A Study of Transitions in Dynamic Behavior of Physical Systems

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Abstract

This paper develops a classification of transition behaviors for hybrid dynamic systems. Physical system behaviors often comprise of multiple temporal and spatial scales. A hybrid system model abstracts the operational domain of a continuous system into patches of similar behaviors. Behaviors in each patch are described by simpler models. When state variables cross threshold values, the system makes discrete transitions from one patch to another. We assign physical semantics to the transition behaviors in order to develop a compositional modeling framework and self-consistent simulators for hybrid systems.

Introduction

Physical systems, governed by the principles of *conservation of energy* and *continuity of power* (Mosterman & Biswas 1998b), exhibit continuous behaviors made up of multiple temporal and spatial scales. Some of the behavioral phenomena can be attributed to fast, nonlinear effects that are hard to characterize. However, these phenomena may not require detailed analysis if one is only interested in the gross, overall system behaviors. For example, consider the cam follower system in Fig. 1. In automobiles the cam-axis translates rotational motion into a linear movement. A pushing rod driven by the linear displacement opens the engine valves. For a valve to close, it is important that the pushing rod follow the cam base. This is ensured by a spring that presses the rod against the cam. However, due to wear of the spring, reduced lubrication in the passages, and the fast revolutions of the engine, the rod may exhibit bouncing behavior that is governed by the small elasticity effects in the rod and cam. From a system modeler's viewpoint, the detailed collision process is not of much importance in terms of the overall engine behavior, therefore, it can be abstracted and replaced by a phenomena where the rod undergoes instantaneous reversals in velocity. Previous work (Mosterman & Biswas 1996; 1998b) developed a formal hybrid modeling formalism to analyze such phenomena in a number of different types of dynamic physical systems.

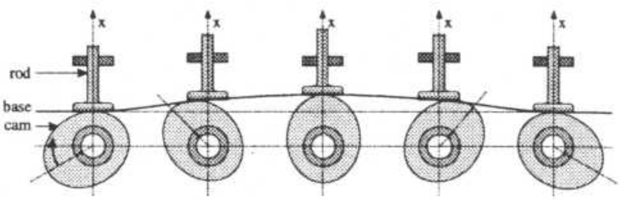

Figure 1: **A cam mechanism opens a valve.**

Hybrid modeling techniques simplify complex continuous nonlinear behaviors to piecewise continuous behaviors interspersed with discrete transitions. Nonlinear effects associated with fast behaviors are replaced by discrete transitions, and this alleviates the numerical problems caused by steep gradients. Model generation is also simplified because parasitic parameters have been abstracted away. Discrete transitions are linked to *configuration changes* in the system model, and result in the system operating in a number of different *modes*. To achieve compositionality, discrete changes are modeled as local switching functions defined in terms of system variables crossing threshold values. A local transition can trigger additional, instantaneous changes which continue till no further local transition functions are active, and the system behavior resumes continuous evolution in time. In a global modeling framework, these sequences of instantaneous changes can be replaced by direct transitions. However, for large systems composed of many primitive switching elements this is a daunting task

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that is complicated because mode transitions depend on the present mode as well as the state values. Therefore, to support compositional modeling approaches (e.g., *hybrid bond graphs* (Mosterman & Biswas 1996; 1998b)), it is pragmatic to develop a semantics that preserves the physical correctness of these instantaneous transitions.

Sometimes, the new continuous mode achieved is departed in an infinitesimally small time interval, and the system goes through a new sequence of discrete changes. This occurs when system variables in the new mode are exactly at their threshold values (Mosterman & Biswas 1997a; 1997c). In certain situations, this may cause the system to exhibit quick oscillations between two modes of operation, i.e., the system behavior *chatters* (Mosterman, Zhao, & Biswas 1997; Zhao & Utkin 1996).

More formally, *hybrid systems*[1] (Alur *et al.* 1994; Deshpande & Varaiya 1995; Guckenheimer & Johnson 1995; Mosterman & Biswas 1996; 1997a; 1998a) combine piecewise continuous behaviors with discrete mode transitions. Continuous behaviors in individual modes, governed by ordinary differential equations (ODEs) or differential and algebraic equations (DAEs), evolve in regions of real space. At well-defined points, discrete transitions defined by finite state machines or Petri nets, invoke instantaneous mode changes that may produce discontinuities in the system variables. Hybrid modeling techniques are becoming increasingly popular in analyzing embedded systems (physical systems with discrete controllers) and complex physical systems that exhibit fast and slow behaviors.

This paper develops a taxonomy of transitions in hybrid models of complex physical systems and a unified semantics that combines the normal continuous modes with mythical, pinnacle, and sliding behavior modes. The semantics forms the basis for developing a consistent simulation algorithm for hybrid systems. Simulation results have demonstrated the effectiveness of this approach. This work builds on top of two existing strands of work that studied individual mode transitions: (i) analysis of hybrid systems models with instantaneous mode and state vector changes (Mosterman & Biswas 1996; 1997a), and (ii) analysis of hybrid system models which exhibit *chattering*, i.e., fast transitions between two modes of operation (Mosterman, Zhao, & Biswas 1997; Zhao 1995; Zhao & Utkin 1996). The unique contribution of this paper lies in the systematic treatment of transitions between different modes. Defining the precise guards for each transition type and its corresponding semantics enables the implementation of a unifying simulation algorithm for hybrid system models.

[1] See also the Hybrid Dynamic Systems Web site at http://www.op.dlr.de/~pjm/hds.

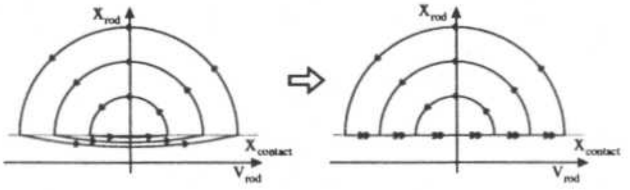

Figure 2: **Bouncing rod phase space.**

Mode Transitions in Hybrid Systems

Behaviors of a dynamic system can be conveniently described by an n-dimensional space spanned by the state vectors of the system called the phase space. As the system evolves over time, the consecutive states that define system behavior appear as a trajectory in phase space.

Discontinuities in Phase Space

Building simulation models for analyzing physical systems exhibiting behaviors at multiple scales is computationally complex. An approach to simplifying this task is to approximate a complex trajectory by piecewise continuous segments, where the system of equations describing the behavior in each segment is simpler than the original set of nonlinear equations. This requires additional mechanisms in the simulation scheme to detect transition boundaries and switch system models when transitions occur. Transitions may cause discontinuous changes in systems. The phase space behavior representation of the bouncing rod in the cam-follower system, illustrated in Fig. 2, shows the complex nonlinear behavior upon collision being replaced by an instantaneous transition, i.e., a discontinuous jump in the rod's velocity (indicated by the double arrow-heads).

Hybrid System Definition

Hybrid dynamic system behaviors evolve over time with no *gaps* on the time line, i.e., system behaviors are dense on the time line. Hybrid system behavior consists of three distinct subdomains:

- A *continuous* domain, T, with time, $t \in T$, as the special continuous variable.
- A *piecewise continuous* domain, V_α, where system behavior is governed by well-defined continuous functions f called *fields*, specified by a set of ODEs or DAEs. The set of temporal behaviors in a field is called a *flow*, $\mathcal{F}$, which specifies the state vectors $x_\alpha(t)$ uniquely on the time-line.

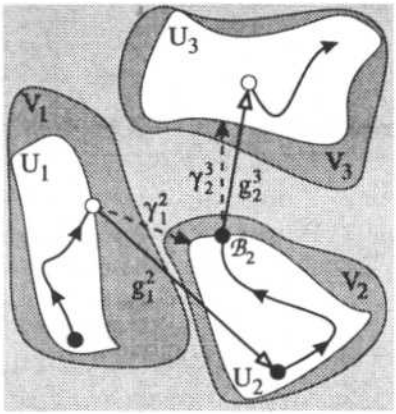

Figure 3: **A planar hybrid system.**

- A *discrete* domain, I, that captures all the operative piecewise continuous domains, V_α.

We adopt a notation similar to Guckenheimer and Johnson (Guckenheimer & Johnson 1995) and specify I to be a discrete indexing set, where $\alpha \in I$ represents the *mode* of the system. $\mathcal{F}_\alpha$ is a continuous C^2 flow on a possibly open subset V_α of $\Re^n$, called a *chart*, with a corresponding field f_α (Fig. 3). The sub-domain of V_α where a continuous flow in time occurs is called a *patch*, $U_\alpha \subset V_\alpha$. The flows constitute the piecewise continuous part of the hybrid system. The trajectory of the system is specified by the state vector $x_\alpha(t)$. The set of valid state vectors for an operational patch corresponding to mode α defines the state space X_α for that mode. The discrete switching function γ_α^β is defined as a *threshold function* on V_α. If $\gamma_\alpha^\beta \leq 0$ in mode α, the system transitions to β, defined by the mapping $g_\alpha^\beta : V_\alpha \rightarrow V_\beta$. The piecewise continuous level curves $\gamma_\alpha^\beta = 0$, denoted as S_α^β, define patch boundaries. If a flow $\mathcal{F}_\alpha$ includes the level curve, S_α^β, it contains the *boundary point*, $\mathcal{B}_\alpha$ (see Fig. 3). A hybrid dynamic system is defined by the 5-tuple[2]

$$H = < I, X_\alpha, f_\alpha, \gamma_\alpha^\beta, g_\alpha^\beta > . \quad (1)$$

A *trajectory* in the system starts at an initial point $x_{\alpha_1}(t)$ and if $\forall_{\alpha_2}\ \gamma_{\alpha_1}^{\alpha_2} > 0$, the point flows in α_1 as specified by $\mathcal{F}_{\alpha_1}$ until the minimal time t_s at which $\gamma_{\alpha_1}^{\alpha_2}(x_{\alpha_1}(t)) = 0$ for some α_2. Computing $x_{\alpha_1}(t_s^-) = \lim_{t \uparrow t_s} \mathcal{F}_{\alpha_1}(t)$ the transformation $g_{\alpha_1}^{\alpha_2}$ takes the trajectory from $x_{\alpha_1}(t_s^-) \in V_{\alpha_1}$ to $x_{\alpha_2}(t_s) \in V_{\alpha_2}$. The point $x_{\alpha_2}(t_s) = g_{\alpha_1}^{\alpha_2}(x_{\alpha_1}(t_s^-))$ is regarded as a new initial point.

[2]Guckenheimer and Johnson refer to the respective parts as $< V_\alpha, X_\alpha, \mathcal{F}_\alpha, h_\alpha^\beta, T_\alpha^\beta >$ (Guckenheimer & Johnson 1995; Mosterman & Biswas 1998a).

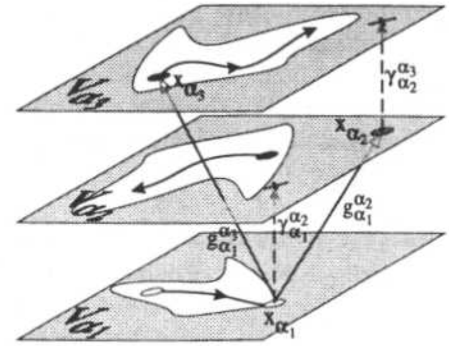

Figure 4: **Redirected trajectory because the transported point is not in the domain of the new patch.**

If there exists $\alpha_3 \in I$, such that $\gamma_{\alpha_2}^{\alpha_3}(x_{\alpha_2}(t_s)) \leq 0$, the trajectory is immediately transferred to $g_{\alpha_2}^{\alpha_3}(x_{\alpha_2}(t_s)) \in V_{\alpha_3}$ (see Fig. 4). A characteristic of hybrid systems is the possibility of a number of these immediate changes occurring before a new patch is arrived at, where again a flow defined by a field governs system behavior (Alur *et al.* 1994; Guckenheimer & Johnson 1995; Mosterman & Biswas 1996; 1997b). In general, this situation occurs if $\gamma_{\alpha_k}^{\alpha_{k+1}}$ transports a trajectory to α_{k+1}, and the initial point is transported by $g_{\alpha_k}^{\alpha_{k+1}}$ to a value that results in $\gamma_{\alpha_{k+1}}^{\alpha_{k+2}} \leq 0$, i.e., $g_{\alpha_k}^{\alpha_{k+1}}(x_{\alpha_k}) \notin U_{\alpha_{k+1}}$, and another mode α_{k+2} is instantaneously arrived at. These immediate transitions continue till a mode α_m is arrived at where the initial point is within U_{α_m}. Details of the mathematical model for hybrid systems appear in (Mosterman & Biswas 1997a; 1998a).

An Illustration: The Falling Rod

Consider the ideal rigid body collision between a thin rod and a floor in Fig. 5. Upon collision, small deformation effects may occur which forces the vertical velocity of the rod-tip, A, to quickly become 0. If this phenomenon occurs on a time scale much smaller than the time scale of interest, these effects can be abstracted away. As a result, the model will show a discontinuous change when computing $v_{A,y}$, the vertical component of the rod velocity at the point of contact. Given the existence of Coulomb friction (Lötstedt 1981) between the rod and floor, the rod may stick and rotate around the point of initial contact (mode α_1 in Fig. 5).

Alternately, if the rod-tip exerts a force in the horizontal direction that is larger than the product of the normal force and friction coefficient, i.e., $|F_{A,x}| > \mu F_n$, the rod may slide (mode α_2 in Fig. 5). When the velocity along the surface, $v_{A,x}$ falls below a threshold

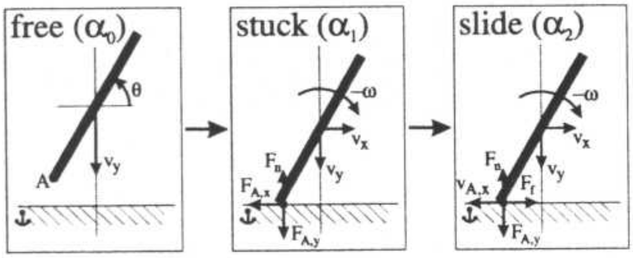

Figure 5: **Possible scenarios after collision.**

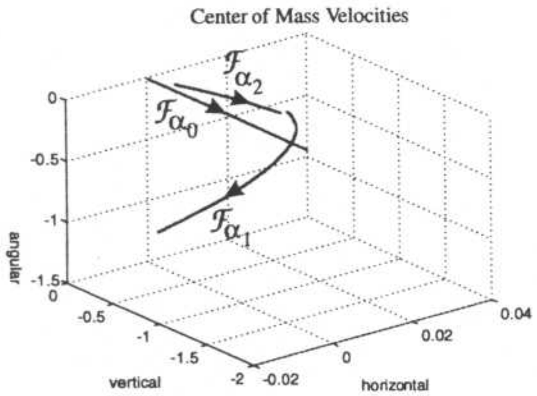

Figure 6: **A trajectory in phase space of the colliding rod, $\mu = 0.004, v_{th} = 0.0015, \theta = 0.862, l = -0.1, y_0 = 0.23$.**

value, v_{th}, the rod may get stuck.

Entry Points in Phase Space After Mode Transitions

A simulated trajectory of the rod in phase space is shown in Fig. 6. The coordinate system for the rod is centered at the middle point of the rod as shown in Fig. 5. The system is initialized with zero angular and linear velocities ($(0,0,0)$). Once the rod is released from height y_0, flow $\mathcal{F}_{\alpha_0}$ applies, and the magnitude of its vertical velocity increases in time. When the rod-tip, point A, touches the floor the rod may start to slide, governed by flow $\mathcal{F}_{\alpha_2}$. The mode α_2 where the rod slides is activated immediately after α_0 because a force balance computation indicates that the stuck mode α_1 is departed instantaneously, i.e., it is beyond the patch of a new mode. Also, the discontinuous jumps between flows are illustrated in Fig. 6.

If the simulation is repeated with a longer rod, initially the rod may slide on hitting the ground, but when sliding starts, the balance of forces indicates that the rod disconnects and lifts off the ground (i.e., the normal force of the floor would become negative $F_n < 0$). In this case the rod is in the sliding mode, α_2, for a point in time, after which it transitions back to

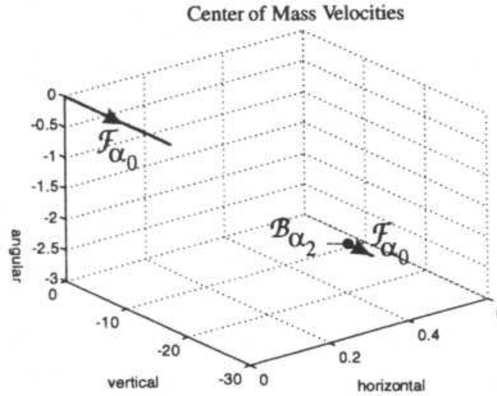

Figure 7: **A boundary in phase space of the colliding rod, $\mu = 0.5, v_{th} = 0.0015, \theta = 0.862, l = -10, y_0 = 23$.**

the free mode of operation, α_0. Note that this occurs even though the collision is modeled to be perfectly non-elastic, i.e., there is no restitution of momentum difference in any of the operational modes ($\epsilon = 0$). A simulation result for this example is shown in Fig. 7. This simulation demonstrates how the boundary point $\mathcal{B}_{\alpha_2}$ changes the state vector for the flow in α_0 (Mosterman & Biswas 1997a). Note that in α_2 a field governs behavior at a point in phase space.

Summary

When a mode change occurs, the system behavior enters a new region of phase space. As discussed earlier, a portion of this region, the patch (region 1 in Fig. 8) defines continuous behavior evolution. Surrounding the patch is a boundary region marked 2 in Fig. 8, and the region outside the boundary is marked 3 in Fig. 8 where no real behaviors can occur. In terms of transitions, three scenarios are possible in behavior evolution:

(1) Transition to patch interior: This occurs when the derived state vector after a mode change does not result in an immediate further transition within an infinitesimal period of time. For convenience, we call this an *interior* mode.

(2) Transition to a boundary point: This occurs when the derived state vector after a mode transition is at the boundary of a patch, and, therefore, a new mode change occurs after an infinitesimal period of time (e.g., mode α_2 in the falling rod example). This point is called a *boundary*.

(3) Transition to a point outside of patch and boundary: A mode change is immediately followed by another transition, because the derived state vector is beyond the patch and boundary of that mode (e.g., mode α_1

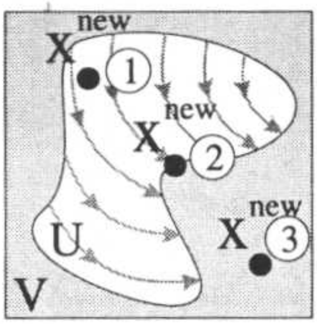

Figure 8: **Possible entry points in phase space after mode transitions: (1) interior; (2) boundary; (3) beyond patch.**

in the falling rod example). In such situations, the mode is called a *mythical* mode.

A Detailed Study

Hybrid system models resulting from systematic abstractions of complex physical systems approximate continuous behaviors as piecewise patches with discrete transitions between them. A study of the different forms of abstraction that can be applied and their links to defining mode transitions in hybrid models are analyzed. Appropriate semantics have to be developed for the different classes of transition behavior (Mosterman & Biswas 1997a; 1998a).

Two Abstraction Types

Discontinuities in behaviors of hybrid models of physical systems have been attributed to two general abstraction techniques: (i) *time scale* and (ii) *parameter* abstraction (Mosterman & Biswas 1997c; Mosterman, Zhao, & Biswas 1997). Time scale abstractions represent complex behaviors over a small time interval by a discontinuous change at a point in time. An example is the model of a bouncing rubber ball, where the ball velocity reverses instantaneously upon collision with the floor. In reality, the the initial kinetic energy of the ball is stored on impact as elastic energy within the ball and the floor for a very small time period, and then returned back as kinetic energy to the ball, which causes it to fly back up. Time scale abstraction reduces the process of energy storage and return to a point in time. In contrast, parameter abstractions eliminate small, parasitic dissipation and storage parameters from the system model. In case of a steel ball, the elasticity coefficient of the ball may be small enough to be abstracted away. The implication is that the collision between the floor and ball becomes non-elastic (there is no energy storage at collision), and the ball comes to rest at the point of contact.

Physical Semantics

Discontinuities may manifest as discontinuous changes in field gradients and as jumps in system variables. The corresponding transitions may generate a number of mode classes that require the development of systematic semantics to define consistent behavior models.

Mythical Modes Parameter abstractions cause loss of behaviors, which may result in transitions through mythical modes, i.e., modes that have no real existence on the time line. Transitions through a sequence of mythical modes do not affect the system state. This is expressed as the principle of *invariance of state* (Mosterman & Biswas 1996; 1997a). In phase space, they correspond to a new computed state vector that is beyond the patch in the mode that is just arrived at. Therefore, there is an immediate further transition out of this mode.

An example of a mythical mode is α_1 in case of the falling rod that starts to slide upon contact (Fig. 5). Because the rod starts to slide, upon contact the system moves to mode α_2. To infer whether the condition for sliding $|F_{A,x}| > \mu F_n$ is satisfied, the balance of forces in mode α_1 has to be computed based on the redistribution of momentum in mode α_1. However, once sliding is inferred, redistribution of momentum in mode α_2 is not affected by the redistribution in the now mythical mode α_1. To make this clear, consider $\mu = 0$. Upon collision the stuck mode α_1 prescribes that $v_{A,x} = 0$, and, therefore, the center of mass has a velocity that cancels the effect of the rotation on $v_{A,x}$, i.e., $v_x = l\omega sin\theta$. Because $\mu = 0$, the system moves to the sliding mode α_2, with no friction force present and there never is a horizontal force in the system. Therefore, v_x should equal 0. This is derived properly from the $v_x = 0$ state immediately before contact with the floor, mode α_0, whereas the mythical mode α_1 would mandate $v_x \neq 0$.

Sliding Mode Parameter abstractions can also cause sliding mode behavior (Mosterman, Zhao, & Biswas 1997; Zhao 1995; Zhao & Utkin 1996). Consider two adjacent modes (α and β), and assume the system of interest transitions between these two modes (Fig. 9). If a transition leads the system to the boundary of the adjoining mode, and the direction of the field vector is toward the first mode, the system may switch from the second mode back to the boundary of the first. If the gradient of the field is again toward the second mode the first transition may re-

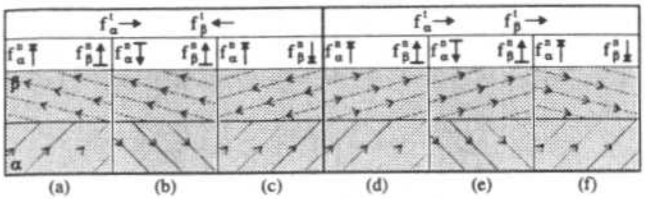

Figure 9: **Types of phase space behaviors near a switching surface.**

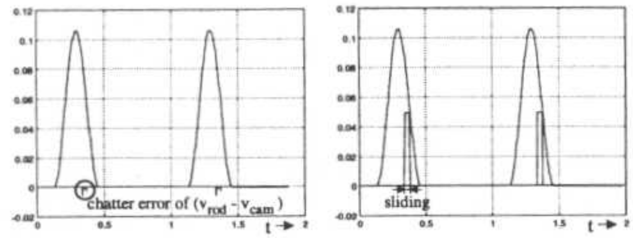

Figure 10: **Sliding mode simulation during an interval of time.**

peat (see Fig. 9c and Fig. 9f). If this phenomenon continues, one observes *chattering* behavior (e.g., the behavior of the anti-lock braking system in automobiles). Chattering is best-handled by introducing *sliding mode* behavior on the surface at the boundary of the two modes. Mathematically, the switching surface is a singularity, and behavior on the surface is undefined. In reality, small parameters are present that prevent the system from immediately switching back and forth, which induces a smooth motion along the switching surface (Mosterman, Zhao, & Biswas 1997; Zhao & Utkin 1996). In the limit as these parameter values tend to 0, the system moves exactly along the switching surface, which is called a *sliding regime*.

Sliding mode behavior is illustrated for the cam-follower system in Fig. 1. As discussed earlier, wear and tear in the rod and cam system may cause the rod to disconnect from the cam during deceleration, but they may reconnect within an infinitesimal period of time. This results in chattering behavior, i.e., fast back and forth transitions between two modes, in the simulation process. This causes computational inefficiency in that it slows down behavior generation. As $\Delta T \rightarrow 0$, the system would move smoothly along the sliding surface, $v_{rod} = v_{cam}$, and, therefore, a larger simulation step can be taken along this surface.

The sliding mode algorithm replaces chattering behavior by an *equivalent dynamics* to derive the time-varying nonlinear behavior from the linearized phase space under the assumption of small physical inertial and hysteresis effects (Mosterman, Zhao, & Biswas

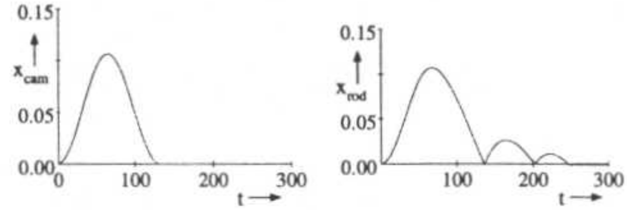

Figure 11: **Simulation of the cam-follower system.**

1997). Fig. 10 illustrates its application to a cam-follower mechanism. The simulation results on the left do not apply equivalent dynamics. The cam and rod alternately have equal and nonequal velocities. When the rod disconnects from the cam the velocity difference builds up. However, because the cam decelerates, at the next simulation time step a nonelastic collision occurs and the rod and cam velocities are instantaneously forced to be equal. The simulation on the right applies equivalent dynamics to remove this simulation artifact. The system slides on the switching surface and there is no error due to chattering. This conforms with true physical behavior, where unmodeled higher order physical phenomena such as adhesive forces between the rod and cam would result in the rod and cam having the same velocity.

Pinnacles Time scale abstractions lead to compression of behaviors in small intervals to a single point in time, i.e., they give rise to *pinnacles* (Mosterman & Biswas 1997a; 1998a). In Fig. 1, after the cam-base has reached its highest point, it starts to move down. In extreme cases, the rod may disconnect and bounce back onto the cam-base (Fig. 11). The collision can be modeled by Newton's elastic collision rule which states

$$v_{rod}^{+} = -\epsilon v_{rod}, \tag{2}$$

where ϵ represents the coefficient of restitution. It equals 1 in case of a perfect elastic collision. This collision rule captures the elasticity parameters that are active for a short period of time on collision, and collapses that effect (the collision) into a point in time. Change in the state vector is governed by algebraic equations, which hold only for that point in time, called a *pinnacle*. A pinnacle shows up as a jump in the phase space diagram, shown on the right in Fig. 2. After momentum transfer, v_{rod} has the same magnitude but opposite sign, and the system moves back to a continuous mode of operation.

Notice that pinnacles are different from boundaries. Pinnacle behaviors, caused by time scale abstraction, are governed by algebraic equations that cause jumps in phase space. As soon as the *a priori* state vec-

Mode Class	Semantics
mythical mode	invariance of state
sliding mode	equivalence of dynamics
pinnacle	no continuous evolution

Table 1: **Semantics governing each transition mode behavior.**

tor is updated, the pinnacle is departed and no continuous behavior evolution should take place in that mode. Boundary behaviors can be attributed to parameter abstraction and are governed by a gradient of continuous flow. After the state vector is updated, the boundary is active. It is departed after behavior evolves over an infinitesimal amount of time. Also note the difference between mythical modes and pinnacles. Unlike mythical modes, energy redistributions can occur within the system at pinnacles, and, therefore, the state vector can be modified, whereas mythical mode behaviors are governed by the principle of invariance of state (i.e., they do not affect the state vector).

Summary Behavior in interior and boundary modes are governed by the gradient of the flow in the particular mode of operation. In addition, our work has identified three other transition modes that occur in behavior analysis of hybrid systems: (i) mythical modes, (ii) sliding modes, and (iii) pinnacles. A mythical mode results in an immediate new transition, governed by the principle of invariance of state (Table 1). When the system switches back and forth between two boundary modes, i.e., the switches between the two boundary modes occur within an infinitesimal point in time, chattering behavior occurs, and this is defined as a sliding mode. In our work, sliding mode behavior is governed by the equivalence dynamics methodology (Mosterman, Zhao, & Biswas 1997). At a pinnacle, system behavior is defined by algebraic relations, therefore, there is no continuous evolution of behavior within this mode. An instantaneous mode change occurs and the new state vector is derived by the algebraic relations.

A Formal Representation and an Application

We have identified three types of transition behavior: (i) immediate transitions through mythical modes, (ii) transitions from pinnacles, governed by algebraic relations, and (iii) continuous behavior evolution. Fig. 12 shows how the components of a hybrid dynamic system model defined in Eq. (1) interact to mathematically establish system behaviors.

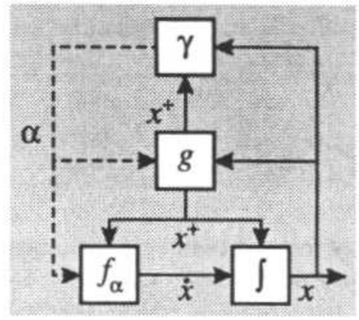

Figure 12: **A mathematical model based on a physical semantics.**

The Mathematical Model

The mathematical model defines a switching function, γ_α^β, with parameters the state vector x_α, prior to the jump and x_α^+, the state vector just after the jump computed by the mapping $x_\alpha^+ = g_\alpha^\beta(x_\alpha)$.

Initially the state vector evolves in V_{α_1} as specified by $\mathcal{F}_{\alpha_1}$ until the minimal time t_s at which $\gamma_{\alpha_1}^{\alpha_2}(x_{\alpha_1}(t)) = 0$ for some α_2. Then, the new state x_{α_2} is derived by $x_{\alpha_2} = g_{\alpha_1}^{\alpha_2}(x_{\alpha_1})$. If x_{α_2} is not in U_{α_2}, $\gamma_{\alpha_2}^{\alpha_3}(x_{\alpha_2}) \leq 0$ and x_{α_2} is not considered to have an actual representation on the time-line, and does not affect the mapping of x_{α_1} to x_{α_3}. Therefore, the new x_{α_3} is derived by applying $g_{\alpha_1}^{\alpha_3}$ to the original point x_{α_1}, $x_{\alpha_3} = g_{\alpha_1}^{\alpha_3}(x_{\alpha_1})$. In this case, x_{α_2} is mythical, i.e., it has no real existence in time for this behavior trajectory. Using $x_{\alpha_1}^+ = g_{\alpha_1}^{\alpha_i}(x_{\alpha_1})$ results in the expressions of the form $\gamma_{\alpha_1}^{\alpha_2}(x_{\alpha_1}^+) \leq 0$ and $\gamma_{\alpha_2}^{\alpha_3}(x_{\alpha_1}^+) \leq 0$.

In general, γ_α^β is also a function of x_α, the state value prior to the jump, e.g., in the case of pinnacles, a further mode change is triggered when the *a priori* value is updated (Mosterman & Biswas 1998a). With values x_α, prior to the jump and values x_α^+ after the jump, the semantics of transitions are specified by the recursive relation between γ_α^β and g_α^β (Mosterman 1997)

$$\begin{cases} x_{\alpha_k}^+ = g_{\alpha_k}^{\alpha_i}(x_{\alpha_k}) \\ \gamma_{\alpha_i}^{\alpha_i+1}(x_{\alpha_k}, x_{\alpha_k}^+) \leq 0 \end{cases} \tag{3}$$

The recursion terminates in a mode α_m when $\forall_{\alpha_n}$ $\gamma_{\alpha_m}^{\alpha_n}(x_{\alpha_k}, x_{\alpha_k}^+) > 0$. In this case α_k and α_m are real modes (pinnacle or continuous). Note the α_k subscript of x_{α_k} in $g_{\alpha_k}^{\alpha_i}$.

In physical systems, continuous behavior is completely specified by the state. Therefore, the state mapping is independent of the departed mode, i.e., g_α^β is independent of α. This results in the general sequence

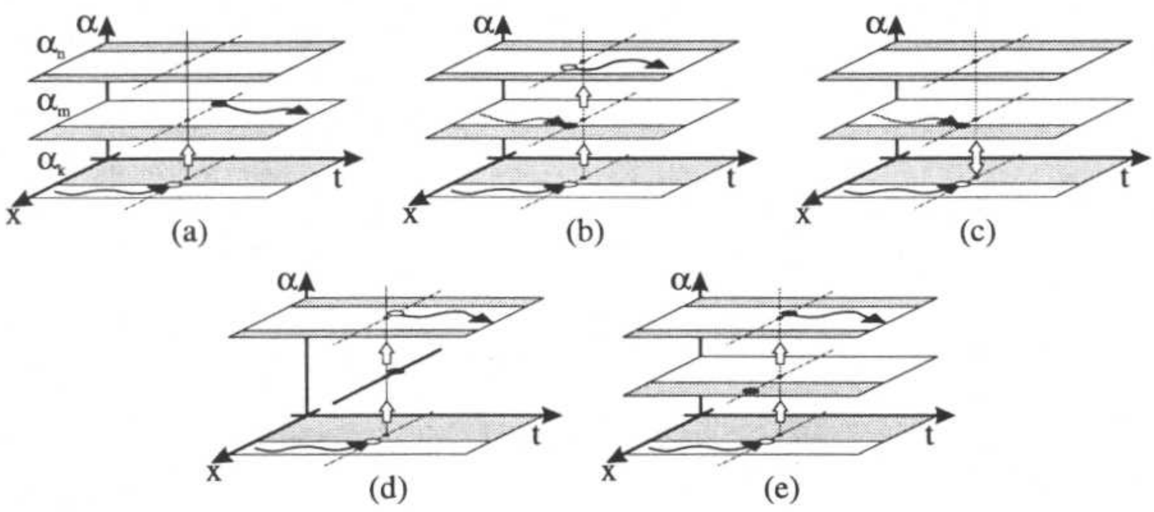

Figure 13: **Transition classes: (a) interior; (b) boundary; (c) sliding; (d) mythical; (e) pinnacle.**

$$\underbrace{\left\{\begin{array}{l} x^+ = g^{\alpha_1}(x) \\ x = x^+ \\ \dot{x} = f_{\alpha_1}(x,t) \end{array}\right.}_{\alpha_1} \overset{\gamma_{\alpha_1}^{\alpha_2}(x,x^+)}{\longrightarrow} \underbrace{\left\{\begin{array}{l} x^+ = g^{\alpha_2}(x) \\ x = x^+ \\ \dot{x} = f_{\alpha_2}(x,t) \end{array}\right.}_{\alpha_2} \overset{\gamma_{\alpha_2}^{\alpha_3}(x,x^+)}{\longrightarrow} \cdots \overset{\gamma_{\alpha_{m-1}}^{\alpha_m}(x,x^+)}{\longrightarrow} \underbrace{\left\{\begin{array}{l} x^+ = g^{\alpha_m}(x) \\ x = x^+ \\ \dot{x} = f_{\alpha_m}(x,t) \end{array}\right.}_{\alpha_m} \quad (4)$$

In this sequence, each mode, α, may be departed when any of the three assignment statements is executed.

Classifying Mode Transitions

The resultant computational model handles the types of mode transition behavior that may occur as illustrated in Fig. 13 (Note that **x** represents the state vector and α the mode of the system in this figure). This figure shows a sequence of possible mode changes from mode α_k to mode α_m and possibly α_n at a given point in time. Fig. 13a shows the situation that the mode transition moves the state vector into the interior of mode α_m and behavior evolution is governed by a continuous field, f_{α_m}. In Fig. 13b the trajectory transitions onto the boundary of α_m which causes a further change to α_n within an infinitesimal period of time. When the system is moved back to mode α_k, and the field gradients in both modes are directed towards each other, and chattering occurs. This is indicated by the double arrow on the discrete indexing axis, α, in Fig. 13c. Another case where the system resides in α_m for an infinitesimal period of time is when behavior in α_m is governed by an algebraic equation Fig. 13e. In this mode, no continuous behavior evolution is specified. Finally, if the trajectory is transported to a mode that has no representation for the given state and time, an immediate further transition follows, shown in Fig. 13d. In this case, α_m does not affect the state vector, and, therefore, is called mythical.

A classification of the transition behavior for the five classes can be made based on the nature of the transitions, illustrated by the mathematical model in Fig. 14:

(a) *Transition to continuous mode:* In this case after the transition and update $x = x^+$, $\forall_{\alpha_n} \gamma_{\alpha_m}^{\alpha_n}(x, x^+) > 0$. Therefore, f_{α_m} is active. Three situations may occur:

(i) *Interior mode:* Behavior evolution is continuous governed by field, f_{α_m} (see Fig. 13a).

(ii) *Boundary:* A transition occurs after an infinitesimal period of time, which indicates a patch boundary was reached (see Fig. 13b).

(iii) *Sliding mode:* A transition occurs after an infinitesimal period of time and the newly established mode switches back to the current one within an infinitesimal period of time (see Fig. 13c).

(b) *Transition to pinnacle:* This occurs when updating the state vector $x = x^+$ results in $\gamma_{\alpha_i}^{\alpha_{i+1}}(x, x^+) \leq 0$. Updating x stored in the $\int$ element causes a mode transition. Therefore, mode α_i only exists at a point in time but the state vector can change with the transition (see Fig. 13e).

(c) *Transition to mythical mode:* This occurs when $x^+ = g^{\alpha_i}(x)$ leads to $\gamma_{\alpha_i}^{\alpha_{i+1}}(x, x^+) \leq 0$. The immedi-

Mode Class	Criteria
mythical mode	$\exists \alpha_n(\gamma_{\alpha_m}^{\alpha_n}(x_{\alpha_k}(t), x_{\alpha_k}(t)^+) \leq 0)$
pinnacle	$\exists \alpha_n(\gamma_{\alpha_m}^{\alpha_n}(x_{\alpha_k}(t)^+, x_{\alpha_k}(t)^+) \leq 0)$
sliding mode	$\exists \delta t_1(\delta t_1 < \epsilon)(\gamma_{\alpha_m}^{\alpha_k}(x_{\alpha_m}(t+\delta t)) \leq 0) \wedge$ $\exists \delta t_2(\delta t_2 < \epsilon)(\gamma_{\alpha_k}^{\alpha_m}(x_{\alpha_k}(t+\delta t)) \leq 0)$

Table 2: **Classification scheme and guards.**

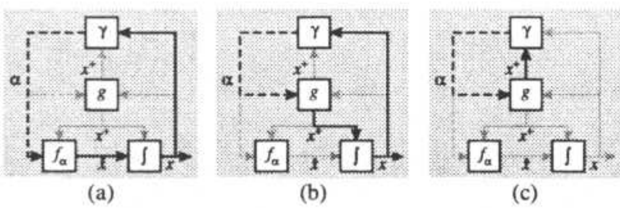

Figure 14: **Classes of modes of operation.**

ate transition bypasses the integrator ($\int$), therefore, the state vector x remains unchanged through the transition (see Fig. 13d).

Given α_k was an interior mode, and after a period of time a transition occurred from α_k to α_m with $x_{\alpha_k}(t)^+ = g^{\alpha_m}(x_{\alpha_k})$, Table 2 specifies the conditions that have to be satisfied for each of the classified modes based on the semantics described above.[3] Pinnacles and continuous modes are referred to as *real* modes because they change the state vector x stored in the $\int$ element. Pinnacles do so by a direct change and continuous modes by integration against time. Interior modes, boundary modes and sliding modes are called *continuous* because a field defines the behavior evolution process.

Note that the described mode transitions may appear in combination with one another. For example, in the cam-follower system, collision effects occur between the cam and the pushing rod that opens valves. These collisions introduce pinnacles in phase space that are traversed in between sliding modes. Likewise, mythical modes may be part of the transition between the chattering real modes.

Results

The mode transition classification and their physical semantics have formed the basis for developing a hybrid dynamic system simulation algorithm. The algorithm has been implemented and used to produce the simulation results shown in Figures 6, 7, 10, and 11. Continuous behavior evolution, identification of mode transitions, and updating the mode transitions when mode transitions occur have been implemented using the semantics developed in the earlier sections of this paper.

A high level description of the simulation algorithm appears as Algorithm 1. The input to the simulator is a mathematical hybrid system model (see (Mosterman & Biswas 1998a) for details) and an initial state vector. The output of the algorithm is a behavior trajectory that includes piecewise continuous behavior evolution plus mode transitions. A forward Euler numerical approximation function, *timeStep*(α, x), evolves behavior along field gradients in the continuous mode of operation. When a transition condition is detected ($\gamma_\alpha^\beta \leq 0$), this function generates behavior up to the transition event using a bi-sectional root search. Mode transitions are handled by the function *recursion*(α, x) which implements the recursive relations in Eq. (3). When recursion terminates, the state vector is updated ($x = x^+$). This may cause a further change implying a pinnacle. The pinnacle may be followed by mythical modes. When mode changes terminate in a new continuous mode, the sliding mode condition in Table 1 is checked by the function *slide*(α, x). If satisfied, equivalence dynamics approximates system behavior until behavior moves away from the switching surface. The system continues to evolve until a new transition condition is detected. Applications to the cam-follower and the falling rod systems were illustrated earlier.

Algorithm 1 Hybrid Simulation Algorithm

```
Require: α, x, f_α, γ_α^β, g_α^β
  while time < end time do
    x = timeStep(α, x)
    [α⁺, x⁺] = recursion(α, x)
    if α⁺ ≠ α then
      repeat
        α = α⁺
        x = x⁺
        [α⁺, x⁺] = recursion(α, x)
      until α⁺ = α
      [α, x] = slide(α, x)
    end if
  end while
```

Details of the simulation algorithm have been presented elsewhere (Mosterman & Biswas 1998a; Mosterman, Zhao, & Biswas 1997; Zhao & Utkin 1996).

Conclusions

A systematic study of abstractions in physical system models provides a formal methodology for defining hybrid systems and the semantics of temporal behavior evolution for these models. Model behaviors can be described by multiple piecewise-continuous patches in phase space. Transitions between patches give rise to additional modes of behavior: (i) interior, (ii) boundary, (iii) sliding, (iv) pinnacle, and (v) mythical. This paper has developed a systematic classification scheme

[3] The function $\gamma_\alpha^\beta(x, x^+)$ is replaced for clarity reasons by $\gamma_\alpha^\beta(x^+)$ for sliding modes because $x^+ = x$.

to define modes and transitions between modes. The classification scheme is developed into a consistent and efficient simulator for hybrid system behaviors. The simulator has been tested on a number of physical examples, such as diode-inductor circuits, the secondary sodium cooling loop of a fast breeder reactor, the falling rod, and the cam-follower system. It handles the idiosyncrasies of each transition type as well as combinations of transitions.

Future research will focus on applying this framework to verification and analysis tasks for control problems. The primary challenges in a more extensive analysis include:

- Verification mechanisms to study mode transition guards for complex physical systems with much higher dimensional phase spaces,
- A more formal characterization of the pinnacle mode where the change in the state vector is governed by algebraic equations. Some preliminary work in this regard has been presented in (Mosterman & Biswas 1997b; 1998b; 1998a), and
- Development of a systematic methodology for compositional modeling from individual component models, while ensuring that the time scale and parameter abstractions are consistent across models. Preliminary work in this area has been presented in the bond graph framework with switched junctions (Mosterman & Biswas 1997b; 1998b).

References

Alur, R.; Courcoubetis, C.; Halbwachs, N.; Henzinger, T.; Ho, P.-H.; Nicollin, X.; Olivero, A.; Sifakis, J.; and Yovine, S. 1994. The algorithmic analysis of hybrid systems. In *Proceedings of the 11th International Conference on Analysis and Optimization of Discrete Event Systems*, 331–351. Springer-Verlag. Lecture Notes in Control and Information Sciences 199.

Deshpande, A., and Varaiya, P. 1995. Viable control of hybrid systems. In Antsaklis, P.; Kohn, W.; Nerode, A.; and Sastry, S., eds., *Hybrid Systems II*, volume 999, 128–147. Springer-Verlag. Lecture Notes in Computer Science.

Guckenheimer, J., and Johnson, S. 1995. Planar hybrid systems. In Antsaklis, P.; Kohn, W.; Nerode, A.; and Sastry, S., eds., *Hybrid Systems II*, volume 999, 202–225. Springer-Verlag. Lecture Notes in Computer Science.

Lötstedt, P. 1981. Coulomb friction in two-dimensional rigid body systems. *Z. angew. Math. u. Mech.* 61:605–615.

Mosterman, P. J., and Biswas, G. 1996. A Formal Hybrid Modeling Scheme for Handling Discontinuities in Physical System Models. In *AAAI-96*, 985–990.

Mosterman, P. J., and Biswas, G. 1997a. Formal Specifications for Hybrid Dynamical Systems. In *IJCAI-97*, 568–573.

Mosterman, P. J., and Biswas, G. 1997b. Hybrid Modeling Specifications for Dynamic Physical Systems. In *1997 International Conference on Bond Graph Modeling and Simulation (ICBGM '97)*, 162–167. Phoenix, AZ: Society for Computer Simulation.

Mosterman, P. J., and Biswas, G. 1997c. Principles for Modeling, Verification, and Simulation of Hybrid Dynamic Systems. In *Fifth International Conference on Hybrid Systems*, 21–27.

Mosterman, P. J., and Biswas, G. 1998a. Modeling and simulation semantics for hybrid dynamic physical systems. In *Hybrid Systems V*. Springer-Verlag. in review.

Mosterman, P. J., and Biswas, G. 1998b. A theory of discontinuities in dynamic physical systems. *Journal of the Franklin Institute* 335B(6):401–439.

Mosterman, P. J.; Zhao, F.; and Biswas, G. 1997. Model semantics and simulation for hybrid systems operating in sliding regimes. In *AAAI Fall Symposium on Model Directed Autonomous Systems*, 48–55.

Mosterman, P. J. 1997. *Hybrid Dynamic Systems: A hybrid bond graph modeling paradigm and its application in diagnosis.* PhD dissertation, Vanderbilt University.

Zhao, F., and Utkin, V. I. 1996. Adaptive simulation and control of variable-structure control systems in sliding regimes. *Automatica: IFAC Journal* 32(7):1037–1042.

Zhao, F. 1995. Qualitative reasoning about discontinuous control systems. In *Proceedings of IJCAI-95 Engineering Problems for Qualitative Reasoning Workshop*, 83–93.