

On the Complexity of Qualitative Spatial Reasoning: A Maximal Tractable Fragment of RCC-8 *

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Abstract

The computational properties of qualitative spatial reasoning have been investigated to some degree. However, the question for the boundary between polynomial and NP-hard reasoning problems has not been addressed yet. In this paper we explore this boundary in the "Region Connection Calculus" RCC-8. We extend Bennett's encoding of RCC-8 in modal logic. Based on this encoding, we prove that reasoning is NP-complete in general and identify a maximal tractable subset of the relations in RCC-8 that contains all base relations. Further, we show that for this subset path-consistency is sufficient for deciding consistency.

Introduction

When describing a spatial configuration or when reasoning about such a configuration, often it is not possible or desirable to obtain precise, quantitative data. In these cases, qualitative reasoning about spatial configurations may be used.

One particular approach in this context has been developed by Randell, Cui, and Cohn (1992), the so-called *Region Connection Calculus* (RCC), which is based on binary topological relations. One variant of this calculus, RCC-8, uses eight mutually exhaustive and pairwise disjoint relations, called base relations, to describe the topological relationship between two regions (see also Egenhofer (1991)).

Some of the computational properties of this calculus have been analyzed by Grigni et al. (1995) and Nebel (1995). However, no attempt has yet been made to determine the boundary between polynomial and NP-hard fragments of RCC-8, as it has been done for Allen's (1983) interval calculus (Nebel and Bürckert, 1995). We address this problem and identify a maximal fragment of RCC-8 that is still tractable and contains all base relations.

As in the case of qualitative temporal reasoning, this proof relies on a computer generated case-analysis that cannot be reproduced in a research paper.¹ Further, we show that for this fragment path-consistency is sufficient for deciding consistency.² We also give an estimation of how the efficiency of the general reasoning problem can be improved when using the maximal tractable fragment.

Qualitative Spatial Reasoning with RCC

RCC is a topological approach to qualitative spatial representation and reasoning where *spatial regions* are subsets of topological space (Randell et al., 1992). Relationships between spatial regions are defined in terms of the relation $C(a, b)$ which is true iff the closure of region a is connected to the closure of region b , i.e. if they share a common point. Regions themselves do not have to be internally connected, i.e. a single region may consist of different disconnected parts. The domain of *spatial variables* (denoted as X, Y, Z) is the whole topological space.

In this work we will focus on RCC-8, but most of our results can easily be applied to RCC-5, a subset of RCC-8 (Bennett, 1994). RCC-8 uses a set of eight pairwise disjoint and mutually exhaustive relations, called *base relations*, denoted as DC, EC, PO, EQ, TPP, NTPP, TPP^{-1} , and $NTPP^{-1}$, with the meaning of *DisConnected*, *Externally Connected*, *Partial Overlap*, *EQual*, *Tangential Proper Part*, *Non-Tangential Proper Part*, and their converses. Examples for these relations are shown in Figure 1. In RCC-5 the boundary of a region is not taken into account, i.e. one does not distinguish between DC and EC and between TPP and NTPP. These relations are combined to the RCC-5 base relations DR for *DiscRete* and PP for *Proper Part*, respectively.

Sometimes it is not known which of the eight base relations holds between two regions, but it is possible to restrict to some of them. In order to represent

¹The programs can be obtained from the authors.

²Due to space limitations, some of our proofs will only be sketched while others will be left out. Full proofs can be found in our technical report (Renz and Nebel, 1997).

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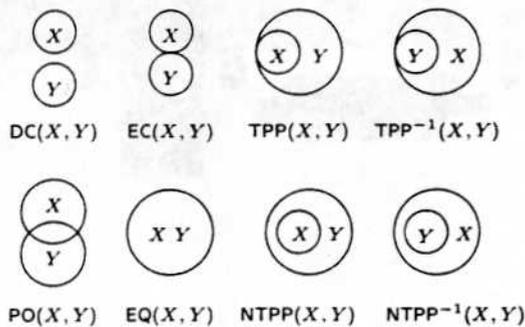


Figure 1: Two-dimensional examples for the eight base relations of RCC-8

this, unions of base relations can be used. Since base relations are pairwise disjoint, this results in 2^8 different relations, including the union of all base relations, which is called *universal relation*. In the following we will write sets of base relations to denote these unions. Using this notation, DR, e.g., is identical to $\{DC, EC\}$. *Spatial formulas* are written as XRY , where R is a spatial relation. Apart from union (\cup), other operations are defined, namely, converse (\sim), intersection (\cap), and composition (\circ) of relations. The formal definitions of these operations are:

$$\begin{aligned} \forall X, Y : X(R \cup S)Y &\leftrightarrow XRY \vee XSY, \\ \forall X, Y : X(R \cap S)Y &\leftrightarrow XRY \wedge XSY, \\ \forall X, Y : XR \sim Y &\leftrightarrow YRX, \\ \forall X, Y : X(R \circ S)Y &\leftrightarrow \exists Z : (XRZ \wedge ZSY). \end{aligned}$$

The compositions of the eight base relations are shown in Table 1. Every entry in the composition table specifies the relation obtained by composing the base relation of the corresponding row with the base relation of the corresponding column. Composition of two arbitrary RCC-8 relations can be obtained by computing the union of the composition of the base relations.

A *spatial configuration* can be described by a set Θ of spatial formulas. One important computational problem is deciding *consistency* of Θ , i.e. deciding whether it is possible to assign regions to the spatial variables in a way that all relations hold. We call this problem RSAT. When only relations of a specific set S are used in Θ , the corresponding reasoning problem is denoted RSAT(S). In the following \hat{S} denotes the closure of S under composition, intersection, and converse.

Encoding of RCC-8 in Modal Logic

In this work we use Bennett's (1995) encoding of RCC-8 in propositional modal logic. A concise introduction to modal logic is given in the appendix. Bennett obtained this encoding by analyzing the relationship of regions to the universe \mathcal{U} . He restricted his analysis to closed regions that are connected if they

share a point and overlap if they share an interior point. If, e.g. X and Y are disconnected, the complement of the intersection of X and Y is equal to the universe. Further, both regions must not be empty, i.e. the complements of both X and Y are not equal to the universe. In this way the eight base relations can be represented by constraints of the form ($m = \mathcal{U}$), called *model constraints*, and ($m \neq \mathcal{U}$), called *entailment constraints*, where m is a set-theoretic expression containing perhaps the topological interior operator i . Any model constraint must hold, whereas no entailment constraint must hold (Bennett, 1994).

The model and entailment constraints can be encoded in modal logic, where spatial variables correspond to propositional atoms and the interior operator i to a modal operator I (see Table 2).

The axioms for i must also hold for the modal operator I , which results in the following axioms (Bennett, 1995):

$$IX \rightarrow X \quad (1)$$

$$IIX \leftrightarrow IX \quad (2)$$

$$IT \leftrightarrow \top \text{ (for any tautology } T) \quad (3)$$

$$I(X \wedge Y) \leftrightarrow IX \wedge IY. \quad (4)$$

Axioms 1 and 2 correspond to the modal logics T and 4, axioms 3 and 4 already hold for any modal logic K, so I is a modal S4-operator.

The four axioms specified by Bennett are not sufficient to exclude non-closed regions. In order to account for that, we add two formulas for each atom, which correspond to topological properties of closed regions. A closed region is the closure of an open region and the complement of a closed region is an open region:

$$X \leftrightarrow \neg I\neg IX \quad (5)$$

$$\neg X \leftrightarrow I\neg X. \quad (6)$$

In order to combine the different model and entailment constraints, Bennett (1995) uses another modal operator \square . $\square m$ is interpreted as $m = \mathcal{U}$ and $\neg \square m$ as $m \neq \mathcal{U}$. Any model constraint m can be written as $\square m$ and any entailment constraint as $\neg \square m$. If $\square X$ is true in a world w of a model \mathcal{M} , written as $(\mathcal{M}, w \models \square X)$, then X must be true in any world of \mathcal{M} . So \square is an S5-operator with the constraint that all worlds are mutually accessible. Therefore Bennett (1995) calls it a strong S5-operator. The encoding of RCC-8 in modal logic is a multi-modal logic with an S4-operator and a strong S5-operator.

Let Θ be a set of RCC-8 formulas and $Reg(\Theta)$ be the set of spatial variables used in Θ , then $m(\Theta)$ specifies the multi-modal encoding of Θ , where

$$m(\Theta) = \left(\bigwedge_{XRY \in \Theta} m_1(XRY) \right) \wedge \left(\bigwedge_{X \in Reg(\Theta)} m_2(X) \right)$$

o	DC	EC	PO	TPP	NTPP	TPP ⁻¹	NTPP ⁻¹	EQ
DC	*	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC	DC	DC
EC	DC,EC PO,TPP ⁻¹ NTPP ⁻¹	DC,EC PO,TPP TPP ⁻¹ ,EQ	DC,EC PO,TPP NTPP	EC,PO TPP NTPP	PO TPP NTPP	DC,EC	DC	EC
PO	DC,EC PO,TPP ⁻¹ NTPP ⁻¹	DC,EC PO,TPP ⁻¹ NTPP ⁻¹	*	PO TPP NTPP	PO TPP NTPP	DC,EC PO,TPP ⁻¹ NTPP ⁻¹	DC,EC PO,TPP ⁻¹ NTPP ⁻¹	PO
TPP	DC	DC,EC	DC,EC PO,TPP NTPP	TPP NTPP	NTPP	DC,EC PO,TPP TPP ⁻¹ ,EQ	DC,EC PO,TPP ⁻¹ NTPP ⁻¹	TPP
NTPP	DC	DC	DC,EC PO,TPP NTPP	NTPP	NTPP	DC,EC PO,TPP NTPP	*	NTPP
TPP ⁻¹	DC,EC PO,TPP ⁻¹ NTPP ⁻¹	EC,PO TPP ⁻¹ NTPP ⁻¹	PO TPP ⁻¹ NTPP ⁻¹	PO,EQ TPP TPP ⁻¹	PO TPP NTPP	TPP ⁻¹ NTPP ⁻¹	NTPP ⁻¹	TPP ⁻¹
NTPP ⁻¹	DC,EC PO,TPP ⁻¹ NTPP ⁻¹	PO TPP ⁻¹ NTPP ⁻¹	PO TPP ⁻¹ NTPP ⁻¹	PO TPP ⁻¹ NTPP ⁻¹	PO,TPP ⁻¹ TPP,NTPP NTPP ⁻¹ ,EQ	NTPP ⁻¹	NTPP ⁻¹	NTPP ⁻¹
EQ	DC	EC	PO	TPP	NTPP	TPP ⁻¹	NTPP ⁻¹	EQ

Table 1: Composition table for the eight base relations of RCC-8, where * specifies the universal relation

Relation	Model Constraints	Entailment Constraints
DC(X, Y)	$\neg(X \wedge Y)$	$\neg X, \neg Y$
EC(X, Y)	$\neg(\text{IX} \wedge \text{IY})$	$\neg(X \wedge Y), \neg X, \neg Y$
PO(X, Y)	—	$\neg(\text{IX} \wedge \text{IY}), X \rightarrow Y, Y \rightarrow X, \neg X, \neg Y$
TPP(X, Y)	$X \rightarrow Y$	$X \rightarrow \text{IY}, Y \rightarrow X, \neg X, \neg Y$
TPP ⁻¹ (X, Y)	$Y \rightarrow X$	$Y \rightarrow \text{IX}, X \rightarrow Y, \neg X, \neg Y$
NTPP(X, Y)	$X \rightarrow \text{IY}$	$Y \rightarrow X, \neg X, \neg Y$
NTPP ⁻¹ (X, Y)	$Y \rightarrow \text{IX}$	$X \rightarrow Y, \neg X, \neg Y$
EQ(X, Y)	$X \rightarrow Y, Y \rightarrow X$	$\neg X, \neg Y$

Table 2: Encoding of the base relations in modal logic (Bennett, 1995)

$m_1(XRY)$ is a disjunction of the conjunctively connected model and entailment constraints for the base relations in R . m_2 results from the axioms of the I-operator and the additional properties of closed regions:

$$m_2(X) = \square(\text{IX} \rightarrow X) \wedge \square(\text{IX} \rightarrow \text{IIX}) \\ \wedge \square(\neg X \rightarrow \text{I}\neg X) \wedge \square(X \rightarrow \text{I}\neg \text{IX}).$$

$\square(\text{IIX} \rightarrow \text{IX})$, $\square(\text{I}\neg X \rightarrow \neg X)$ and $\square(\neg \text{I}\neg \text{IX} \rightarrow X)$ are entailed by the other formulas and can be ignored. As follows from the work by Bennett (1995), Θ is consistent iff $m(\Theta)$ is satisfiable.

In order to refer to the single model and entailment constraints, we will introduce some abbreviations.

Definition 1 *Abbreviations for the model constraints:*

$$\alpha_{xy} \equiv \square(\neg(X \wedge Y)) \\ \beta_{xy} \equiv \square(X \rightarrow Y) \\ \gamma_{xy} \equiv \square(Y \rightarrow X) \\ A_{xy} \equiv \square(\neg(\text{IX} \wedge \text{IY})) \\ B_{xy} \equiv \square(X \rightarrow \text{IY}) \\ C_{xy} \equiv \square(Y \rightarrow \text{IX}).$$

As the entailment constraints are negations of the model constraints, they will be abbreviated as negations of the above abbreviations. When it is obvious which atoms are used, the abbreviations will be written without indices. The abbreviations can be regarded as “propositional atoms”. Then it is possible to write the modal encoding $m_1(XRY)$ of every relation R of RCC-8 as a “propositional formula” of abbreviations. We will call this formula the *abbreviated form* of R . In the remainder we will use the encoding of $m_1(XRY)$ such that the abbreviated form is in conjunctive normal form (CNF).

Computational Properties of RCC-8

In this section we prove that reasoning with RCC-8 as well as RCC-5 is NP-hard. A similar but weaker result has been proven by Grigni et al. (1995) (see Related Work).

In this paper NP-hardness proofs for different sets S of RCC-8 relations will be carried out. All of them use a reduction of a propositional satisfiability problem to $RSAT(S)$ by constructing a set of spatial formulas Θ for every instance \mathcal{I} of the propositional problem, such that Θ is consistent iff \mathcal{I} is a positive instance. These satisfiability problems include 3SAT, NOT-ALL-EQUAL-3SAT where every clause has at least one true and one false literal, and ONE-IN-THREE-3SAT where exactly one literal in every clause must be true (Garey and Johnson, 1979).

The reductions have in common that every literal as well as every literal occurrence L is reduced to two spatial variables X_L and Y_L and a relation $R = R_t \cup R_f$, where $R_t \cap R_f = \emptyset$ and $X_L R Y_L$. L is true iff $X_L R_t Y_L$ holds and false iff $X_L R_f Y_L$ holds. Additional "polarity" constraints have to be introduced to assure that for the spatial variables X_{-L} and Y_{-L} , corresponding to the negation of L , $X_{-L} R_t Y_{-L}$ holds iff $X_L R_f Y_L$ holds, and vice versa. Using these polarity constraints, spatial variables of negative literal occurrences are connected to the spatial variables of the corresponding positive literal, and likewise for positive literal occurrences and negative literals. Further, "clause" constraints have to be added to assure that the clause requirements of the specific propositional problem are satisfied in the reduction.

Theorem 2 $RSAT(RCC-5)$ is NP-hard.

Proof Sketch. Transformation of NOT-ALL-EQUAL-3SAT to $RSAT(RCC-5)$ (see also Grigni et al. (1995)). $R_t = \{PP\}$ and $R_f = \{PP^{-1}\}$. Polarity constraints:

$$X_L \{PP, PP^{-1}\} X_{-L}, Y_L \{PP, PP^{-1}\} Y_{-L}, \\ X_L \{PO\} Y_{-L}, Y_L \{PO\} X_{-L}.$$

Clause constraints for every clause $c = \{i, j, k\}$:

$$X_i \{PP, PP^{-1}\} X_j, X_j \{PP, PP^{-1}\} X_k, X_k \{PP, PP^{-1}\} X_i, \\ X_i \{PO\} Y_k, X_j \{PO\} Y_i, X_k \{PO\} Y_j. \quad \blacksquare$$

Since RCC-5 is a subset of RCC-8, this result can be easily applied to RCC-8.

Corollary 3 $RSAT(RCC-8)$ is NP-hard.

In order to identify the borderline between tractability and intractability, one has to examine all subsets of RCC-8. We limit ourselves to subsets containing all base relations, because these subsets still allow to express definite knowledge, if it is available. Additionally, we require the universal relation to be in the subset, so that it is possible to express complete ignorance. This reduces the search space from 2^{256} subsets to 2^{247} subsets. We proved a property

that has likewise been used in identifying the maximal tractable subset of Allen's calculus (Nebel and Bürkert, 1995) that can be used to further reduce the search space.

Theorem 4 $RSAT(\widehat{S})$ can be polynomially reduced to $RSAT(S)$

Corollary 5 Let S be a subset of RCC-8.

1. $RSAT(\widehat{S}) \in P$ iff $RSAT(S) \in P$.
2. $RSAT(S)$ is NP-hard iff $RSAT(\widehat{S})$ is NP-hard.

The first statement of Corollary 5 can be used to increase the number of elements of tractable subsets of RCC-8 considerably. With the second statement of Corollary 5, NP-hardness proofs of $RSAT$ can be used to exclude certain relations from being in any tractable subset of RCC-8. The NP-hardness proof of Theorem 2, e.g., only uses the relations $\{PO\}$ and $\{PP, PP^{-1}\}$. So for any subset S with the two relations contained in \widehat{S} , $RSAT(S)$ is NP-hard. The following NP-hardness results can be used to exclude more relations.

Lemma 6 Let S be a subset of RCC-8 containing all base relations. If any of the relations $\{TPP, NTPP, TPP^{-1}, NTPP^{-1}\}$, $\{TPP, TPP^{-1}\}$, $\{NTPP, NTPP^{-1}\}$, $\{NTPP, TPP^{-1}\}$ or $\{TPP, NTPP^{-1}\}$ is contained in \widehat{S} , then $RSAT(S)$ is NP-hard.

Proof Sketch. When $R_f \cup R_t$ is replaced by $\{TPP, NTPP, TPP^{-1}, NTPP^{-1}\}$, $\{TPP, TPP^{-1}\}$ or $\{NTPP, NTPP^{-1}\}$, the transformation of Theorem 2 can be applied. For $\{NTPP, TPP^{-1}\}$ and $\{TPP, NTPP^{-1}\}$ ONE-IN-THREE-3SAT can be reduced to $RSAT$ using the same transformation steps. \blacksquare

By computing the closure of all sets containing the eight base relations together with one additional relation, the following lemma can be obtained.

Lemma 7 $RSAT(S)$ is NP-hard for any subset S of RCC-8 containing all base relations together with one of the 72 relations of the following sets:

$$\mathcal{N}_1 = \{R \mid \{PO\} \not\subseteq R \text{ and } (\{TPP, TPP^{-1}\} \subseteq R \text{ or } \{NTPP, NTPP^{-1}\} \subseteq R)\},$$

$$\mathcal{N}_2 = \{R \mid \{PO\} \not\subseteq R \text{ and } (\{TPP, NTPP^{-1}\} \subseteq R \text{ or } \{TPP^{-1}, NTPP\} \subseteq R)\}.$$

Transformation of $RSAT$ to SAT

For transforming $RSAT$ to propositional satisfiability (SAT) we will transform every instance Θ of $RSAT$ to a propositional formula in CNF that is satisfiable iff Θ is consistent. We will start from $m(\Theta)$, the multimodal encoding of Θ , and show that whenever $m(\Theta)$ is satisfiable it has a Kripke model of a specific type.

This model will then be used to transform $m(\Theta)$ to a propositional formula.

$m(\Theta)$ is satisfiable if it is true in a world w of a Kripke model $\mathcal{M} = \langle W, \{R_1 = W \times W, R_2 \subseteq W \times W\}, \pi \rangle$, where W is a set of worlds, R_1 the accessibility relation of the \Box -operator, R_2 the accessibility relation of the \mathbf{I} -operator, and π a truth function that assigns a truth value to every atom in every world. The truth conditions for $\mathcal{M}, w \models m(\Theta)$ can be specified as a combination of truth conditions of the single atoms according to the form of $m(\Theta)$. In this way $\mathcal{M}, w \models \mathbf{I}\varphi$, e.g., can be written as $(\forall u : wR_2u. \mathcal{M}, u \models \varphi)$ and $\mathcal{M}, w \models \neg\mathbf{I}\varphi$ as $(\exists u : wR_2u. \mathcal{M}, u \models \neg\varphi)$. We will call this form of writing $\mathcal{M} \models m(\Theta)$ the *explicit form* of $m(\Theta)$.

Before transforming $m(\Theta)$ to a propositional formula, we have to show that there is a Kripke model of $m(\Theta)$ that is polynomial in the number of spatial variables n .

Definition 8 Let $u \in W$ be a world of the model \mathcal{M} .

- u is a world of level 0 if vR_2u only holds for $v = u$.
- u is a world of level $l + 1$ if vR_2u holds for a world v of level l and there is no world $v \neq u$ of level $> l$.

We assume that every occurrence of a sub-formula of $m(\Theta)$ of the form $\neg\Box\varphi$, where φ contains no \Box operators, introduces a new world of level 0. As these sub-formulas correspond to entailment constraints, the number of worlds of level 0 is polynomial in n .

For every spatial variable X and every world w there might be sub-formulas that force the existence of a world u with wR_2u where X is true or where $\neg X$ is true. Because there are n different spatial variables, $2n$ different worlds u with wR_2u are sufficient for each world w .

Definition 9 An RCC-8-frame $\mathcal{F} = \langle W, \{R_1, R_2\} \rangle$ has the following properties:

1. W contains only worlds of level 0, 1 and 2.
2. For every world w of level k ($k = 0, 1$) there are exactly $2n$ worlds u of level $k + 1$ with wR_2u .
3. For every world w of level k there is exactly one world u for every level $0 \leq l \leq k$ with uR_2w .

An RCC-8-model is based on an RCC-8-frame.

Lemma 10 $m(\Theta)$ is satisfiable iff $\mathcal{M}, w \models m(\Theta)$ for an RCC-8-model \mathcal{M} with polynomially many worlds.

Now it is possible to transform the explicit form of $m(\Theta)$ to a propositional formula $p(m(\Theta))$ in CNF such that $p(m(\Theta))$ is satisfiable iff $m(\Theta)$ is satisfiable in a polynomial RCC-8-model \mathcal{M} . For this purpose, for every world w and every atom X a propositional atom $X_{f(w)}^{g(w)}$ is introduced which stands for the truth of atom X in world w of the RCC-8-model \mathcal{M} . The

functions f and g are necessary to preserve the structure of the RCC-8-frame in propositional logic. $f(w)$ determines the world of level 0 with $f(w)R_2w$. $g(w)$ determines the level and the position of w in the frame with respect to $f(w)$.

Universally quantified truth conditions are transformed into conjunctions over the corresponding truth conditions. In order to cope with existentially quantified truth conditions, we use the property of RCC-8-frames that for every world w of level l there are exactly $2n$ worlds u of level $l + 1$ with wR_2u . Each of these $2n$ worlds will be reserved for a different positive or negative literal. Suppose, e.g., w is a world of level 1 and u is the world of level 2 with wR_2u that is reserved for $\neg X$. If u holds X , then all $2n$ worlds v of level 2 with wR_2v hold X . If any of the $2n$ worlds holds $\neg X$, then u also holds $\neg X$. To assure that these properties hold, additional propositional formulas have to be specified for each atom X . These formulas are Horn formulas. Using this property of RCC-8-frames, existentially quantified truth conditions are transformed into truth conditions on worlds w of a particular level and position $g(w)$.

Theorem 11 RSAT(RCC-8) can be polynomially reduced to SAT.

Together with Corollary 3 this leads to the following corollary.

Corollary 12 RSAT(RCC-8) is NP-complete.

Tractable Subsets of RCC-8

In order to identify a tractable subset of RCC-8, we analyze which relations can be expressed as propositional Horn formulas, as satisfiability of Horn formulas (HORNSAT) is tractable.

Proposition 13 Applying the transformation p to the model and entailment constraints, to the axioms for \mathbf{I} , and to the properties of closed regions leads to Horn formulas.

Since the model constraints α and A are transformed to indefinite Horn formulas, the transformation of any disjunction of these constraints with any other constraint is also Horn. All relations with an abbreviated form using only abbreviations or disjunctions of abbreviations transformable to Horn formulas can be transformed to Horn formulas. In this way 64 different relations can be transformed to Horn formulas. We call the subset of RCC-8 containing these relations \mathcal{H}_8 (see Appendix B).

Theorem 14 RSAT(\mathcal{H}_8) can be polynomially reduced to HORNSAT and therefore RSAT($\hat{\mathcal{H}}_8$) $\in P$.

Theorem 15 $\hat{\mathcal{H}}_8$ contains the following 148 relations:

$$\hat{\mathcal{H}}_8 = \text{RCC-8} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$$

with \mathcal{N}_1 and \mathcal{N}_2 as defined in Lemma 7 and

$$\mathcal{N}_3 = \{R \mid \{\text{EQ}\} \subseteq R \text{ and } ((\{\text{NTPP}\} \subseteq R, \{\text{TPP}\} \not\subseteq R) \\ \text{or } (\{\text{NTPP}^{-1}\} \subseteq R, \{\text{TPP}^{-1}\} \not\subseteq R))\}.$$

For proving that $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8, we have to show that no relation of \mathcal{N}_3 can be added to $\widehat{\mathcal{H}}_8$ without making RSAT intractable.

Lemma 16 *The closure of every set containing $\widehat{\mathcal{H}}_8$ and one relation of \mathcal{N}_3 contains the relation $\{\text{EQ}, \text{NTPP}\}$.*

Therefore it is sufficient to prove NP-hardness of $\text{RSAT}(\widehat{\mathcal{H}}_8 \cup \{\text{EQ}, \text{NTPP}\})$ for showing that $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8.

Lemma 17 $\text{RSAT}(\widehat{\mathcal{H}}_8 \cup \{\text{EQ}, \text{NTPP}\})$ is NP-hard.

Proof Sketch. Transformation of 3SAT to $\text{RSAT}(\widehat{\mathcal{H}}_8 \cup \{\text{EQ}, \text{NTPP}\})$. $R_t = \{\text{NTPP}\}$ and $R_f = \{\text{EQ}\}$. Polarity constraints:

$$X_L \{\text{EC}, \text{NTPP}\} X_{-L}, Y_L \{\text{TPP}\} Y_{-L}, \\ X_L \{\text{TPP}, \text{NTPP}\} Y_{-L}, Y_L \{\text{EC}, \text{TPP}\} X_{-L},$$

Clause constraints for each clause $c = \{i, j, k\}$:

$$Y_i \{\text{NTPP}^{-1}\} X_j, Y_j \{\text{NTPP}^{-1}\} X_k, Y_k \{\text{NTPP}^{-1}\} X_i \blacksquare$$

Theorem 18 $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8.

It has to be noted that there might be other maximal tractable subsets of RCC-8 that contain all base relations.

As $\widehat{\mathcal{H}}_8$ is tractable, the intersection of RCC-5 and $\widehat{\mathcal{H}}_8$ is also tractable. We will call this subset $\widehat{\mathcal{H}}_5$.

Theorem 19 $\widehat{\mathcal{H}}_5$ is the only maximal tractable subset of RCC-5 containing all base relations.

Applicability of Path-Consistency

As shown in the previous section, $\text{RSAT}(\widehat{\mathcal{H}}_8)$ can be solved in polynomial time by first transforming a set of $\widehat{\mathcal{H}}_8$ formulas to a propositional Horn formula and then deciding it in time linear in the number of literals. This way of solving RSAT does not appear to be very efficient.

As RSAT is a Constraint Satisfaction Problem (CSP) (Mackworth, 1987), where variables are nodes and relations are arcs of the constraint graph, algorithms for deciding consistency of a CSP can also be used. A correct but in general not complete $O(n^3)$ algorithm for deciding inconsistency of a CSP is the path-consistency method (Mackworth, 1977) that makes a CSP path-consistent by successively removing relations from all edges using

$$\forall k : R_{ij} \leftarrow R_{ij} \cap (R_{ik} \circ R_{kj}),$$

where i, j, k are nodes and R_{ij} is the relation between i and j . If the empty relation occurs while performing

this operation, the CSP is not path-consistent, otherwise it is.

In this section we will prove that path-consistency decides $\text{RSAT}(\widehat{\mathcal{H}}_8)$. This is done by showing that the path-consistency method finds an inconsistency whenever positive unit resolution (PUR) resolves the empty clause from the corresponding propositional formula. As PUR is refutation-complete for Horn formulas, it follows that the path-consistency method decides $\text{RSAT}(\mathcal{H}_8)$. The only way to get the empty clause is resolving a positive and a negative unit clause of the same variable. Since the Horn formulas that are used contain only a few types of different clauses, there are only a few ways to resolve unit clauses using PUR.

Definition 20

- R_K denotes the set of relations of \mathcal{H}_8 with the conjunct K appearing in their abbreviated form.
- R_{K_1, K_2} denotes $R_{K_1} \cup R_{K_2}$.
- R_Γ denotes $R_\gamma \cup R_{\alpha\vee\gamma} \cup R_{\alpha\vee\gamma} \cup R_C \cup R_{\alpha\vee C} \cup R_{\alpha\vee C}$.
- An R_K -chain $R_K^*(X, Y)$ is a path from region X to region Y , where all relations between successive regions are from R_K .

Proposition 21 Let Θ be a set of \mathcal{H}_8 -formulas and $p(m(\Theta))$ be the corresponding Horn formula.

- A positive unit clause $\{X_{f(w)}^{g(w)}\}$ can only be resolved from $\{Y_{f(w)}^{g(w)}\}$ and a clause of $p(m(\Theta))$ resulting from $X R_\Gamma Y \in \Theta$. When such a resolution is possible, $X R_{\alpha, A} Y$ cannot hold so $X R_{\gamma, C} Y$ must hold.
- A negative unit clause $\{\neg X_{f(w)}^{g(w)}\}$ can only be resolved from $\{Y_{f(w)}^{g(w)}\}$ and a clause of $p(m(\Theta))$ resulting from $X R_{\alpha, A} Y \in \Theta$.

Lemma 22 *If the positive unit clause $\{X_{f(w)}^{g(w)}\}$ can be resolved with PUR using an R_Γ -chain from X to Y , the path-consistency method results in $X R_{\gamma, C} Y$.*

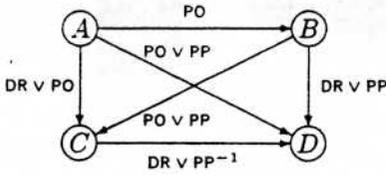
Using Lemma 22, it can be proven that the path-consistency method decides $\text{RSAT}(\mathcal{H}_8)$. Using the proof of Theorem 4, it is possible to express every relation of $\widehat{\mathcal{H}}_8$ as a Horn formula. Then the following theorem can be proven.

Theorem 23 *The path-consistency method decides $\text{RSAT}(\widehat{\mathcal{H}}_8)$.*

Another interesting question is whether and for which sets of relations the path-consistency method also decides the minimal-label problem RMIN. As the following proposition shows, this is not the case even for the set \mathcal{H}_5 , the intersection of \mathcal{H}_8 with RCC-5.

Proposition 24 *The path-consistency method is not sufficient for deciding the minimal-label problem $\text{RMIN}(\mathcal{H}_5)$.*

Proof. The following figure shows a constraint graph that is path-consistent but not minimal. The relation between A and D can be refined to PO but not to PP.



Related Work

Nebel (1995) showed that $\text{RSAT}(\widehat{\mathcal{B}})$ can be decided in polynomial time, where \mathcal{B} is the set of the RCC-8 base relations. Since $\mathcal{B} \subseteq \mathcal{H}_8$, our result is more general. Further, $\widehat{\mathcal{B}}$ contains only 38 relations, whereas $\widehat{\mathcal{H}}_8$ contains 148 relations, i.e. about 58% of RCC-8.

Grigni et al. (1995) proved NP-hardness of problems similar to RSAT. For instance, they considered the problem of *relational consistency*, which means that there exists a path-consistent refinement of all relations to base relations, and showed that this problem is NP-hard. While our NP-hardness result on RSAT implies their result, the converse implication follows only using the above cited result by Nebel (1995).

In addition to this syntactic notion of consistency, Grigni et al. (1995) considered a semantic notion of consistency, namely, the *realizability* of spatial variables as internally connected planar regions. This notion is much more constraining than our notion of consistency. It is also computationally much harder.

Applicability of $\widehat{\mathcal{H}}_8$

In this section we will discuss some practical advantages of the theoretical results obtained so far. One obvious advantage of the maximal tractable subset $\widehat{\mathcal{H}}_8$ is that the path-consistency method can now be used to decide RSAT when relations of $\widehat{\mathcal{H}}_8$ are used and not only when base relations are used.

As in the case of temporal reasoning, where the usage of the maximal tractable subset ORD-HORN has been extensively studied (Nebel, 1997), $\widehat{\mathcal{H}}_8$ can also be used to speed up backtracking algorithms for the general NP-complete RSAT problem. Previously, every spatial formula had to be refined to a base relation before the path-consistency method could be applied to decide consistency. In the worst case this has to be done for all possible refinements. Supposing that the relations are uniformly distributed, the average branching factor, i.e. the average number of different refinements of a single relation to relations of \mathcal{B} is 4.0.

Using our results it is sufficient to make refinements of all relations to relations of $\widehat{\mathcal{H}}_8$. Except for four relations, every relation not contained in $\widehat{\mathcal{H}}_8$ can be expressed as a union of two relations of $\widehat{\mathcal{H}}_8$, the four relations can only be expressed as a union of three

$\widehat{\mathcal{H}}_8$ relations. This reduces the average branching factor to 1.4375. Both branching factors are of course worst-case measures because the search space can be considerably reduced when path-consistency is used as a forward checking method (Ladkin and Reinefeld, to appear).

The following table shows the worst-case running time for the average branching factors given above. All running times are computed as $b^{(n^2-n)/2}$ where b is the average branching factor and n the number of spatial variables contained in Θ . We assumed that 100.000 path-consistency checks can be performed per second.

#spatial variables	\mathcal{B} (4.0)	$\widehat{\mathcal{H}}_8$ (1.4375)
5	10sec	3msec
7	500days	20msec
10	10^{14} years	2min

Summary

We analyzed the computational properties of the qualitative spatial calculus RCC-8 and identified the boundary between polynomial and NP-hard fragments. Using a modification of Bennett's encoding of RCC-8 in a multi-modal propositional logic, we transformed the RCC-8 consistency problem to a problem in propositional logic and isolated the relations that are representable as Horn clauses. As it turns out, the fragment identified in this way is also a maximal fragment that contains all base relations and is still computationally tractable. Further, we showed that for this fragment path-consistency is sufficient for deciding consistency.

As in the case of qualitative temporal reasoning (Nebel, 1997), our result allows to check whether the relations that are used in an application allow for a polynomial reasoning algorithm. Further, if the application requires an expressive power beyond the polynomial fragment, it can be used to speed up backtracking algorithms.

Appendix A: Basics on Modal Logic

Propositional modal logic (Fitting, 1993; Chellas, 1980) has the same syntax as standard propositional logic except for an additional unary operator \Box . One common approach to interpret modal logical formulas are the Kripke semantics, where models \mathcal{M} are build upon so-called frames $\mathcal{F} = \langle W, R \rangle$ that consist of a set of worlds W together with an accessibility relation $R \subseteq W \times W$ defined on worlds, and a truth function π that assigns truth values to all the propositional atoms in every world.

The truth of a modal formula ϕ in a world w of a model \mathcal{M} , written as $\mathcal{M}, w \models \phi$, is defined on the inductive structure of ϕ :

$\mathcal{M}, w \Vdash a$	iff	$\pi(w, a) = \text{true}$
$\mathcal{M}, w \Vdash \neg\phi$	iff	$\mathcal{M}, w \not\Vdash \phi$
$\mathcal{M}, w \Vdash \phi \wedge \psi$	iff	$\mathcal{M}, w \Vdash \phi$ and $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \phi \vee \psi$	iff	$\mathcal{M}, w \Vdash \phi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \phi \rightarrow \psi$	iff	$\mathcal{M}, w \not\Vdash \phi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \Box\phi$	iff	$\forall u : wR_u.\mathcal{M}, u \Vdash \phi$

It can be seen that a modal operator is closely connected to the corresponding accessibility relation.

A modal logic is determined by a set of frames, while frames can be determined either by specifying particular relations or defining axioms that all corresponding frames must hold. Some well-known modal logics are given in the following table:

Name	Relation	Axiom
K	-	$\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
T	reflexive	$\Box\phi \rightarrow \phi$
4	transitive	$\Box\phi \rightarrow \Box\Box\phi$
5	euclidian	$\neg\Box\phi \rightarrow \Box\neg\Box\phi$

K is the set of all frames. Other modal logics can be obtained by combining these logics. Two of them, which are of particular interest in this paper, are S4 and S5. S4 are all frames that are K and T as well as 4. S5-frames are both S4 and 5, they have the special property that worlds are clustered. All worlds of a particular cluster are mutually accessible.

In a multi-modal logic it is possible to use more than one modal operator, where every operator \Box_i has its own accessibility relation R_i , defined on the same set of worlds. The \Box -rule defined above has to be changed to different \Box_i -rules:

$$\mathcal{M}, w \Vdash \Box_i\phi \quad \text{iff} \quad \forall u : wR_iu.\mathcal{M}, u \Vdash \phi.$$

Appendix B: Abbreviated Form of \mathcal{H}_8

Relation	Abbreviated Form
{DC}	α
{EC}	$\neg\alpha \wedge A$
{DC, EC}	A
{PO}	$\neg\beta \wedge \neg\gamma \wedge \neg A$
{DC, PO}	$\neg\beta \wedge \neg\gamma \wedge (\alpha \vee \neg A)$
{EC, PO}	$\neg\alpha \wedge \neg\beta \wedge \neg\gamma$
{DC, EC, PO}	$\neg\beta \wedge \neg\gamma$
{TPP}	$\beta \wedge \neg\gamma \wedge \neg B$
{DC, TPP}	$\neg\gamma \wedge \neg B \wedge (\alpha \vee \beta)$
{EC, TPP}	$\neg\alpha \wedge \neg\gamma \wedge \neg B \wedge (A \vee \beta)$
{DC, EC, TPP}	$\neg\gamma \wedge \neg B \wedge (A \vee \beta)$
{PO, TPP}	$\neg\gamma \wedge \neg A \wedge \neg B$
{DC, PO, TPP}	$\neg\gamma \wedge \neg B \wedge (\alpha \vee \neg A)$
{EC, PO, TPP}	$\neg\alpha \wedge \neg\gamma \wedge \neg B$

{DC, EC, PO, TPP}	$\neg\gamma \wedge \neg B$
{NTPP}	$\neg\gamma \wedge B$
{DC, NTPP}	$\neg\gamma \wedge (\alpha \vee B)$
{EC, NTPP}	$\neg\alpha \wedge \neg\gamma \wedge (A \vee B)$
{DC, EC, NTPP}	$\neg\gamma \wedge (A \vee B)$
{TPP, NTPP}	$\beta \wedge \neg\gamma$
{DC, TPP, NTPP}	$\neg\gamma \wedge (\alpha \vee \beta)$
{EC, TPP, NTPP}	$\neg\alpha \wedge \neg\gamma \wedge (A \vee \beta)$
{DC, EC, TPP, NTPP}	$\neg\gamma \wedge (A \vee \beta)$
{PO, TPP, NTPP}	$\neg\gamma \wedge \neg A$
{DC, PO, TPP, NTPP}	$\neg\gamma \wedge (\alpha \vee \neg A)$
{EC, PO, TPP, NTPP}	$\neg\alpha \wedge \neg\gamma$
{DC, EC, PO, TPP, NTPP}	$\neg\gamma$
{TPP ⁻¹ }	$\neg\beta \wedge \gamma \wedge \neg C$
{DC, TPP ⁻¹ }	$\neg\beta \wedge \neg C \wedge (\alpha \vee \gamma)$
{EC, TPP ⁻¹ }	$\neg\alpha \wedge \neg\beta \wedge \neg C$
	$\wedge (A \vee \gamma)$
{DC, EC, TPP ⁻¹ }	$\neg\beta \wedge \neg C \wedge (A \vee \gamma)$
{PO, TPP ⁻¹ }	$\neg\beta \wedge \neg A \wedge \neg C$
{DC, PO, TPP ⁻¹ }	$\neg\beta \wedge \neg C \wedge (\alpha \vee \neg A)$
{EC, PO, TPP ⁻¹ }	$\neg\alpha \wedge \neg\beta \wedge \neg C$
{DC, EC, PO, TPP ⁻¹ }	$\neg\beta \wedge \neg C$
{NTPP ⁻¹ }	$\neg\beta \wedge C$
{DC, NTPP ⁻¹ }	$\neg\beta \wedge (\alpha \vee C)$
{EC, NTPP ⁻¹ }	$\neg\alpha \wedge \neg\beta \wedge (A \vee C)$
{DC, EC, NTPP ⁻¹ }	$\neg\beta \wedge (A \vee C)$
{TPP ⁻¹ , NTPP ⁻¹ }	$\neg\beta \wedge \gamma$
{DC, TPP ⁻¹ , NTPP ⁻¹ }	$\neg\beta \wedge (\alpha \vee \gamma)$
{EC, TPP ⁻¹ , NTPP ⁻¹ }	$\neg\alpha \wedge \neg\beta \wedge (A \vee \gamma)$
{DC, EC, TPP ⁻¹ , NTPP ⁻¹ }	$\neg\beta \wedge (A \vee \gamma)$
{PO, TPP ⁻¹ , NTPP ⁻¹ }	$\neg\beta \wedge \neg A$
{DC, PO, TPP ⁻¹ , NTPP ⁻¹ }	$\neg\beta \wedge (\alpha \vee \neg A)$
{EC, PO, TPP ⁻¹ , NTPP ⁻¹ }	$\neg\alpha \wedge \neg\beta$
{DC, EC, PO, TPP ⁻¹ , NTPP ⁻¹ }	$\neg\beta$
{EQ}	$\beta \wedge \gamma$
{DC, EQ}	$(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
{EC, EQ}	$\neg\alpha \wedge (A \vee \beta)$
	$\wedge (A \vee \gamma)$
{DC, EC, EQ}	$(A \vee \beta) \wedge (A \vee \gamma)$
{TPP, NTPP, EQ}	β
{DC, TPP, NTPP, EQ}	$(\alpha \vee \beta)$
{EC, TPP, NTPP, EQ}	$\neg\alpha \wedge (A \vee \beta)$
{DC, EC, TPP, NTPP, EQ}	$(A \vee \beta)$
{TPP ⁻¹ , NTPP ⁻¹ , EQ}	γ
{DC, TPP ⁻¹ , NTPP ⁻¹ , EQ}	$(\alpha \vee \gamma)$
{EC, TPP ⁻¹ , NTPP ⁻¹ , EQ}	$\neg\alpha \wedge (A \vee \gamma)$
{DC, EC, TPP ⁻¹ , NTPP ⁻¹ , EQ}	$(A \vee \gamma)$
{PO, TPP, NTPP, TPP ⁻¹ , NTPP ⁻¹ , EQ}	$\neg A$
{DC, PO, TPP, NTPP, TPP ⁻¹ , NTPP ⁻¹ , EQ}	$(\alpha \vee \neg A)$
{EC, PO, TPP, NTPP, TPP ⁻¹ , NTPP ⁻¹ , EQ}	$\neg\alpha$

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