j) \text{ times.}$$

D.4.5.2: The root ($\dot{\overline{)}}$) of an element is the interpretation of the entire division of its rank by the rank of the root:

$$\forall\ q_i \in QS_n(x),\ \forall\ q_j \neq q_m,$$

$$\overset{q_j}{\overline{)q_i}} = \begin{cases} q_k \text{ iff: } \mathcal{R}(q_i) = \mathcal{R}(q_j)\cdot\mathcal{R}(q_k) \\ (q_k, q_{k'}) \text{ iff: } \mathcal{R}(q_i) = \mathcal{R}(q_j)\cdot\mathcal{R}(q_k) + \tau \\ \qquad = \mathcal{R}(q_j)\cdot\mathcal{R}(q_{k'}) + \sigma \end{cases}$$

for $\tau, \sigma \in Z$, $\tau > 0$, $\sigma < 0$ and $d(q_k, q_{k'}) = 1$ (closest neighbors of the exact result).

P.4.5.2: Properties of $\dot{\overline{)}}$.

- Identity element: $\forall\ q_i,\quad \overset{I(1)}{\overline{)q_i}} = q_i$
- Self-inverse: $\forall\ q_i \neq q_m,\ \overset{q_i}{\overline{)q_i}} = I(1)$
- Absorption: $\forall\ q_i \neq q_m,\ \overset{q_i}{\overline{)q_m}} = q_m$
- Compatibility with the power:

$$\forall\ q_j \neq q_m,\ \left[\overset{q_j}{\overline{)q_i}}\right]^{q_j} = q_i$$

- Distributivity: $\forall\ q_j \neq q_m$,

— over $\otimes$: $\quad \overset{q_j}{\overline{)q_i \otimes q_k}} = \overset{q_j}{\overline{)q_i}} \otimes \overset{q_j}{\overline{)q_k}}$

— over $\varnothing$: $\quad \overset{q_j}{\overline{)q_i \varnothing q_k}} = \overset{q_j}{\overline{)q_i}} \varnothing \overset{q_j}{\overline{)q_k}}$

4.6 Operations Semantics

We can set some semantic equivalence between: (i) I and exponentiation, and (ii) the ranking function $\mathcal{R}$ and logarithms (see Table 1). With this respect, a quality space has to some extend the semantic value of a geometric progression, whereas the rank space stands for an arithmetic progression (with ratio 1). This is why we call multiplication, division, power and root, operations on QS based on +, –, ×, /, in Z, respectively.

Table 1: Corresponding notions between the *DuAl* and exponentiation as a model.

	Quality Spaces	Model	Substitution Rules
Values	$q_i, q_j, q_k \in QS_n(x)$	$y_i, y_j, y_k \in \Re^+$	
Ranks	$z_i = \mathcal{R}(q_i) \in RS_n(x) \subset Z$	$x_i = \log_a y_i \in \Re$	
Interpretation	$q_i = I(z_i) = I(\mathcal{R}(q_i))$	$y_i = a^{xi}$	
Special elements	q_{min} q_m	0 1	$0 ::= q_{min}$ $1 ::= q_m$
Addition	$q_i \oplus q_j = max(q_i, q_j)$	$y_i + y_j \approx a^{\max(xi,\ xj)}$	$+ ::= \oplus$
Subtraction	$q_i \ominus q_j = q_i$	$y_i - y_j \approx a^{xi}$	$- ::= \ominus$
Mult. inverse	$inv(q_i) = I(-\mathcal{R}(q_i))$	$(y_i)^{-1} = a^{-xi}$	$(.)^{-1} ::= inv(.)$
Multiplication	$q_i \otimes q_j = I(\mathcal{R}(q_i) + \mathcal{R}(q_j))$	$y_i\, y_j = a^{xi}a^{xj} = a^{xi + xj}$	$\times ::= \otimes$
Division	$q_i \varnothing q_j = I(\mathcal{R}(q_i) - \mathcal{R}(q_j))$	$y_i / y_j = a^{xi}/a^{xj} = a^{xi - xj}$	$/ ::= \varnothing$
Power	$[q_i]^{q_j} = I(\mathcal{R}(q_i)\cdot\mathcal{R}(q_j))$	$(y_i)^{xj} = (a^{xi})^{xj} = a^{xi\ .xj}$	$(.)\cdot ::= [.]\cdot$
Root	$\overset{q_j}{\overline{)q_i}}$	$(y_i)^{1/xj} = a^{xi\ /xj}$	$(.)^{1/.} ::= \dot{\overline{)}}$

5 Two Examples of Application

5.1 Analysis of a Non-Linear Model

We take from De Kleer and Brown (1984) the example of the Cochin's law:

$$Q = CA\sqrt{\frac{2P}{\rho}} \qquad \text{(Eq. c1)}$$

where Q is the flowrate through a valve, C the discharge coefficient through the orifice of area A, P the pressure across the valve and ρ the mass density of the fluid. Since C, A, P and ρ are all positives, using the sign algebra-based formalism yields $[Q] = +$. Instead of that, one would like to assess the influence of the magnitude of any of the variables C, A, ρ and P, on the value of Q. That is, answering questions such as: "What if C is low, A small, P much above normal and ρ is more than very heavy ?". Similarly: "keeping C, A, and P values constant, what would be the influence of ρ ?".

Transforming the Cochin's expression into the *DuAl* formalism yields:

$$Q = C \otimes A \otimes \sqrt[I(2)]{(P \oplus P) \varnothing \rho}$$

since $P \oplus P = P$ (idempotence), the expression simplifies to:

$$Q = C \otimes A \otimes \sqrt[I(2)]{P \varnothing \rho} \qquad \text{(Eq. c2)}$$

Quality spaces we may choose (this is our choice) for the variables are:

$QS_1(A)$ = {small, regular, large}
$QS_2(Q)$ = {very_small, small, OK, big, very_big}
$QS_2(C)$ = {very_low, low, normal, high, very_high}
$QS_2(P)$ = {m_below, below, normal, above, m_above}
$QS_3(\rho)$ = {VVL, VL, L, M, H, VH, VVH}

Answering the first question is achieved after substitution in Eq. c2 of known variables for their values:

$$Q = \text{low} \otimes \text{small} \otimes \sqrt[I(2)]{\text{m\_above} \varnothing VVH}$$

However to perform a consistent calculation, those values must be expressed in terms of their equivalent in $QS_2(Q)$ using transformation functions (Def. 3.3): c: $q_i \mapsto c(q_i) = [q_i]_{\approx 2/QS2(Q)}$.

So, the initial values of known variables to be used in the calculus are given by:

C = low $\mapsto c$(low) = {small}
A = small $\mapsto c$(small) = {very_small, small}
P = m_above $\mapsto c$(m_above) = {very_big}
ρ = VVH $\mapsto c$(VVH) = {very_big}

Since we have two values for A, we get two possible solutions:

(i) if A = very_small

$$Q = \text{small} \otimes \text{very\_small} \otimes \sqrt[I(2)]{\text{very\_big} \varnothing \text{very\_big}}$$

(ii) if A = small

$$Q = \text{small} \otimes \text{small} \otimes \sqrt[I(2)]{\text{very\_big} \varnothing \text{very\_big}}$$

However, since:

(small ⊗ very_small) = (small ⊗ small) = very_small,

solutions (i) and (ii) sum up to only one solution:

$$Q = \text{very\_small} \otimes \sqrt[I(2)]{\text{very\_big} \varnothing \text{very\_big}}$$

$$= \text{very\_small} \otimes \sqrt[I(2)]{\text{OK}} = \text{very\_small} \otimes \text{OK}$$

that is: Q = very_small, which answers the above first question.

To answer the second question, keeping the above values for C, A, and P, let us calculate Q according to all the possible values of ρ by solving the expression derived from instanciation of Eq. c2 by known values:

$$Q = \text{low} \otimes \text{small} \otimes \sqrt[I(2)]{\text{m\_above} \varnothing \rho}$$

using equivalent values and simplifications as above, it is rewritten as:

$$Q = \text{very\_small} \otimes \sqrt[I(2)]{\text{very\_big} \varnothing \rho}$$

Solutions are in table 2. Although the initial model is non-linear, we get a linear response.

Table 2: Calculus of Q according to ρ (for C = low, A = small, P = m_above).

ρ	VVL	VL	L	M	H	VH	VVH
$c(\rho)$	very_small	very_small	small	OK	big	very_big	very_big
Q	OK	OK	OK, small	small	small, very_small	very_small	very_small

Table 3: Calculus of P from C = low, A = small, and values of ρ and Q in table 2.

ρ	VVL	VL	L	M	H	VH	VVH
$c(\rho)$	m_ below	m_ below	below	normal	above	m_ above	m_ above
Q	OK	OK	OK, small	small	small, very_small	very_small	very_small
$c(Q)$	normal	normal	normal, below	below	below, m_below	m_below	m_below
P	m_ above	m_ above	m_ above, above	m_above	m_above, above	m_above	m_above

In order to check the stability of calculus with respect to a transformation of the initial expression, let us calculate P from the initial values of C and A (i.e. C = low, A = small), and all the configurations of ρ and Q values expressed in table 2.

The transformation is performed as follows:

$$Q = C \otimes A \otimes \overline{)P \varnothing \rho}^{\,I(2)}$$

$$\Rightarrow Q \varnothing (C \otimes A) = \overline{)P \varnothing \rho}^{\,I(2)}$$

$$\Rightarrow [Q \varnothing (C \otimes A)]^{I(2)} = P \varnothing \rho$$

$$\Rightarrow P = \rho \otimes [Q \varnothing (C \otimes A)]^{I(2)}.$$

The calculus must lead to P = m_above. The results are summarized in table 3, which shows that stability is reasonably insured since all the solutions cover the expected one.

5.2 Relative Orders of Magnitude

The O(M) formalism provides a set of primitive relations to compare quantities one to each other (Mavrovouniotis & Stephanopoulos, 1988).

The semantics of a relation is the comparison of the ratio of both quantities with respect to 1, say $x < y$ iff: $x/y < 1$.

We can compute this using the *DuAl* formalism by considering the ratio x/y as a single variable depending on x and y values such that:

$x/y = x \varnothing y$, and taking as a QS:

$QS_3(x/y)$ = {<<, -<, ~<, ==, >~, >-, >>},

the primitive relations of O(M), ordered with respect to strict equality == ($x/y = q \in QS_3(x/y)$, means $x\ q\ y$).

For sake of clarity, let x and y quality spaces be the same:

$QS_3(x) = QS_3(y)$ = {tiny, v_small, small, normal, tall, v_tall, giant}.

The respective values of x/y are given in table 4.

Applied to the counter current heat exchanger example described in Mavrovouniotis and Stephanopoulos (1988) we can derive conclusions consistent with O(M).

The model is based on two equations:

- Temperatures differences:

  DTH – DT1 – DTC + DT2 = 0 (Eq. o1)

- Energy balance:

  DTH.KH.FH = DTC.KC.FC (Eq. o2)

in which DTx are temperature differences (H and C stand for hot and cold flows, respectively), KH and KC are the molar-heats, FH and FC are the molar flowrates.

Given initial conditions:

DT2/DT1 = (-<),
DT1/DTH = (<<),
KH/KC = (>~),

let us derive: DT2/DTH, DTC/DTH, DT1/DTC, and FC/FH.

- DT2/DTH = DT2/DT1 . DT1/DTH

  ::= DT2/DT1 ⊗ DT1/DTH

  = (-<) ⊗ (<<) = (<<),

solution found by O(M).

- By eq. o1 we get:

1 – DTC/DTH – DT1/DTH + DT2/DTH = 0, then: DTC/DTH = 1 – DT1/DTH + DT2/DTH

::= (==) ⊖ (<<) ⊕ (<<) = (==),

consistent with: (~<..>~) = {~<, ==, >~} the solution derived by O(M).

- DT1/DTC = (DT1/DTH) / (DTC/DTH)

  ::= (<<) ⌀ (==) = (<<),

solution found by O(M).

- By eq. o2:

FC/FH = KH/KC . (DTC/DTH)$^{-1}$

::= KH/KC ⌀DTC/DTH = (>~) ⌀ (==) = (>~),

partially consistent with:

(~<..>~) = {~<, ==, >~},

solution found by O(M).

For all these solutions one can find in table 4 the set of possible values of x and y satisfying the constraint expressed by the ratio x/y .

Table 4: $x/y = x \varnothing y$ where x/y has the O(M) primitive relations as codomain.

$x\rightarrow$ $c(x)\rightarrow$ $y\downarrow$ $c(y)\downarrow$	tiny <<	v_small ≺	small ~<	normal ==	tall >~	v_tall ≻	giant >>
tiny <<	==	>~	≻	>>	>>	>>	>>
v_small -<	~<	==	>~	≻	>>	>>	>>
small ~<	≺	~<	==	>~	≻	>>	>>
normal ==	<<	≺	~<	==	>~	≻	>>
tall >~	<<	<<	≺	~<	==	>~	≻
v_tall >-	<<	<<	<<	≺	~<	==	>~
giant >>	<<	<<	<<	<<	≺	~<	==

6 Conclusions and Perspectives

The *DuAl* formalism states that two elements of same rank are "equal", not because they are represented by the same label or their real magnitude are equal, but because their distance with respect to their own norm within some order are equal. This captures the intuitive notion of assessment or preference scales. Let apple pie be an element of the quality space for cakes, and roasted beef be an element for meat meals; saying: "I like apple pie as much as roasted beef" comes from the fact they are both interpreted as "good meals" in my own scale of preference for meals, although they are different things. Pushing more, I may even say that an apple pie is greater than America's cup watched at the television. This may seem unintuitive and somewhat scruffy, unless I add that if I had to choose between both activities I would prefer eating an apple pie than watching TV. Though, the idea is that apple pie, roasted beef, watching the America's cup, are objects ordered with respect to some common preference, utility, or assessment scale. This scale is the distance of objects to norms and all norms have the same utility (zero) value of being a reference for some relevant objects. Stating an equivalence between two objects having the same interpretation in a common quality space is thus the underlying idea of our equivalence relation at level k.

Therefore, reasoning about quantities, may also lead to deal with ordinal instead of cardinal values. That is why we resolved to choose the term *quality space* instead of *quantity space* to denote the set of possible values for any variable. According to Forbus' definition (1984), a quantity space is a finite set of distinguishable values for signs (taking values on $\{-1, 0, +1\}$), magnitudes (taking values on $\Re^+ \cup \{0\}$) and finally numbers, composed of both sign and magnitude (taking values on $\Re$). In fact, quantity spaces are collections of numerical values, eventually interpreted as linguistic labels (e.g., Davis, 1987). Therefore, operations of comparison and combination on them are well defined, using the classical algebraic operations or adaptation of them (as in the sign or interval algebras). This is also the case in Raiman's (1991) order of magnitude reasoning (e.g. coarse values are sets of numbers). If our QS values q_i stand for some kind of magnitudes, they are not numbers since the sign component is lacking. More generally, Hayes' concept of quality spaces denotes the set of all possible values of a quantity, whatever their nature be. In those, as in our QS, may exist notions such as distance between values, tolerance, non density, as well as functions to map qualities with some linear order. *DuAl* is closer to this concept.

As emphasized by Struss (1990), most of the work in qualitative reasoning has been directed towards the *qualitative* interpretation of *quantitative* equations (i.e. real-valued). Our approach is a bit different. Starting from qualitative, local, poorly structured knowledge, our aim is to provide a representation framework enabling one to derive unknown *values* from existing ones. The *DuAl* approach came from difficulties encountered in modeling complex systems, such as ecological or biological, in which it is often useful to compare and combine what is not comparable or combinable. Human reasoning does so, but classical models do not.

In terms of applications, a preliminary approach, using mainly *ad hoc* combination tables and some primitive operations, was proved useful to heuristic knowledge representation (Guerrin, 1991). In the current work, we put the emphasis on operations. Although the task of representing knowledge in the form of equations is not easy, the use of operations with known properties guarantees sound calculations and provide verification possibilities, especially when many possible values exist. However we think this will not refrain heuristic representation if: (i) the semantics of operations is clear, (ii) the modeler can check several combinations before choosing the adequate one, (iii) identification techniques are set up to help find the right model from partial *ad hoc* tables. Verification of these aspects is a perspective. However, dealing with first principles equations, and comparison to other QR systems, is also possible, as shown in section 5 examples. Particularly, *DuAl* fits the general definition given by Williams (1991) of a qualitative algebra, as an abstraction of a quantitative one (here integers arithmetic). The *DuAl* approach was also checked against Raiman's example (1991) of colliding masses, with consistent analytical results except in one case. This shortcoming comes from the lack of additive inverse (in the sense of our ⊕) and the brute force definition of addition and subtraction, which is a limitation. In fact, our quality spaces, made of positive elements, are more likely to the *Rough* set (not closed under subtraction) of Raiman's approach, whereas the *Small* set (including zero) is lacking. A means to connect our *Rough* sets to *Small* sets to improve calculation is another perspective.

The refinement process for expanding a QS is, till now, based on refining its extremities. This is because greatest and least elements are very often considered as right and left open intervals respectively (e.g. any value above or below numerical thresholds, respectively). We are currently generalizing this notion of expanding a QS by refining all the elements, be they extremities or intermediate elements. This may lead also to define more gradual operations and certainly improve the modeling power of the formalism by improving its flexibility.

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