Reasoning about Structure of Interval Systems: An Approach by Sign Directed-Graph

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Abstract

This paper deals with qualitative and structural reasoning on linear interval systems whose parameters are specified by intervals. We formalize the systems of reasoning about structures of interval systems by the qualitative perturbation principle: the interval system would have the interval property when its underlying sign structure include the component that has the corresponding sign property and the norm of the rest of component (considered qualitative perturbation) is small enough. Several interval properties of interval matrix such as nonsingularity, rank and inverse stability will be discussed by applying the principle to the graphical conditions for the corresponding sign properties. The Klein model in economics is used as an illustrative example.

Introduction

In *Qualitative Reasoning* about dynamical systems, it appears that depending upon only qualitative values such as signs or landmarks may not suffice due to the *ambiguity explosion.* Recently, semi-quantitative approach using intervals has been paid attention (Kuipers and Berleant 1988; Berleant and Kuipers 1992; Kay and Kuipers 1993; Vescovi et al. 1995; Nogi & Ishida 1995) aiming at analysis and synthesis on real problems which are large-scale and complex. However, using intervals as system parameters often causes the *computation explosion.* This paper presents another avenue of research: using qualitative analysis (pure qualitative approach in the sense of using sign directed graph) for reasoning about interval systems.

Qualitative and structural analysis have been studied in many different areas: economic systems, ecosystems, and system theory to mention only a few. Analysis and synthesis on systems with parameter uncertainty has been studied in system theory (Maybee 1981; Ishida et al. 1981; Maybee et al. 1989; Rohn 1990; Mansour 1992; Rohn 1993a). The level of uncertainty may be divided into four; numerical (most specific) specification, interval specification, sign specification, and zero/nonzero specification of parameters as shown in Figure 1. We call numerical, interval, sign and structured systems respectively, corresponding to these level of parameter specification. The motivations for the study of systems with uncertain parameters are as follows: (1) modeling errors are inevitable in modeling real systems which are often complex and large-scale, and (2) early phase of system design must allow undetermined values in parameters.

These results have several implications in *Qualitative Reasoning* about systems with qualitatively expressed interactions and states. In this paper, we present principles which can be used to reason about interval systems based on its sign structure. These principles have been implicitly used in system theory, however, we formalize them as principles upon which theorems and propositions in this paper depend. We briefly present some theoretical results to demonstrate how the principle can be applied to the static properties of the system matrix. This research aims to apply the results for one level to the other level in Figure 1. Particularly, this paper focuses on the results for sign matrices that can be used to deduce the corresponding results for interval matrices by the structural perturbation principle (stated in the next section).

The next section presents qualitative perturbation principle and some preliminary concepts required in the subsequent sections. In the section of *Sign Singularity Analysis by SDG*, sign nonsingularity conditions by signed directed graph (called SDG hereafter) are stated. The section *Singularity Analysis on Interval Matrices* presents an algorithm for determining the max/min values of the determinants of the interval matrices. Some vertex matrices [1] realizing max/min of the determinants of the interval matrices can be fixed based on the structural properties of signed digraph. The section *Properties of Static Systems: Rank, Nonsingurality, and Inverse Stability* extends several properties such as nonsingularity, rank, and inverse stability to sign and interval matrices. Several results associating sign properties with corresponding interval properties are also presented. The section *The Klein Model as an Illustrative Example* demonstrates how the qual-

[1] We use the word vertex for interval matrices and the word node for graphs.

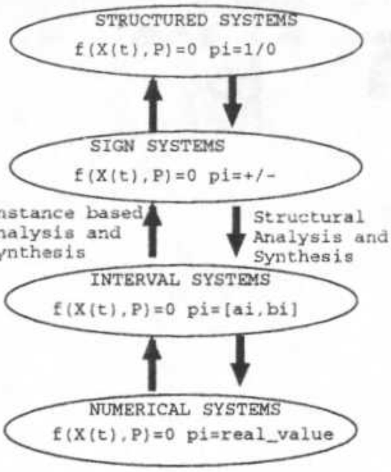

Figure 1: Levels of Systems based on Parameter Specification

itative perturbation principle together with the results stated in section *Properties of Static Systems: Rank, Nonsingurality, and Inverse Stability* can be applied to a specific system, taking the Klein model as an illustrative example.

Preliminaries

Sign matrix A^S is a matrix whose elements are $+, -, 0$. Interval matrix A^I is a matrix whose elements are specified by interval $[a, b]$ where a and b are two terminal values such that $a \leq b$. An interval matrix can be considered a set of matrices whose elements are in the intervals specified by the interval matrix.

For simplicity, we focus on interval matrices whose intervals can be identified as $+, 0$, or $-$. That is, we do not consider such interval as $[-2, 5]$. This may not be a strong limitation, since at least sign of system parameters could be identified in most cases except time varying systems. Further, when the matrix is reducible [2], it can be reduced to irreducible components to each of which the analysis in this paper can be applied independently. Thus, the matrix is assumed to be irreducible in the rest of this paper. Graph theory has been extensively used in search algorithms (e.g. (Tarjan 1972; Rose & Tarjan 1978)). We also use graph for expressing the sign structure of systems.

Definition 1 *(Signed Digraph for Matrices)*
SDG (signed directed graph) of a matrix $A \in R^{n \times n}$ is a graph with n nodes and arcs directed from node i to node j with sign +(-) when $a_{ij} > 0 (< 0)$ (i.e. the interval is lying within $0, \infty$).

Example 1

The SDG of the following interval matrix is shown in Figure 2.

$$\begin{pmatrix} [-2,-1] & [-2,-1] & [0,0] \\ [-2,-1] & [-4,-3] & [-5,-4] \\ [-3,-1] & [2,4] & [-2,-1] \end{pmatrix}$$

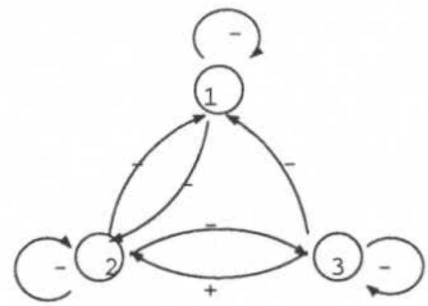

Figure 2: SDG of the Interval Matrix

Definition 2 *(Sign Nonsingularity and Interval Nonsingularity)*

Sign matrix is called sign nonsingular if all the matrices having the sign structure is nonsingular. Interval matrix is called interval nonsingular if all the matrices whose elements are lying within the intervals are nonsingular.

As presented in the section of *Properties of Static Systems: Rank, Nonsingurality, and Inverse Stability*, (Maybee 1981) proved that sign structure with with all the negative loop (with all the diagonal elements negative) and with no positive cycle is sign nonsingular.

In the following, we present a principle that is used to reason about interval systems based on their sign structures. First, *Qualitative Inclusion Principle* may be stated as follows:

> For a system matrix to have an interval property, the system must contain a subsystem that has the corresponding sign property.

This principle can apply to such properties as stability, observability, nonsingularity, inverse stability (that will be defined in the next section), solvability, *etc.* of system matrix. For example, application of the above qualitative inclusion principle to a system matrix for a property of nonsingularity results in the following statement; for an interval matrix to be interval nonsingular, its SDG must contain the arc-subgraph [3] that is

[2]Matrix A is called reducible if it can be decomposed into upper (or lower) diagonal blocks by permutation matrix P:

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

[3]Arc-subgraph is the graph obtained by deleting some arcs. Similarly, node-subgraph is the graph obtained by deleting some nodes and all the arcs associated with the nodes.

sign nonsingular.

Due to the above qualitative inclusion principle, sign structures that generically [4] have the property may be decomposed into two subsystems: subsystem (arc-subgraph of the sign structure) that satisfies a sign property and a set of interactions (arcs) whose removal result in the system having the sign property. The set of interactions being removed is considered a qualitative perturbation imposed on the sign structure having the sign property. With this decomposition, *Qualitative Perturbation Principle* may be stated as:

> If a sign structure generically have the property but the interval system having the sign structure fails to have the corresponding interval property due to the qualitative perturbation, then the interval system would have the interval property if the qualitative perturbation is made small enough relative to the rest of system.

Again, applied to nonsingularity of matrix, the next statement follows; if the sign structure of an interval matrix is generically nonsingular, then the interval matrix can be made interval nonsingular by making the qualitative perturbation (the set of arcs whose removal leaves sign nonsingular graph) small.

This principle can be used not only in the analysis on whether the given interval system satisfy an interval property but also in the synthesis of parameters to satisfy the interval property based on its sign structure as will be discussed in the section of *Related Work and Discussions.* Although we explained the principle applying from sign systems to interval systems, similar principle can be used from structured systems to sign systems and from interval systems to numerical systems.

We will be more specific about how the principle can be used in the following sections by applying it to such properties as nonsingularity and inverse stability.

In the next section, we will present more informative measure than nonsingularity for sign and interval matrices. For singular matrices, rank would give information of how singular the matrix is. For nonsingular matrices, information of how many submatrices of order $n-1$ are nonsingular (and recursively for submatrices with the order $n-2,\cdots,1$) would indicate how nonsingular the matrix is. The former singularity measure can be used to reason which arcs or nodes should be removed (in case of sign matrix) or made small (in case of interval matrix) to attain nonsingularity. The latter nonsingularity measure can be used to assess the properties stronger than nonsingularity such as inverse stability and solvability.

Sign Singularity Analysis by SDG

Definition 3 *(Cycle and G[n]-cycle)*
The cycle of length k denoted by $c[i_1,i_2,\cdots,i_k]$ is a path connecting the nodes $i_1,i_2,\cdots,i_k$ and i_1 sequentially[5]. The set of disjoint cycles is called G[n]-cycle if the total length of these cycles is n.

All the possible G[n]-cycles for $A \in R^{n\times n}$ correspond to all the terms in the expansion of determinant of A^I. Figure 3 shows all the possible G[3]-cycle for the graph shown in Figure 2.

Let $p[c_i]$ denote the product of all the elements in the cycle $c_i = c[i_1,i_2,\cdots,i_k]$. That is, $p[c_i] = a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_k}$. Let $C_i = \{c_{i1},c_{i2},\cdots,c_{iq_i}\}$ be a G[n]-cycle. Then a term in the expansion of the determinant A can be written as follows by (Goldberg 1958):

$$(-1)^n(-1)^{q_i}p[c_{i1}]p[c_{i2}]\cdots p[c_{iq_i}].$$

Definition 4 *(Admissible Qualitative Operations (Lancaster 1962))*
(1) multiplying the sign in any rows by (-1).
(2) multiplying the sign in any column by (-1).
(3) interchanging any two rows.
(4) interchanging any two columns.

For any sign matrix, sign solvability (hence sign nonsingularity) is known to preserve under any combination of above admissible qualitative operations (Lancaster 1962).

The admissible qualitative operations, however, do not preserve the structure of graph.

Lemma 1 *A cycle $c[i_1,i_2,\cdots,i_k]$ of length k can be transformed into a set of k loops (i.e. cycle with length one) $\{c[i_1],c[i_2],\cdots,c[i_k]\}$ by the admissible qualitative operations.*

Sign structure of sign matrices has been studied extensively by (Maybee 1981; Maybee et al. 1989).

The following is a graphical characterization for a sign matrix to be sign nonsingular quoted from (Maybee 1981).

Theorem 2 *(Sign Nonsingularity Condition (Maybee 1981))*
Let $A \in R^{n\times n}$ be a matrix with $a_{ii} < 0$ for $i = 1,\cdots n$. Then all terms in the expansion of det A are weakly of the same sign if and only if all cycles of A are nonpositive.

In fact, the condition that all the diagonal elements are negative can be considered necessary.

Lemma 3 *If a sign matrix is nonsingular then by the admissible qualitative operations it can be put into the form where all the diagonal elements are negative. Such form is called* **standard form** *hereafter.*

Theorem 4 *By the admissible qualitative operations, if an interval matrix can be put into the form:*
(1) All the diagonal elements are negative, and
(2) There are no positive cycle.

[4] An interval (sign, or structured) system is said to have a property generically when at least one instance of the interval (sign, or structured) system has the property.

[5] We use the word cycle to mean a simple cycle. That is, the nodes $i_1,i_2,\cdots,i_k$ are different.

In the rest of paper, matrix is assumed to be transformed into the form where all the diagonal elements are negative.

Corollary 5 *The sign of of a term in the expansion of the determinant of the matrix $A \in R^{n\times n}$ is invariant if any negative cycle of length k $c[i_1, i_2, \cdots, i_k]$ is replaced with corresponding k disjoint negative loops: $\{c[i_1], c[i_2], \cdots, c[i_k]\}$.*

Definition 5 *(Sign Conflict)*
If all the cofactors of the element a_{ij} of the matrix A when its determinant is expanded is not of the same sign, then the element a_{ij} (or its corresponding arc in SDG) is called **sign conflict**. *Otherwise, it is called* **sign non-conflict**.

The next lemma follows directly from the definition of sign conflict and G[n]-cycle.

Lemma 6 *If an element of the matrix is both in the G[n]-cycle consisting of only negative cycles and in the G[n]-cycle consisting of at least one positive cycle then the element is* **sign conflict**.

Whether or not the element is sign conflict can be known in the SDG without decomposing it into G[n]-cycle. The following is a graph theoretical condition for an element to be **sign conflict** as follows.

Theorem 7 *An element a_{ij} is* **sign conflict**, *if and only if (1) the arc a_{ij} is both in a positive cycle and a negative cycle, or (2) the arc a_{ij} is in the cycle disjoint with a positive cycle.*

Sign nonsingular matrix is such a matrix that does not have any sign conflict element.

The element a_{12} of the matrix of Example 1 is sign conflict, since it satisfies the condition (1) of theorem 7 (it is included in the positive cycle $c[1,2]$ and the negative cycle $c[1,2,3]$). The element a_{33} is also known to be sign conflict, since it satisfies the condition (2) of theorem 7 (there is a positive cycle e $c[1,2]$ disjoint with it). These are also known to be sign conflict by applying above Lemma 6 to the G[3]-cycle decomposition shown in Figure 3.

Singularity Analysis on Interval Matrices

Min/Max of the determinant of interval matrices

Since sign nonsingularity requires that all the non-zero terms of the expansion of determinant must be of the same sign, the next lemma follows immediately.

Lemma 8 *If the SDG of an interval matrix is sign nonsingular, then the vertex that realizes the minimum absolute value of determinant of the interval matrix is that with smaller(greater) absolute value of two terminal values for each interval.*

Since all the diagonal elements of the interval matrices under consideration are assumed to be negative, the determinant of the interval matrices have the term $p[c_1]p[c_2]\cdots p[c_n]$ in the expansion. We call the sign of the term $sgn(p[c_1]p[c_2]\cdots p[c_n]) = (-1)^n$ standard sign, since all the other non-zero terms in the determinant expansion of sign nonsingular matrices have the same sign as this.

Even if an interval matrix is not sign nonsingular, the terminal value that realizes the maximum or minimum absolute value can be determined if the element is sign non-conflict.

Theorem 9 *The vertex that realizes the minimum absolute value of determinant of the interval matrix is that with smaller(greater) absolute value for the sign non-conflict element when it is in the negative(positive) cycle.*

Obviously, results similar to lemma 8 and theorem 9 stating the vertex realizing the maximum absolute value can be obtained with the word "smaller" and "greater" exchanged.

Algorithm for finding minimum value of the determinant

Based on the above theorem 9, the following algorithm for finding the vertex that realizes determinant with minimum absolute value is proposed.

Algorithm 10

STEP 1: Assign the terminal values to the elements of sign non-conflict based on the theorem 9.

STEP 2: Find the element of sign conflict whose cofactor does not have the element of sign conflict. If found, assign the appropriate terminal value to the element of sign conflict depending on the sign of the cofactor. Continue this step until there is no element of sign conflict whose cofactor does not have the element of sign conflict.

STEP 3: Find the element of sign conflict whose cofactor has the elements of sign conflict, but the sign of the cofactor does not change which terminal value the element of sign conflict may take. If found, assign the appropriate terminal value to the element of sign conflict depending on the sign of the cofactor, and go back to the STEP 2. If not found, proceed to the next step.

STEP 4: Carry out a combinatorial search for the remained elements of sign conflict.

Example 2

In the following, the above algorithm is demonstrated for the same example as 1

In STEP 1 of the algorithm, a_{12} is the sign conflict element by the condition (1) of theorem 7. a_{33} is also

Table 1: Sample matrices

No.	Size	Nonzero elements	Cycles	Conflicts	Nonsingularity
1	5	25	89	25	nonsingular
2	5	15	10	5	singular
3	8	22	15	0	nonsingular
4	8	39	104	9	nonsingular
5	14	42	30	0	nonsingular
6	14	42	30	37	nonsingular
7	14	82	69	0	nonsingular
8	14	118	105	0	nonsingular
9	14	109	105	0	nonsingular
10	14	44	35	18	nonsingular
11	15	54	486	53	singular
12	20	60	42	40	nonsingular
13	20	62	47	59	nonsingular
14	20	94	47	94	nonsingular
15	20	72	1178	71	singular
16	20	82	4802	81	nonsingular

Table 2: Processing time

No.		(sec) A	B
1		7,360	0
2	F	0	0
3		0.1	0.1
4		0.1	0.1
5		0.2	5.5
6		0.2	6.8
7		8.9	9.0
8		11.9	5.7
9		7.3	5.1
10		0.3	6.4
11		1.3	17.4
12		0.4	904
13		0.4	1,051
14	F	60.2	1,174
15		7.4	1,383
16		81.5	930

the sign conflict element by the condition (2) of the theorem. All the other elements are sign non-conflict and their terminal value can be determined. Sign conflict element is indicated by * symbol in the matrix.

Since a_{33} has the sign conflict element a_{12} in its disjoint cycle, and a_{12} has the sign conflict element a_{33} in its disjoint cycle, there is no sign conflict element specified in the STEP 2. The terminal values so far determined for the sign non-conflict elements are underlined in the matrix. For example, a_{21} takes the the terminal value -2.

In STEP 3, the cofactor of a_{12}, i.e., $(-4) \times (-1) - (-2) \times [-2, -1]$ does not change the sign whichever the terminal value $a_{33} = [-2, -1]$ may take. Hence the terminal value of a_{12} can be determined. Since there is no more sign conflict other than a_{33}, the terminal value of a_{33} can be assigned in the STEP 2. Thus, the terminal value of -1 at a_{12} is taken for the minimum absolute value of the determinant. Then, this will again determine the terminal value of -1 at a_{33}. Thus the vertex that realize minimum absolute value of determinant is obtained. The value the determinant is -13, hence the interval matrix is nonsingular (the determinant does not cross 0).

$$\begin{pmatrix} [-2, \underline{-1}] & [-2, -1]^* & [0, 0] \\ [\underline{-2}, -1] & [-4, \underline{-3}] & [-5, \underline{-4}] \\ [-3, \underline{-1}] & [\underline{2}, 4] & [-2, -1]^* \end{pmatrix}$$

Checking interval nonsingularity

We have implemented the algorithm of checking interval nonsingularity in the previous subsection or (Nogi & Ishida 1995) in detail We will compare it with so far known best algorithm. Table 1 lists the structural character of 16 sample interval matrices used for the comparison. Table 2 compares the processing time between A (our algorithm) and B (algorithm by (Rohn 1990) which uses solvability of a systems of linear interval equations). In table 2, F indicates when the algorithm fails to check the interval nonsingularity due to many sign conflict (and number there indicating time until the failure).

Clearly the processing time of our algorithm A heavily depends on the structure of the system, however algorithm B does not. Compare the sample matrices 5 and 8. The number of nonzero elements and other structural features greatly differ, however the processing time for these two samples by the algorithm B does not differ much. Further, our algorithm seems to be more efficient in most cases, however it fails to check when many sign conflicts.

Properties of Static Systems: Rank, Nonsingurality, and Inverse Stability

Sign matrices after put into the standard form with all the diagonal elements negative is sign nonsingular if the corresponding SDG does not have any positive cycle [6] (Maybee 1981). Thus, making all the positive cycles small[7] would make the interval matrix interval nonsingular by the qualitative perturbation principle.

Although it is necessary to cut all the positive cycles by deleting at least one arc from each positive arc to modify sign matrix which is not sign nonsingular but generically nonsingular (i.e. the SDG has

[6] Sign of a cycle is defined to be sign of the product of all the elements in the cycle.

[7] Cycle is said to be small(big) when the absolute value of the product of all the elements in the cycle is small but not zero(big). More specifically for interval matrices, when $b \le cycle - product \le a$, the cycle is said to be small(big) when $|a|, |b| \neq 0$ are small(big).

arc-subgraph that is sign nonsingular) into sign nonsingular one, only some set of disjoint cycles stated in the following proposition 11 must be handled to control the interval nonsingularity.

Proposition 11 *(Control of Interval Nonsingularity)* *Interval matrix that is generically nonsingular can be made interval nonsingular by making the elements corresponding to the following arcs big enough; the arcs whose removal will cut all the cycles in the G[n]-cycle* $\{C_{i1}, \cdots, C_{ip}\}$ *such that the sign* $(-1)^p sgn(C_{i1} \cdots \cdots C_{ip})$ *is positive.*

One trivial set of cycles satisfying the conditions above is the set of all the negative loops (diagonal elements). Thus, making absolute value of all the diagonal elements big results in interval nonsingular matrices. This result also comes from well-known nonsingularity condition of diagonal dominance.

Nonsingularity indicates whether or not information is preserved in the mapping by the matrix from a domain space to an image space. However, nonsingularity does not provide any further information about singular matrices. Rank of matrices is more informative property than nonsingularity, since it can provide information about a kind of distance from nonsingularity.

Definition 6 *(Sign Rank and Interval Rank)*

Sign (interval) rank of a sign (interval) matrix is the order of its maximal submatrix that is sign(interval) nonsingular.

Obviously, the interval rank of an interval matrix is greater than or equal to sign rank of the sign structure of the interval matrix. Sign rank can be obtained by SDG of the sign matrix as will be stated after proposition 12.

Although rank is more informative than nonsingularity, it cannot tell any further information for nonsingular (i.e. full rank) matrices. As it will be made clear through this section, information of nonsingularity of original matrix as well as submatrices of order $n-1$ is needed to characterize the concepts such as inverse stability discussed in this section.

Proposition 12 *(Sign Nonsingularity of Submatrix)* *Let N be a set of integers* $\{1, 2, \cdots, n\}$ *and I, J be a subset of N. If, for all the positive cycles and paths of opposite sign, there is at least one arc from the node* $i \in I$ *or to the node* $j \in J$ *that will cut the positive cycle or the paths of opposite sign, then the submatrix* $A_{\bar{I}\bar{J}}$ *is sign nonsingular where* $\bar{I}$ *and* $\bar{J}$ *are complement set of I and J. Here,* $A_{\bar{I}\bar{J}}$ *denotes the submatrix obtained by deleting all the* i^{th} *row in* $i \in I$ *and all the* j^{th} *column in* $i \in J$ *from A.*

The order of the maximal submatrix obtained by this proposition is sign rank.

In order to analyze the relation between a matrix and its submatrices, the relation between the graph corresponding to the original matrix and the subgraph corresponding to the submatrices may be used. A property of graph is said to be hereditary (Berge 1965) if the property holds for its subgraph. For example, the property of having no cycle is hereditary. Graphical condition characterizing sign nonsingular matrix, that is having no positive cycle, is also hereditary (in the sense of node-subgraph). Thus, sign nonsingularity (for irreducible matrices) is hereditary as in the next theorem.

Theorem 13 *(Sign Nonsingularity of Matrix and Submatrix)*
If a sign matrix of order n is sign nonsingular then the submatrix A_{ij} *of order* $n-1$ *are sign nonsingular for* i, j *such that* $a_{ij} \neq 0$.

The above theorem 13 holds for only irreducible sign nonsingular matrices. If a reducible sign matrix of order n is sign nonsingular then its submatrix A_{ij} corresponding to $a_{ij} \neq 0$ of order $n-1$ are either sign nonsingular (i.e. sign rank is $n-1$) or structurally singular (i.e. term rank[8] $< n-1$). Although the number of sign nonsingular submatrices of order $n-1$ is at least the number of nonzero elements $a_{ij} \neq 0$ for sign nonsingular irreducible matrix of order n as known from the above theorem 13, that for sign nonsingular reducible matrix is at least n. This fact for reducible sign nonsingular matrix can be known from the cofactor expansion of A:

$$detA = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} A_{ij}.$$

An example that has exactly n sign nonsingular submatrices of order $n-1$ would be the sign matrix with all the diagonal elements negative and others zero.

This recursiveness of nonsingularity stated in theorem 13 does not hold for interval matrix with interval nonsingularity. That is, even if an interval matrix is interval nonsingular its submatrices may not be interval nonsingular as demonstrated in the next section.

The following condition for interval nonsingularity also follows from the determinant expansion.

Proposition 14 *(Interval Nonsingularity Condition by Submatrix)*
If there exists i^{th} *row (or* j^{th} *column) in the interval matrix such that* $a_{ij} \neq 0$ *always implies that* A_{ij} *is interval nonsingular and that* $(-1)^{i+j} a_{ij} A_{ij}$ *are of the same sign for* $j = 1, \cdots, n$ *(or* $i = 1, \cdots, n$*), then the original interval matrix is interval nonsingular.*

The following concept of inverse stability is related to sign(interval) nonsingularity of submatrices with order $n-1$.

[8] Term rank for structured matrix where only zero/non-zero pattern is specified for each element is the maximal rank that the matrix satisfying the zero/non-zero pattern of the elements can attain. In other words, term rank fails to be full means that there is no non-zero term in its expansion of the determinant.

Definition 7 *(Inverse Stability)*
A sign matrix A^S (interval matrix A^I) is called sign (interval) inverse stable when A^S (A^I) is sign (interval) nonsingular, and $|\{A^{-1}\}_{ij}| > 0$ for all $A \in A^S(A^I)$. That is, all the element of the inverse of the given matrix does not become zero. A sign matrix A^S (interval matrix A^I) is called sign (interval) inverse semi-stable when when A^S (A^I) is sign (interval) nonsingular, and $\{A^{-1}\}_{ij} \geq 0$ or $\{A^{-1}\}_{ij} \leq 0$ for all $A \in A^S(A^I)$. Here, $\{B\}_{ij}$ denotes ij-element of the matrix B.

This inverse stability, originally defined on interval matrices (as done in (Rohn 1990)), can be defined on sign matrices as above. Obviously, the inverse stability of sign (interval) matrices is equal to that all the submatrices of order $n-1$ as well as the original matrix of order n is sign (interval) nonsingular. The inverse semi-stability of sign (interval) matrix is equal to that the original matrix of order n are sign (interval) nonsingular and that all the submatrices of order $n-1$ are either sign(interval) positive semi-definite[9] or sign(interval) negative semi-definite. Condition for inverse stability of interval matrices has been already obtained by (Rohn 1993b). For sign matrices, the following result directly follows from theorem 13.

Corollary 15 *A (reducible) sign matrix is inverse (semi-)stable if it is sign nonsingular and all the paths from node i to node j corresponding to $a_{ji} = 0$ are of the same sign.*

Result similar to corollary 15 does not hold for interval nonsingularity and inverse stability of interval matrices. Inverse stability condition for interval matrix consists of interval nonsingularity of original interval matrix and that of all the submatrices of order $n-1$. Inverse stability of interval matrices plays an important role for specifying intervals of inverse of interval matrices as well as solving interval systems of linear equations (Rohn 1990; Rohn 1993b).

As known from the above corollary 15, irreducibility is necessary for inverse stability of sign matrices. In order to make the given sign matrix inverse stable, cutting the positive cycles and the paths of opposite sign from node i to node j such that $a_{ji} = 0$ does not always suffice (it does, however, for inverse semi-stability).

Proposition 16 *Irreducible sign matrix with positive cycles (hence not sing nonsingular) can be made inverse stable by deleting a set of arcs whose removal*

1. *leaves no positive cycle,*
2. *leaves no pair of paths of opposite sign from node i to node j such that $a_{ji} = 0$, and*
3. *preserves irreducibility.*

[9] Sign matrix A^S is sign positive (semi-)definite when all matrices $A \in A^S$ are positive (semi-)definite. Interval positive (semi-)definite can be defined in the same manner for interval matrices.

When applying the qualitative perturbation principle to the above proposition 16, irreducibility need not to be taken care of. Interval inverse stability can be attained by making positive cycles and paths of opposite sign from node i to node j such that $a_{ji} = 0$ small, since irreducibility is always preserved by the operation of making the absolute values of matrix elements small. Table 3 summarizes the principle of perturbations presented in this section.

The Klein Model as an Illustrative Example

We use the Klein model in economics as an illustrative example. Figure 4 shows the SDG of the irreducible component of the Klein model corresponding to the following interval matrix. Intervals are assigned arbitrary according to the sign structure of the Klein model.

$$\begin{pmatrix} [-3,-1] & [0,0] & [2,3] & [0,0] & [-5,-3] \\ [0,0] & [-7,-4] & [0,0] & [0,0] & [-1,-2] \\ [0,0] & [0,0] & [-8,-3] & [10,13] & [0,0] \\ [2,4] & [2,5] & [0,0] & [-7,-5] & [0,0] \\ [0,0] & [0,0] & [7,9] & [-5,-3] & [-4,-1] \end{pmatrix}$$

Table 4 lists all the cycles of length greater than one

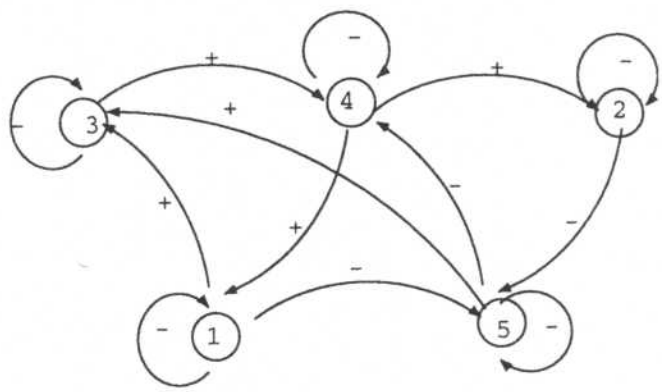

Figure 4: The SDG of the Irreducible Component of the Klein Model

and the critical paths: paths from node i to node j for i,j such that $a_{ji} = 0$. In order to make the given sign matrix (interval matrix) sign nonsingular (interval nonsingular), all the positive cycles must be cut (made small). In order to further make the sign nonsingular (interval nonsingular) matrix sign inverse stable (interval inverse stable), some arcs in the critical paths listed must be cut (made small) to make critical paths have the same sign, preserving irreducibility.

First, it is known that the sign structure of the interval matrix is not sign nonsingular due to the existence of positive cycles: $a_{25}a_{54}a_{42}, a_{13}a_{34}a_{41}, a_{15}a_{54}a_{41}$. Among the cut set of arcs whose removal will cut all these positive cycles, the sets

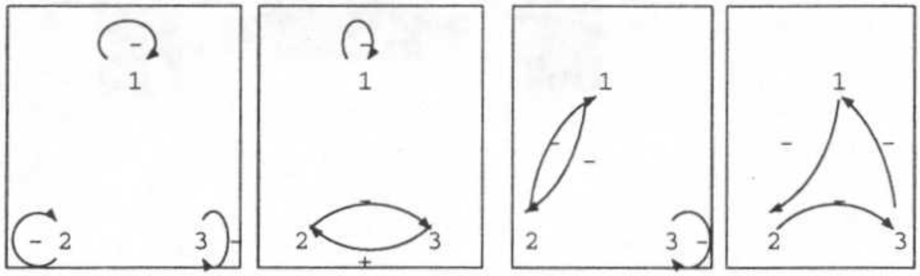

Figure 3: G[3]-Cycle Decomposition of SDG

Table 3: Summary of Effects of Deleting or Weakening Perturbations

Deleting or Weakening Perturbations	Effect of Deleting or Weakening
Deletion of all the Positive Cycles(Maybee 1981)	Not Sign nonsingular → Sign nonsingular
Making Positive Cycles Small	Not interval nonsingular → Interval nonsingular
Making Arcs satisfy condition stated in Proposition 11 big	Not interval nonsingular → Interval nonsingular
Deletion of all the Paths of Opposite Sign of length smaller than k+1 (Proposition 12)	Sign rank $<$ k → Sign rank $\geq$ k
Addition of Paths of Opposite Sign satisfying the condition of Proposition 16	Sign inverse stable → Not sign inverse stable
Making Positive Cycles and Paths of Opposite Sign satisfy the condition of Proposition 16 small	Not interval inverse stable → Interval inverse stable
Deletion of Arcs resulting in Reducible Matrix (Corollary 15)	Sign inverse stable → Not sign inverse stable

Table 4: Cycles and Critical Paths

	POSITIVE	NEGATIVE
Cycles of length greater than one	$a_{13}a_{34}a_{41}, a_{15}a_{54}a_{41}, a_{25}a_{54}a_{42}$	$a_{25}a_{53}a_{34}a_{42}, a_{15}a_{53}a_{34}a_{41}$
Paths from 1 to 2	$a_{15}a_{54}a_{42}, a_{13}a_{34}a_{42}$	$a_{15}a_{53}a_{34}a_{42}$
Paths from 1 to 3	a_{13}	$a_{15}a_{53}$
Paths from 2 to 1	$a_{25}a_{54}a_{41}$	$a_{25}a_{53}a_{34}a_{41}$
Paths from 2 to 3	$a_{25}a_{54}a_{41}a_{13}$	$a_{25}a_{53}$
Paths from 5 to 3	a_{53}	$a_{54}a_{41}a_{13}$
Paths from 5 to 4	$a_{53}a_{34}$	a_{54}

$\{a_{54}, a_{41}\}$,$\{a_{54}, a_{34}\}$,$\{a_{54}, a_{13}\}$,$\{a_{41}, a_{42}\}$,$\{a_{25}, a_{41}\}$ are the minimal set. Thus, making either of these values small would make the interval matrix interval nonsingular by the qualitative perturbation principle. There is an arc contained only in the negative cycles; a_{53}. For example, the interval matrix obtained by replacing a_{53} of the above interval matrix with the interval: $[1,9]$is not interval nonsingular. However, by proposition 11, we can make this interval matrix interval nonsingular by changing a_{53} from the interval: $[1,9]$ to $[5,9]$.

Although this strategy can work to control interval nonsingularity, it does not work to control interval inverse stability. In fact, a_{53} does not appear in the submatrix A_{52} that is not sign nonsingular, hence may not be interval nonsingular.

By proposition 12, the following submatrices of order four turned out to be sign nonsingular; $A_{13}, A_{14}, A_{23}, A_{24}, A_{43}, A_{44}, A_{51}, A_{53}, A_{54}$. Thus, the sign rank of the given sign structure is known to be four.

Since A_{44} and A_{54} are sign nonsingular, the interval stability of the submatrix A_{34} implies interval nonsingularity of the given interval matrix by proposition 14. After A_{34} is known to be interval nonsingular either by the combinatorial search of min/max of determinant or recursively analyzing the sign structure of the submatrix A_{34}, the given interval matrix is judged to be interval nonsingular. The status of entire submatrices of order four follows:

$$\begin{pmatrix} INS & INS & SNS & SNS & INS \\ NotINS & INS & SNS & SNS & INS \\ INS & NotINS & NotINS & INS & NotINS \\ NotINS & INS & SNS & SNS & INS \\ SNS & NotINS & SNS & SNS & NotINS \end{pmatrix}$$

where i^{th} row and j^{th} column indicates the status of the submatrix A_{ij}. NotINS, INS, and SNS, respectively indicate not interval nonsingular, interval nonsingular and sign nonsingular.

Among the cut set of positive cycles, the removal of only the set $\{a_{54}, a_{13}\}$ will leave the graph irreducible and no pair of paths of opposite sign from node i to node j such that $a_{ji} = 0$ as known from Table 4. Hence, by proposition 16, the removal of these two arcs will make the sign structure not only sign nonsingular but also sign inverse stable. By the qualitative perturbation principle, making these parameters small would result in the interval matrix interval nonsingular and interval inverse stable.

As known from the status of submatrices A_{ij} above, the given interval matrix is not interval inverse stable. However, changing intervals from $a_{13} = [2,3], a_{54} = [-5,-3]$ to $a_{13} = [1/10, 1/5], a_{54} = [-1/5, -1/10]$ would make it interval inverse stable. In this case, sign of all the elements of A^{-1} can be identified by the signs of paths from node i to node j and the sign of $detA$ as follows [10]:

$$\begin{pmatrix} - & + & + & + & + \\ + & - & + & + & + \\ - & - & - & - & + \\ - & - & + & - & + \\ - & - & - & - & - \end{pmatrix}$$

Related Work and Discussions

In qualitative reasoning of dynamical systems, reasoning about behavior especially simulation that allows parameters and/or initial values to be specified by interval have been studied.

Since the envelope that contains possible evolutions within the interval tends to be pessimistically large when time progresses, several techniques are introduced to restrict the upper and the lower bounds of the envelope (Kay and Kuipers 1993; Vescovi et al. 1995).

In contrast to these approaches, our methods try to explore the relation between parameters specified by intervals and several system properties rather than evolution through time. Thus our approach cannot directly deal with the question; what the next state will be, given the current state and the parameter ranges? Rather, our approach is meant to deal with the question; what are the qualitative property (such as stability, solvability, controllability, etc.) of the behavior specified by the system with some parameter ranges?

Our approach can be understood the interval version of the qualitative and structural analysis on systems by sign concepts defined on SDG (Ishida 1989; Ishida 1990; Ishida 1991; Ishida 1992; Ishida 1993).

Other than the field of QR, interval arithmetic or more generally interval analysis is indispensable in dealing with interval systems. Interval analysis provides several iterative procedures that will give the solution specified by intervals and contain the exact solution within it (Moore 1979; Alwfeld & Herzberger 1983; Neumaier 1990). However, many of these procedures work under non-trivial conditions such as interval nonsingularity of the given matrix, which in turn can be checked or reasoned by our method. Our approach seems to be complementary to interval analysis; our result may provide conditions under which the procedure of interval analysis works.

The most relevant work has been done by (Brouwer et al. 1987) that proposed two approaches for mixed sign analysis: (1) the top-down approach: starting from sign equations, then try to identify which and how many equations have to be estimated numerically, (2) the bottom-up approach: starting from fully estimated numerical equations, then try to identify which and how many equations can be specified in sign equations preserving sign properties.

We have demonstrated an alternative approach in the previous section for the same example (i.e. the Klein model) as (Brouwer et al. 1987). That is, (3) the interval approach: starting from interval systems

[10]Sign of ij-element of A^{-1} is $sgn(detA)sgn((-1)^{i+j}A_{ji})$ where $sgn(detA)$ is $(-1)^{order-of-A}$ and $sgn((-1)^{i+j}A_{ji})$ is the sign of the path from node i to node j.

whose intervals are set rather wide, then try to make the intervals narrower.

The qualitative perturbation principle discussed in this paper can be applied to any sign properties whose graphical conditions are known. The principle would indicate which elements of interval systems are critical to attain the interval properties.

Conclusion

It is shown that qualitative analysis on SDG can reduce the computation required for reasoning about interval systems. One example of reducing the reasoning about interval linear static system has been demonstrated. We presented the qualitative perturbation principle that can be used to reason about interval properties of interval systems based on their sign structures. By the principle, it is known which interactions are critical or should be made small (or big) to attain certain interval properties. Future problem includes characterization of interval versions of other important concepts such as weak/strong connectivity; and integrating our approach and interval analysis to enhance both efficiency and flexibility in reasoning.

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