Modeling Multiple Strategies for Solving Geometric Analogy Problems

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Abstract

We present an improved computational model for performing geometric analogy. The model combines two previously modeled strategies and makes explicit claims about when people will use one strategy or the other. We compare the model to human performance on a classic problem set. The model's strategy shifts, along with working memory load, account for most of the variance in human reaction times.

Keywords: geometric analogy; visual problem-solving; structure-mapping

Introduction

Visual problem-solving has long been a popular tool for evaluating people's cognitive abilities (Raven, Raven, & Court, 1998; Dehaene et al., 2006). Problem-solving tasks frequently involve a sequence of images (e.g., Figure 1). Individuals must compare the images, identifying some pattern across them. They must then apply this pattern, finding the answer that best completes (or violates) it.

Visual problem-solving depends on a comparison process for identifying commonalities and differences in images. We have previously argued that structure mapping (Gentner, 1983), a theory of analogical comparison, may also explain concrete visual comparison in humans (Markman & Gentner, 1996; Lovett et al., 2009a; Sagi, Gentner, & Lovett, in press). According to structure mapping, people compare stimuli by aligning the common relational structure in symbolic, qualitative representations. We have posited that structure mapping may play a ubiquitous role, identifying commonalities and differences and estimating similarity. Based on this hypothesis, we have built models of three visual problem-solving tasks: geometric analogy (Lovett et al., 2009b), Raven's Progressive Matrices (Lovett, Forbus, & Usher, 2010), and the oddity task (Lovett & Forbus, 2011a).

Here, we complement our model of visual comparison with a model of *visual inference*. Visual inference explains how individuals apply a set of differences to one image to create a novel image representation. It plays a key role in tasks such as geometric analogy (Figure 1), where participants are asked "A is to B as C is to…?" We show how this leads to a new model for geometric analogy. Rather than assuming participants always infer the correct answer, the model makes explicit claims about when visual inference will succeed and when it will fail, requiring a shift to an alternate strategy. The model's strategic shifts correlate well with human reaction times on a classic geometric analogy problem set (Evans, 1968).

In the following section, we provide some background on the geometric analogy task. We then present our computational model, which utilizes two strategies for performing the task. Afterwards, we compare the model against human performance and discuss the results. We close with related work and conclusions.

Background

Geometric analogy involves comparing images to identify differences. However, researchers disagree on which comparisons people make. The debate encompasses two competing strategies. The first involves inserting each possible answer into the analogy to evaluate it. Consider Figure 1A. The strategy proceeds as follows:

1) Compare A and B to get Δ(A,B), the differences between A and B. Here there is a change from two overlapping objects to one object inside the other.

2) For each possible answer *i*, compare C to *i* to get Δ(C,i), the differences between C and that answer. Then, perform a second-order comparison: measure the similarity of Δ(A,B) to Δ(C,i). Whichever answer produces the most similar set of differences is chosen. Here answer 3 produces an identical set of differences to Δ(A,B), so it is chosen.

The second strategy solves for the answer directly:

I) Compare A and B to get Δ(A,B).

II) Compare A and C to get the corresponding objects. In Figure 1A, the large leftmost shapes correspond, and the small rightmost shapes correspond.

III) Apply the differences in Δ(A,B) to the corresponding objects in C to infer D', a representation of the answer. Here the small rectangle in C would move inside of the larger shape. Pick the answer most similar to D'.

Mulholland, Pellegrino, and Glaser (1980) call the first strategy *infer-infer-compare* and the second strategy *infer-map-apply*. However, this assumes that different processes are used to compare images in steps 1), 2), and II). We believe structure-mapping can determine differences, identify correspondences, *and* measure similarity. Therefore, we instead call the strategies *second-order comparison* and *visual inference*.

Sternber (1977) argued that people use visual inference to perform geometric analogy. However, Mulholland, Pellegrino, and Glaser (1980) found evidence that second-order comparison was being used. Bethell-Fox, Lohman, and Snow (1984) suggested that individuals may adjust their strategy, depending on problem difficulty. Their eye-tracking data demonstrated that people typically use visual inference, solving directly for the answer. However, as problems become more difficult, people may abandon this

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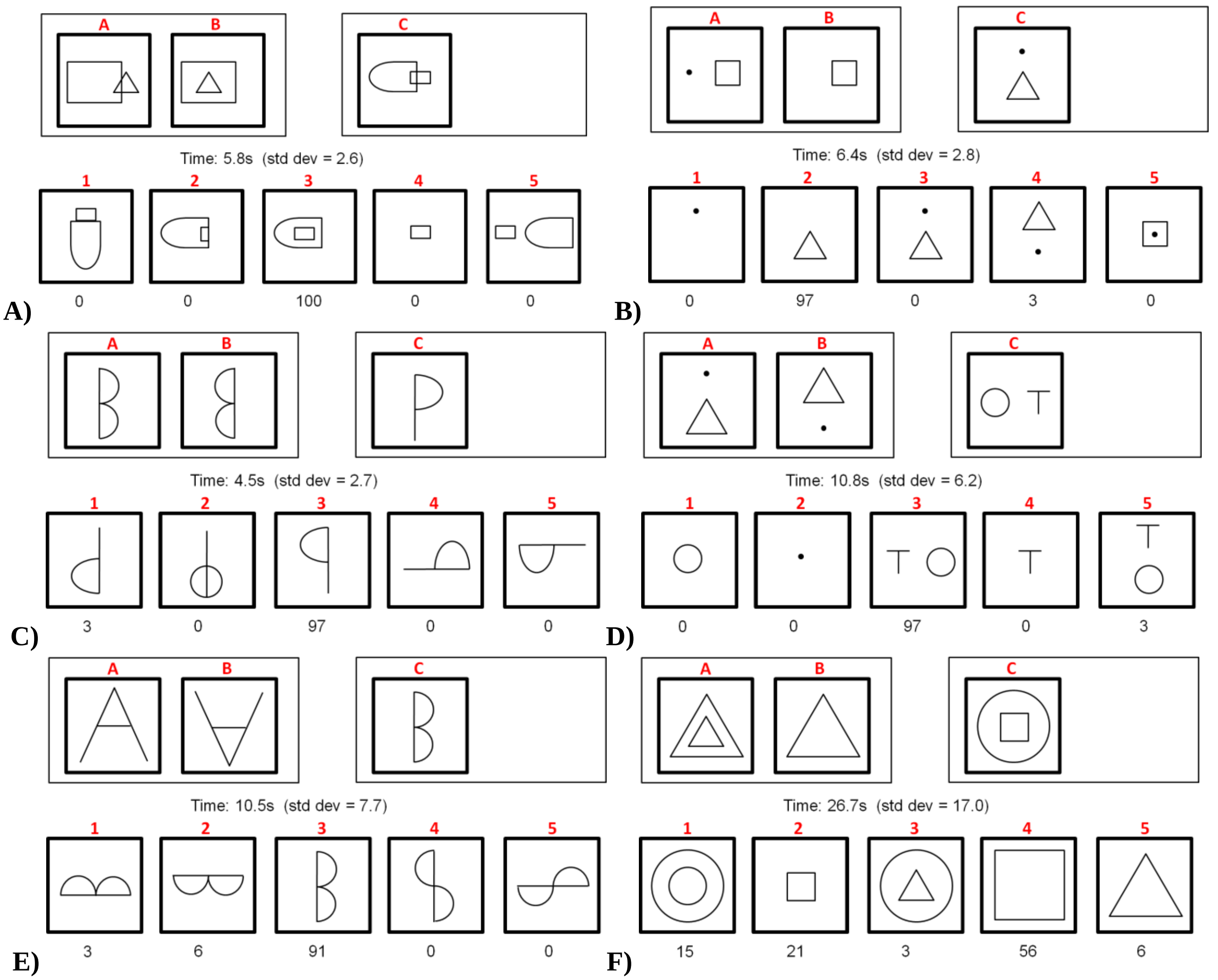

Figure 1: Geometric analogy problems with average response times and percent selecting each answer (Lovett et al., 2009b).

approach, instead trying out each possible answer in the analogy.

One might conclude that people use visual inference when problems are easy and second-order comparison when problems are relatively difficult. However, this is not the full story. Sternberg (1977) found that participants used visual inference on multiple choice problems, where they had to consider several possible answers. Mulholland, Pellegrino, and Glaser (1980) found that participants used second-order comparison on true/false problems, where they saw a completed analogy and simply judged whether it was correct. The Sternberg problems appear more difficult—at the very least, they put more load on working memory, as there are multiple answers to consider. Why, then, would people use second-order comparison on the easier Mulholland et al. problems?

We believe the answer lies in the algorithmic complexity of the two strategies. In the analysis below, we compute each strategy's complexity by counting the number of comparisons necessary to solve a problem. We concede that this may be a simplification; for example, second-order comparisons may require more time and effort than others. However, our model predicts that a common process (structure-mapping) is used for all comparisons. Thus the number of times this process repeats should provide a reasonable approximation of a strategy's complexity.

Consider second-order comparison. If there are *n* possible answers, then the number of comparisons is 1 (A to B) + n (C to each answer) + n (Δ(A,B) to Δ(C,n) for each answer) = 2n + 1.

Now consider visual inference. The number is 1 (A to B) + 1 (A to C) + n (D' to each answer) = n + 2. However, this strategy also requires a non-comparison operation: inferring D' by applying Δ(A,B) to the corresponding objects in C.

Suppose we have a true/false problem. Then the number of answers n = 1. The number of comparisons is 3 for both strategies. In this case, one might prefer second-order comparison, as it doesn't requiring inferring D'. This explains why Mulholland et al. found that participants used second-order comparison.

Suppose we have a multiple-choice problem. The number of answers n > 1. Now, there will be fewer comparisons for visual inference, and this should be the preferred strategy, as Sternberg found. However, if a problem is particularly complex, participants may be unable to perform the inference operation. In this case, they may fall back on the second-order comparison strategy, which requires only comparison operations.

Our model, described below, attempts to solve problems via visual inference. When it fails, either because it cannot perform the inference or because the inferred image doesn't match any of the answers, the model reverts to second-order comparison. Thus, the model can help explain why some problems take longer to solve than others: certain problems require a shift from the default visual inference strategy to a less efficient second-order comparison strategy.

Model

Our model depends on three core processes: perception, visual comparison, and visual inference. Below, we describe each process and then show how the model combines these processes to implement different problem-solving strategies.

Perception

Our model uses the CogSketch sketch understanding system (Forbus et al., 2011) to generate a qualitative representation for each image. Qualitative, or *categorical*, representations are abstract, describing features like relative position (**right of**) or relative orientation (**parallel**), rather than exact numerical sizes and orientations. There is abundant psychological evidence that people are sensitive to such features (e.g., Huttenlocher, Hedges, & Duncan, 1991; Rosielle & Cooper, 2001).

CogSketch performs sketch understanding, rather than full vision. It requires users to draw separate line drawings of each object in a visual scene (e.g., the rectangle and triangle in image A of Figure 1A). Given these objects, CogSketch automatically computes spatial relations between objects and attributes for individual objects.

Our model takes the process a step further. It can automatically segment an object into edges and build an edge-level representation, describing qualitative spatial relations between the edges. Alternatively, it can group several objects together based on similarity to build a group-level representation. See Lovett and Forbus (2011b) for details on this process, as well as the full vocabulary of qualitative terms at the edge, object, and group levels.

Visual Comparison

We model visual comparison using the Structure-Mapping Engine (SME) (Falkenhainer, Forbus, & Gentner, 1989), a computational model based on Genter's (1983) structure-mapping theory. Given two cases described in predicate calculus, it computes a mapping between them by aligning their common relational structure. SME is biased to prefer aligning deep structure. For example, at the edge level, a first-order relation might indicate that there is a convex corner between two edges. A second-order relation might indicate that two convex corners are adjacent. These higher-order relations, which take other relations as their arguments, receive priority during mapping.

SME computes up to 3 mappings between the compared cases. Each mapping contains: A) a similarity score based on the breadth and depth of aligned structure; B) correspondences between the entities and expressions in the two cases; and C) *candidate inferences* (CIs), inferences based on expressions in one case that failed to align with the other. For example, consider the A/B comparison in Figure 1A. A forward CI (A->B) would indicate that the two shapes no longer partially overlap. A backward CI (B->A) would indicate that one shape no longer contains the other. CIs are useful for identifying differences between the cases.

The model compares two images via the following steps:

1) Compare the qualitative image representations using SME. This produces a mapping indicating the corresponding objects, commonalities, and differences.

2) For each pair of corresponding objects, compare the objects' shapes to identify a shape transformation. This is done by comparing the objects' edge-level representations to get corresponding edges, and then using those correspondences to compute a transformation, such as a rotation or reflection (see Lovett & Forbus, 2011b).

Sometimes there are multiple valid transformations. For example, in Figure 1C, there is both a rotation and a reflection between the 'B' shapes. In such cases, the model picks the simplest transformation, according to the following rankings: identity, reflection, rotation. Objects can also change scale, becoming larger or smaller.

3) Compute the similarity between images. This is primarily based on SME's similarity score, but it is updated according to the shape comparisons: if two objects are identical, the images will be rated more similar.

4) Compute a qualitative, structural representation of the differences between the images. This describes differences of the following types:

A) Spatial relation addition/removals, based on the CIs.

B) Reversals of spatial relations. This is a special case of A) where two objects swap places in a relation. In Figure 1D, the dot and triangle swap places in an **above** relation.

C) Object additions/removals, where objects are added or removed between images (e.g., Figure 1B).

D) Object transformations, where there is a shape transformation between corresponding objects.

Visual Inference

The visual inference operation applies a set of differences to an image to produce a novel image representation. In geometric analogy, the A/B differences are applied to C to produce D', a representation of the answer image. Consider Figure 1A. Inference proceeds as follows:

1) Compare image A to image C to get the corresponding objects.

2) Apply the A/B differences to the corresponding objects in C to produce a new qualitative representation:

A) Add or remove spatial relations.

B) Reverse the arguments of spatial relations.

C) Add or remove objects. If an object is added, create a new object in CogSketch, basing it off some existing object.

If an object is removed, remove any spatial relations referring to that object.

D) For all other objects, apply the appropriate shape transformation to the object in C to create a new object for D'. This might mean leaving the shape unchanged (identity), rotating it, reflecting it, scaling it, etc.

E) Compute shape attributes for all the newly created objects, and add them to the D' representation.

Note that D' contains: a) a qualitative, structural image description; and b) a set of concrete, quantitative objects. Thus, it contains just enough to support visual comparisons between D' and other images. However, D' is not a concrete image: the model lacks exact, quantitative locations for each object.

There are several ways that visual inference can fail:

A) When a spatial relation cannot be added because the objects it describes are not found in C (there are no corresponding objects).

B) When a spatial relation cannot be reversed. In Figure 1D, there is a reversal of **above** in the A/B differences. However, there is no **above** in C to reverse.

C) When an object cannot be removed or transformed because the object is not found in C.

D) When transforming an object doesn't produce the desired effect. On Figure 1E, the model reflects the 'B' shape over the x-axis. However, when it compares the result to the original 'B' shape, they appear identical (recall that identity is ranked before reflection). The model treats this as a failure to transform.

The model is focused on generation: adding expressions to C's representation to produce D'. Thus, visual inference fails when a spatial relation cannot be added to C or reversed in C, but it does *not* fail when a spatial relation cannot be removed from C. For example, in Figure 1B, the A/B differences include removing a **rightOf** relation. There is no such relation in C to be removed. Visual inference succeeds here, whereas it fails in 1D, where there is no **above** relation in C to be reversed. Thus, the model explains why 1D is a harder problem (compare the reaction times).

Geometric Analogy

Our new geometric analogy model solves problems via two strategies: visual inference and second-order comparison. For visual inference, the model compares A and B to get Δ(A,B), the differences between them. It applies Δ(A,B) to C to get D', a representation of the answer image. It compares D' to each possible answer. If an answer is sufficiently similar, it selects that answer.

For second-order comparison, the model compares A and B to get Δ(A,B). For each answer *i*, it compares C and *i* to get Δ(C,i) and then compares Δ(C,i) to Δ(A,B) (again, using SME). If an answer's Δ(C,i) is sufficiently similar to Δ(A,B), it selects that answer.

In each case, an answer is sufficiently close if either a) SME detects no differences; or b) the SME similarity score lies above a *similarity threshold*. We use a similarity threshold of 0.8 (where 1.0 is a perfect match). However, a sensitivity analysis shows that our results would be the same for values ranging from .67 to .87. If multiple answers tie for the best score, this is treated as a failure.

Note that when SME compares Δ's for second-order comparison, it is possible to find a perfect match even for non-identical Δ's. SME supports tiered identicality (Falkenhainer, 1990), where non-identical predicates can align when they are members of a common category. For example, in Figure 1D, Δ(A,B) and Δ(C,3) each involve reversal of a positional relation (**above** and **rightOf**). Thus, Figure 1D is not solvable by visual inference, but it is easily solvable by second-order comparison.

Strategic Shifts The model first attempts to solve a problem via visual inference. This can fail in two ways: either the inference operation may fail (Figures 1D, 1E), or D' may fail to match any of the answers. For example, in Figure 1F, the A/B differences show the inner shape being removed. The model applies these differences to C to infer an image with a large circle, which matches none of the answers.

If visual inference fails, the model reverts to second-order comparison. When the model utilizes this strategy, it must make two other strategic decisions: the comparison mode when comparing A to B, and the comparison mode when comparing C to each answer *i*. The comparison modes are:

A) Normal: Images are compared as described above.

B) Reflection: Instead of preferring identity during shape comparison, the model prefers reflection. In Figure 1E, the C/3 comparison will find a y-axis reflection between the 'B' shapes, instead of treating them as identical.

C) Rotation: Instead of preferring identity during shape comparison, the model prefers rotation.

D) Alternate: The model looks for an alternate mapping between the images. In Figure 1F, an alternate A/B mapping aligns the small triangle in A with the large triangle in B.

The model only considers an alternate mapping when SME finds more than one mapping between the images. It only considers the Reflection/Rotation modes when the images each contain a single object, allowing the model to focus on different ways of comparing that one object.

The model independently varies the mapping mode for A/B and C/*i* comparisons, beginning with Normal for each. It terminates when it identifies a sufficient answer. If no such answer is found, it picks the highest-scoring answer.

Experiment

We evaluated our model on 20 geometric analogy problems from Evans (1968). We recreated each problem in PowerPoint and then imported the problems into CogSketch. This required us to manually segment each problem into images (image A, image B, etc) and segment each image into objects (each object was drawn as a separate shape in PowerPoint). Beyond this, the model automatically segmented each object into edges and generated representations at the edge and object levels—this problem set did not contain any groups of objects.

Our prior behavioral study (Lovett et al., 2009b) provides data on human performance which our simulation models, so we summarize it next.

Behavioral Study

The Evans problems were shown to 34 adult participants. They were given a description of the geometric analogy task followed by two simple example problems (without feedback) before they saw the 20 problems. Both the ordering of the problems and the ordering of the five possible answers were randomized across participants.[1]

Before each problem, participants clicked on a fixation point in the screen's center to indicate readiness. After the problem was presented, participants clicked on the picture that best completed the analogy. Participants were instructed to be as quick as possible without sacrificing accuracy. The two measures of interest were the answer chosen and the reaction time, i.e., the time taken to solve the problem.

Results The results show a high degree of consistency across participants. All participants chose the same answer for 9 of the 20 problems, while over 90% chose the same answer for 7 additional problems. The greatest disagreement was on Figure 1F, where only 56% chose the same answer. Henceforth, we refer to the answer chosen by the majority as the *preferred answer*. In reporting and analyzing reaction times (including Figure 1), we consider only responses with the preferred answer, filtering out minority responses.

Simulation & Analysis

The model chose the preferred answer on all 20 problems. This indicates that our approach—qualitative representation, comparison via structure mapping, and visual inference—is sufficient for matching human performance on the task.

We next asked whether people take longer to solve problems where our model must make a strategy shift. We coded each problem for three factors: Alt-Strategy, Alt-Mapping, and Alt-Transform. Alt-Strategy indicates that our model reverts to second-order comparison to solve the problem. Alt-Mapping indicates that it uses the Alternate image mapping mode. Alt-Transform indicates that it uses the Reflection or Rotation mapping modes. We group these mapping modes together, as our model uses the same mechanism for computing both transformation types.

We also coded each problem for working memory load. Previous research has shown that geometric analogy problems get harder as either the number of elements or the number of transformations increases (Mulholland, Pellegrino, & Glaser, 1980; Bethell-Fox, Lohman, & Snow, 1984). Mulholland et al. found that this effect was non-linear: there was a higher cost when the numbers of both elements and transformations increased. They suggested this was because at some point the problem exceeds people's working memory capacity, requiring a shift in strategy.

We coded for working memory load by counting the number of elements in Δ(A,B), the differences between images A and B. This is a key representation for both visual inference and second-order comparison. Because Mulholland et al. found a non-linear effect of working memory, we discounted the first two elements. Thus, if Δ(A,B) was one or two, the WM Load was coded as zero.

We ran a linear regression to identify the effect of the above factors on human reaction times. Table 1 shows the results. Overall, this model achieves an R^2 of .95 (.93 adjusted), meaning it explains almost all the variance in human reaction times. The grayed cells indicate which factors made a significant contribution to the model ($p < .01$). The intercept of 6.4 indicates that the easiest problems took around 6.4 s, while the various factors increased the time to complete a problem.

Note that with correlations, extreme values can result in an overestimation of the explained variance (the R^2 value). In this case, participants took far longer to solve the two problems requiring the Alt-Mapping shift (e.g., Figure 1F). If we remove these data points and rerun the analysis, Alt-Mapping ceases to be a factor, and R^2 drops to .80 (.76 adjusted). Thus, even discounting these difficult problems, the regression explains most of the variance in performance.

Table 1. Linear model for human reaction times on geometric analogy (grayed cells are significant factors).

Intercept	WM Load	Alt-Strategy	Alt-Transform	Alt-Mapping
6.4 s	5.7 s	4.4 s	- 0.7 s	10.5 s

The only factor that did not contribute significantly was Alt-Transform. Alt-Transform refers to problems like Figure 1E, where the model must switch to a Reflection mode to identify a reflection between the identical 'B' shapes. The analysis suggests there is no increased cost for Alt-Transform problems. However, this does not mean such problems are easy; they are difficult in that the model must make the Alt-Strategy shift to solve them, changing to second-order comparison. Once this strategy shift has been made, there is no additional cost for the Alt-Transform shift.

Related Work

Evans' ANALOGY (1968) was the first computational model of analogy. A ground-breaking system, it solved the same 20 geometric analogy problems as our model using second-order comparison. However, its brute-force comparison processes do not align well with human cognition (see Lovett et al., 2009b for a discussion).

Our own previous model (Lovett et al., 2009b) also solved problems via second-order comparison. The present approach builds on that model by implementing visual inference as a complementary strategy. The previous model explained .56 of the human variance on the Evans problems,

[1] Due to experimenter error, some participants received the same random orderings. As many as five received one ordering, but on average only 1.5 received the same ordering. When we randomly selected one instance of each ordering, the participant number dropped to 22, and the pattern of results remained the same.

whereas the current model explains .95 of the variance. However, the previous analysis did not consider multiple factors or filter out reaction times for minority responses.

Several other approaches have utilized visual inference strategies, but these suffer from important limitations. Some (Schwering et al., 2009; O'Donoghue, Bohan, & Keane, 2006) use hand-coded symbolic inputs, rather than automatically generating representations. This means the models are unable to reason about quantitative spatial information, e.g., shape transformations. Others (Ragni, Schleipen, & Steffenhagen, 2007) are unclear on their comparison processes. Finally, because these models have not been systematically evaluated on a pre-existing problem set, it is unclear how well they match human performance.

Conclusions

We believe our model is the first to combine two established problem-solving strategies: visual inference and second-order comparison. Beyond utilizing both strategies, the model makes explicit claims about when people will abandon visual inference and fall back on second-order comparison. Our analysis shows that these claims help explain human reaction times on the 20 Evans problems: people take longer to solve problems where the model reverts to second-order comparison, and they take even longer when the model must find an alternate mapping.

Importantly, our two problem-solving strategies are not unique to geometric analogy. We recently (Lovett & Forbus, in prep) integrated these strategies into a new model of Raven's Progressive Matrices, a more complex task that is popularly used to evaluate general intelligence. As that model and the present model show, successful problem-solving requires flexibly moving between different comparison strategies. These models, along with our oddity task model (Lovett & Forbus, 2011a), also demonstrate the utility of structural alignment across qualitative representations. In the future, we plan to evaluate the generality of our approach on new problem-solving tasks.

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