Analysis of Control Systems using Qualitative and Quantitative Simulation

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Abstract

Qualitative simulation and the quantitative extensions of this methodology are powerful tools for the analysis of uncertain dynamic systems. This paper shows the application of such methods for the analysis of nonlinear control systems. The simulation program QSIM and its quantitative extensions Q2 and NSIM are used for this purpose. In addition to the simulation of such systems, attention is also drawn to the stability analysis of fixed points. A novel analytic stability test for uncertain systems is given in this paper, since the stability test based on qualitative simulation is shown to be unreliable. Simulation studies in combination with analytic evaluations are used to demonstrate the possibilities and limitations of such an approach to systems analysis.

Introduction

Classical simulation techniques cannot provide sufficient results when analyzing uncertain nonlinear dynamic systems. Especially systems with nonparametric uncertainties impose big difficulties on standard simulation methods. However, such a class of systems can be modeled in an elegant way using qualitative differential equations (QDEs) and simulated using qualitative reasoning methods. The simulation system QSIM [Kuipers, 1994] and its quantitative extensions [Kay, 1996] provide such a framework to deduce the possible behaviors of a system using the structural information given by a qualitative model and uncertain numerical information.

In contrast to a numerical simulation, which returns one behavior (trajectory) for the simulated system, a qualitative simulation gives, due to the uncertain information of the QDE, a set of possible behaviors for the qualitative model. Qualitative simulation can be used under the assumption that the dynamic behavior of the simulated system is of a specific type. The system's trajectories must belong to a specific class of functions which is called *reasonable* in the QSIM literature [Kuipers, 1994, Shults, 1996]. The *guaranteed-coverage theorem* provides the justification for the application of the QSIM algorithm for such systems. It guarantees that the set of predicted behaviors contains the qualitative form of the reasonable trajectory shown by the original system. However, the application of this theorem should be handled with care. The question whether the original dynamic system, which is abstracted by a QDE model, shows a reasonable behavior is non-trivial and no criterion exists for this problem yet! In cases where only a QDE model can be formulated the even more difficult question whether all ordinary differential equation (ODE) models, which are abstracted by a QDE, show only reasonable behaviors must be answered[1]. Despite the progress of qualitative simulation, which was achieved by the application of advanced methods, it is still often the case that a pure qualitative simulation of a model of modest size leads to a vast amount of possible behaviors which are hard to interpret. It is therefore beneficial to use additional uncertain numerical information when available. This information reduces the class of ODE models which are abstracted by the qualitative model and can improve the simulation significantly due to the constraints which can be formed based on the quantitative information. Such problems, where functional relationships and parameter values are known with some uncertainty, can be found often in applications of automatic control. It is therefore not surprising that some applications of qualitative-quantitative simulation for the analysis of control systems can be found in literature. Kuipers and Åström [Kuipers and Åström, 1994] use qualitative simulation to analyze dynamic systems controlled by heterogeneous controllers. Gazi, Seider and Ungar [Gazi et al., 1994] follow this approach in order to analyze controlled chemical systems. Simple PI-controlled systems can be found in [Kuipers, 1994].

This paper shows how qualitative-quantitative sim-

[1] This open questions might also require the redefinition of the *reasonable function* class. The definitions in [Kuipers, 1994, Shults, 1996] are too strong; e.g. using the definition of [Kuipers, 1994] would classify *sin(t)* due to its behavior at $t = \infty$ as *not reasonable* and therefore - strictly speaking - we cannot apply the guaranteed-coverage theorem for a wide range of systems which show oscillating behaviors! However, excluding or redefining the assumptions for reasonable functions at $t = \infty$ can overcome this difficulty.

ulation can be used for the analysis of uncertain nonlinear systems controlled via state-feedback. QSIM and its quantitative extensions Q2 and NSIM are used for this purpose. In addition to the exhaustive simulation of such systems, attention is also drawn to the stability analysis of fixed points. It is possible to show that the stability test based on qualitative simulation, as it is used by the QSIM simulation system, can give wrong results in certain cases where unstable points are labeled stable. It is therefore advisable to use an alternative stability test based on algebraic analysis which is presented in this paper. The application of this stability test and qualitative-quantitative simulation for the analysis of a control system is shown by example and the possibilities and limitations of such an approach to systems analysis are discussed.

Stability Test for Qualitative Models

An important task in systems analysis is the evaluation of the dynamic system's stability character. Global assumptions about the stability character of a qualitative-quantitative model are difficult to state and it is not clear how methods developed for ordinary differential equations (e.g. Ljapunov's direct analysis) can be applied to qualitative-quantitative models. However, it is possible to test the stability of the fixed points (equilibria) of such models even in the presence of incomplete knowledge.

Stability Test using Qualitative Simulation

One possibility to test the stability character of fixed points is to evaluate whether it is possible to simulate forward from them. The existence of an outward trajectory can be used as a criterion for the instability of a fixed point [Kuipers, 1994]. Such a test is rather conservative, since an outward trajectory can be spurious and therefore not representative for the original dynamic system. The simulation program QSIM uses such a test to evaluate the stability character of fixed points which are identified during simulation. One would think that such a test is only too weak since stable fixed points can be labeled unstable due to the existence of spurious behaviors. However, it is possible to show that this test can also lead to wrong results, where unstable fixed points are labeled stable. An example for this difficulty can be given by the following simple feedback system with a linear transfer function of third order:

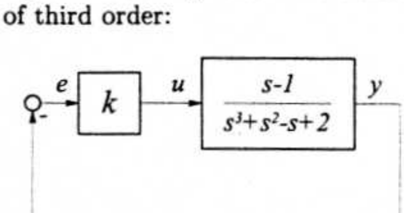

Figure 1: Linear Feedback System

A feedback gain $k \in (1\ 1.5)$ leads to a closed-loop system with one pole $p_1 < 0$ and a pole-pair $p_{2,3} = \delta \pm i\omega,\ \delta > 0$. Although the system has a pole-pair with positive real component, the fixed point $y = 0$ is labeled stable by QSIM's stability test! The reason for the failure can be seen by the stability test's difficulty with unstable spiral fixed points. Since the behavior which leads away from such a point is *not-reasonable* in the QSIM's sense, it is not possible to detect it! Instead, the test relies on the assumption that every unstable spiral fixed point, simulated in QSIM, is accompanied by an infinite family of (spurious) nodal behaviors which converge directly to ∞ after a finite number of cycles and which are detectable by the stability test [Clancy and Kay, 1994]. The above example, however, shows that this assumption does not hold for all systems[2]. In particular, the feedback system (Figure 1), when simulated with QSIM, shows spurious behaviors which converge *to* the unstable fixed point, thus causing the wrong classification. An alternative stability test is therefore given to overcome this difficulty.

Algebraic Stability Test

The stability character of fixed points can be determined by algebraic analysis of the qualitative model. Using symbolic manipulation it is possible to calculate the Jacobian matrix for a QDE model. The eigenvalues of the matrix at the fixed points can be used to evaluate the stability character (Ljapunov's indirect method). The principal method is already given in [Kuipers, 1994]. However, the difficulty lies in the generalization of this method for cases where the resulting Jacobian is not in diagonal form, so that the evaluation of the eigenvalues is non-trivial. The stability test, which is described in the following, is designed for such general cases but requires quantitative information about the system. The test is therefore designed for the analysis of fixed points which are identified by qualitative-quantitative simulation.

Based on the qualitative description of a model in QSIM notation (a set of constraints), it is possible to deduce by symbolic manipulation a model in the form:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \cdots, x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, \cdots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \cdots, x_n) \end{aligned} \tag{1}$$

where the qualitative functions f_i depend on the state variables x_i and can be formulated using the arithmetic operations $+$, $-$, $*$, $/$ and the partially known

[2]Fixed points of systems of the order $n \geq 3$ with a characteristic polynomial of the Jacobian at the point with one root pair $\lambda = \delta \pm i\omega,\ \delta > 0$ and all other roots with positive real component are candidates for such an effect.

functional relationships M^+ and M^-. The Jacobian matrix of this model is given by:

$$J(x_1, \cdots, x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}. \quad (2)$$

Its characteristic polynomial $\Delta(\lambda)$

$$\Delta(\lambda) := det(\lambda I - J(x_f)) = a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n \quad (3)$$

can be used to determine the stability character of the fixed points $x_f := [x_{1f}, \cdots, x_{nf}]'$.

The evaluation of the polynomial $\Delta(\lambda)$ at the fixed points x_f leads, due to the incomplete information, to an interval-polynomial where the coefficients a_i are given by intervals. The values for the upper bound a_i^+ and the lower bound a_i^- of the coefficients can be calculated using interval-arithmetic. Since the Jacobian is calculated by differentiation, it is likely that the expressions for the coefficients of $\Delta(\lambda)$ contain the derivatives of M^+ and M^- functions. The evaluation of these expressions can be done by taking advantage of the fact that $M^+(x_i)$ functions have strictly positive slopes, thus the value of $\frac{dM^+(x_i)}{dx_i}$ can be assumed to lie in the partially open interval $(0\ \infty]$, and $[-\infty\ 0)$ for $\frac{dM^-(x_i)}{dx_i}$ respectively. In cases where there is additional information available about the slope of M^+ or M^- functions it is possible to reduce the interval further.

As the following stability test uses the values of the coefficient's interval bounds a_i^+ and a_i^-, it is beneficial to extend the underlying interval-arithmetic. Standard interval-arithmetic [Moore, 1979, Alefeld and Herzberger, 1983] is based on the assumption that only closed intervals are used. However, for proving asymptotic stability it is useful to extend the arithmetic, so that the character of an interval (closed, partially open and open) is also taken into account. The use of such generalized numerical intervals allows to handle the following situations:

- The intervals for $\frac{dM^+(x_i)}{dx_i}$ and $\frac{dM^-(x_i)}{dx_i}$ can be formulated correctly by excluding 0.
- Numerical intervals can be refined based on qualitative information[3].

In order to prove asymptotic stability for a fixed point it is necessary to show that *all* roots of *all* polynomials, which are defined by the interval-polynomial $\Delta(\lambda)$, have only a negative real component. A finite solution for this problem is given by the Kharitonov-Theorem [Kharitonov, 1978]. Instead of testing all possible polynomials it is sufficient to test whether the following four so called Kharitonov-Polynomials have only roots with a negative real component.

$$\begin{aligned} p_1(\lambda) &= a_0^- + a_1^-\lambda + a_2^+\lambda^2 + a_3^+\lambda^3 + a_4^-\lambda^4 + \ldots \\ p_2(\lambda) &= a_0^+ + a_1^+\lambda + a_2^-\lambda^2 + a_3^-\lambda^3 + a_4^+\lambda^4 + \ldots \\ p_3(\lambda) &= a_0^+ + a_1^-\lambda + a_2^-\lambda^2 + a_3^+\lambda^3 + a_4^+\lambda^4 + \ldots \\ p_4(\lambda) &= a_0^- + a_1^+\lambda + a_2^+\lambda^2 + a_3^-\lambda^3 + a_4^-\lambda^4 + \ldots \end{aligned} \quad (4)$$

One needs not to calculate the roots of the polynomials to prove that their real components are negative. The Routh test provides a criterion based on the values of the polynomial's coefficients. In contrast to the calculation of roots it is further possible to extend the test, so that it can work with polynomial coefficients drawn from the generalized numerical intervals. The Routh test must be modified in two ways:

- The underlying arithmetic must be able to cope with values from the extended real number line.
- The character of the interval bound must be taken into account; e.g. the lower bound of the partially open interval (5 10], which is symbolized by "(5" should be treated ***symbolically*** as $5 + \epsilon$, where ϵ stands for an infinitesimal small, but otherwise undefined positive number.

Arithmetic operations with values drawn from open intervals can lead to ambiguous results; e.g. 3) + (2 can be either 5), 5 or (5. This difficulty is handled in the way, that all possible results of the calculations, which are performed by the Routh test, are considered and their results must pass the test.

Using this stability test it is now possible to analyze the linear feedback system given in Figure 1. Since the system is linear, the Jacobian matrix is equal to the dynamic matrix of the closed-loop system and can be written as:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k-2 & 1-k & -1 \end{bmatrix}. \quad (5)$$

Evaluating the characteristic polynomial $\Delta(\lambda)$ for the feedback gain $k \in (1\ 1.5)$ gives:

$$\Delta(\lambda) = (0.5\ 1) + (0\ 0.5)\cdot\lambda + [1\ 1]\cdot\lambda^2 + [1\ 1]\cdot\lambda^3. \quad (6)$$

Analyzing the first Kharitonov-Polynomial of this interval-polynomial

$$p_1(\lambda) = (0.5 + (0\cdot\lambda + 1]\cdot\lambda^2 + 1]\cdot\lambda^3 \quad (7)$$

gives the following Routh array:

$$\begin{matrix} 1 & (0 & 0 \\ 1 & (0.5 & 0 \\ b_1 & b_2 & 0 \\ c_1 & 0 & 0 \end{matrix} \quad (8)$$

[3]Consider a variable x with a quantity space defined by the ordered list `(0 x1* x2* inf)` and the numerical intervals for the landmarks given by `x1*` = [2 2], `x2*` = [2 10]; based on the quantity space we can deduce that `x1*` < `x2*`, so the numerical interval for `x2*` can be refined to (2 10].

The element b_1 can be calculated in the following way:

$$b_1 = \frac{1 \cdot (0 - 1 \cdot (0.5}{1} = \{-0.5),\ -0.5,\ (-0.5\}. \quad (9)$$

Since the result for the element b_1 contains values $b_{1i} \leq 0$ it is possible to classify the fixed point $y = 0$ to be not asymptotically stable.

The described stability test is implemented as an extension to the QSIM program package and performs the described operations automatically, so that the user must only specify the use of this test when writing the QSIM simulation function or use the test directly via QSIM's menu structure.

Control-System Analysis

QSIM, its quantitative extensions, and the stability test described in the previous section are now used to analyze a 2-tank system which is controlled via state-feedback.

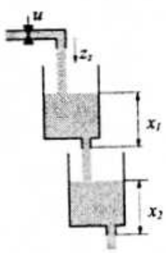

Figure 2: 2-Tank System

The mathematical model of the controlled tank system is given by:

$$\begin{aligned} \dot{x}_1 &= -f_1(x_1) + k_p u \\ \dot{x}_2 &= f_1(x_1) - f_2(x_2) \\ u &= u_f - k_{f1}(x_1 - x_{1f}) - k_{f2}(x_2 - x_{2f}) \end{aligned} \quad (10)$$

where $u = f(x_1, x_2)$ models the control law of the state-feedback controller which adjusts the bias term u_f to give the desired fixed point $x_f := [x_{1f},\ x_{2f}]'$. The uncertainty of the system lies in the inexact knowledge of the functions $f_1(x_1)$ and $f_2(x_2)$, which describe the relationship between the relative outflow rates of the tanks' valves and the fluid levels. It is possible to bound the unknown functions by the static envelopes $f^-(x) := 0.36\sqrt{x}$ and $f^+(x) := 0.41\sqrt{x}$. The variable $k_p = 0.7771$ represents the input gain of the tank system and $k_{f1} = 0.1059$ and $k_{f2} = 0.0567$ are the parameters of the state-feedback controller which was designed for the fixed point x_f. The maximum fluid-level of the tanks lies in the interval [43 45][4].

[4]The numerical values for this tank system are drawn from a laboratory experiment which is used at the Department of Automatic Control - Technical University of Graz

The mathematical model given in equation 10 can be abstracted by the following QDE-model:

$$\begin{aligned} \dot{x}_1 &= -M_1^+(x_1) - M_2^+(x_1) + M_1^-(x_2) \\ \dot{x}_2 &= M_1^+(x_1) - M_3^+(x_2) \end{aligned} \quad (11)$$

where $M_2^+(x_1)$ models the exact functional relationship $k_p k_{f1}(x_1 - x_{1f})$ and $M_1^-(x_2)$ describes the exact relationship $k_p(u_f - k_{f2}(x_2 - x_{2f}))$. $M_1^+(x_1)$ and $M_3^+(x_2)$ model the uncertain relationships.

Evaluation of Fixed Points

QSIM works on the basis of constraint-satisfaction. Thus starting a simulation with all state variables quiescent (`qdir = std`) but otherwise undefined (`qmag = ?`) gives all possible quiescent states (fixed points), which are consistent with the constraints of the QDE-model and the uncertain numerical information. Applying this method to the qualitative model of the tank system gives three qualitatively different fixed points which lie in the intervals:

$$x_{1f} \in [20.0\ 29.9],\quad x_{2f} \in [19.9\ 32.0]. \quad (12)$$

Using symbolic manipulation it is possible to calculate a Jacobian for the QDE model given in equation 11.

$$J = \begin{bmatrix} -\frac{dM_1^+}{dx_1} - \frac{dM_2^+}{dx_1} & \frac{dM_1^-}{dx_2} \\ \frac{dM_1^+}{dx_1} & -\frac{dM_3^+}{dx_2} \end{bmatrix} \quad (13)$$

The characteristic interval-polynomial of the Jacobian can be evaluated to:

$$\Delta(\lambda) = (0\ \infty] + (0.0823\ \infty] \cdot \lambda + [1\ 1] \cdot \lambda^2. \quad (14)$$

As the Jacobian J is independent of the fixed point, the result of the stability test is valid for all identified fixed points of the tank system. The Kharitonov-Polynomials which are defined by the polynomial $\Delta(\lambda)$ can be written as[5]:

$$\begin{aligned} p_1(\lambda) &= (0 + (0.0823 \cdot \lambda + \lambda^2 \\ p_2(\lambda) &= \infty + \infty \cdot \lambda + \lambda^2 \\ p_3(\lambda) &= \infty + (0.0823 \cdot \lambda + \lambda^2 \\ p_4(\lambda) &= (0 + \infty \cdot \lambda + \lambda^2 \end{aligned} \quad . \quad (15)$$

Since all four polynomials pass the extended Routh test it is possible to deduce, that the identified fixed points of the system are asymptotically stable[6].

[5]Since the controlled tank example is of second order, it would be sufficient to test whether all coefficients of the interval polynomial have the same sign. However, the described stability test is designed to work also with higher order systems where this condition is only necessary, but not sufficient.

[6]This example shows also the advantage of the extended interval-arithmetic. Using standard interval-arithmetic would lead to an interval-polynomial $[0\ \infty] + [0.0823\ \infty] \cdot \lambda + [1\ 1] \cdot \lambda^2$. The first Kharitonov-Polynomial $p_1 = 0.0823 \cdot \lambda + \lambda^2$ of this interval-polynomial fails to pass the Routh test, thus suggesting that the fixed point is not asymptotically stable.

be necessary to improve simulation techniques so that more complex systems and systems of order higher than two can be simulated. One direction for future research would be to evaluate how analytic methods developed for ODE systems can be applied to qualitative models. However, also open questions in the theory of qualitative simulation must be solved so that this methodology can be applied to a wide range of real-world problems. The recent progress in qualitative simulation motivates further research in order to provide a novel method for the analysis of uncertain dynamic systems.

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