Approximate Qualitative Temporal Reasoning

Thomas Bittner

Qualitative Reasoning Group, Department of Computer Science, Northwestern University
E-mail: bittner@cs.nwu.edu

We partition the time-line in different ways, for example, into minutes, hours, days, etc. When reasoning about relations between events and processes we often reason about their location within such partitions. For example, x happened yesterday and y happened today, consequently x and y are disjoint. Reasoning about these temporal granularities so far has focussed on temporal units (relations between minute, hour slots). I shall argue in this paper that in our representations and reasoning procedures we need into account that events and processes often lie skew to the cells of our partitions (For example, 'happened yesterday' does not mean that x started at 12 a. m. and ended 0 p. m.) This has the consequence that our descriptions of temporal location of events and processes are often approximate and rough in nature rather than exact and crisp. In this paper I describe representation and reasoning methods that take the approximate character of our descriptions and the resulting limits (granularity) of our knowledge explicitly into account.

Keywords: Approximate Reasoning, Qualitative Reasoning, Temporal Relations, Granularity, Ontology

1. Introduction

Formal systems that support reasoning about calendar units and clock units are called temporal granularities [3]. Such systems have been the subject of intensive research in recent years, e.g., [2,27,12]. These systems provide foundations for task and process management [19,13,18], for work on database systems [20], on (geographic) information systems [23], and they are relevant also in many other domains. Essentially, temporal granularities describe ways of partitioning the time-line and methods for reasoning about cells within the partitions which result. Examples of partitions are: the partition of the time-line into fifteen minute slots produced by your favorite calendar application, or the partition of the time-line created by the succession of update operations of some data-base system. Partitions of the time-line can be rough. Consider, for example, the partition with cells labeled 'before World War 2', 'during World War 2', 'after World War 2'. (In what follows I use the terms *calendar-partition*, *db-partition*, and *WW2-partition* in order to refer to these examples.)

A critical assumption underlying most formal systems dealing with temporal granularities is that the boundaries (beginnings and endings) of the events and processes we want to represent are made to coincide with the boundaries between the partition cells within the representation. For example, if we plan a meeting within the partition created by our calendar application, then the beginning and ending of the meeting must coincide with the beginning and ending of the available fifteen minute slots. I shall argue below that this assumption masks a deep-running problem, which cannot be resolved merely by choosing a finer resolution (e.g., five minute slots). Rather, one must give up the assumption that boundaries of events or processes need to coincide with the boundaries of partition cells[1]. This results in *approximate* rather than exact representations of the temporal locations of events and processes. In this paper I will present formal methods for the approximate representation of events and processes with respect to partitions of the time-line. I also present formal means to derive relations between events and processes captured in approximate representations.

The paper is structured as follows. It starts with a discussion of relationships between events and processes and partitions of the time-line. In Section 3 qualitative relations between temporal regions are defined. These relations are then generalized to relations between approximations of events and processes with respect to an underlying partition in Sections 4-8. In Section 9 the relationships between the notions of approximation and granularity will be discussed as well as limits and potential applications of the proposed formalism.

2. Reference and approximation

Partitions of the time-line are used both as frames of reference and as the basis for approximations. In order to understand the relationships between these two uses we need to understand the relationships between events and processes on the one hand and our partitions of the time-line on the other.

2.1. Temporal granularities

Temporal granularities describe ways of partitioning the time-line and ways in which such partitions can be used as frames of reference. Examples of partitions dealing

[1] Similar points were made, for example, in [17] and [11] from a different perspective.

as frames of reference are the calendar-partition, the db-partition, and the WW2-partition mentioned above. We use them in expressions like 'We will meet on Monday morning at 8 a.m.' or 'The meeting will take one hour', or 'There are were several changes since the last update', or 'Berlin was the cultural center of Europe between the World Wars'.

The time-line, $(\mathcal{T}, \leq_{\mathcal{T}})$, itself is conceived as the totally ordered class of time points, i.e., the class of all possible boundaries of temporal regions, forming a directed one-dimensional space [33,25]. Usually it is assumed that the time-line is isomorphic to the real numbers, reflecting the intuition that boundaries can be located *anywhere*. Given the point-based view of the time-line, temporal intervals (topologically simple one-dimensional regions) can be represented by ordered pairs of boundary points [24]. Temporal intervals, in general, are such that each interval has proper parts. In those domains there are no atomic temporal intervals.

A partition of the time-line is a set of time intervals (cells) that do not overlap but sum to the whole time-line. In opposition to the time-line itself, partition cells are countable, and so human beings can name them. One way of naming them consists in assigning integer numbers to them in such a way that the ordering of the integers corresponds to the ordering of the underlying time-line: for example, your computer internally counts the seconds that have passed by since January 1st 1970 in order to give you the time. Partitions have different granularities and they can be hierarchically organized in virtue of the fact that some partitions include others. This occurs whenever the cells of a finer partition subsume the cells of partitions at a coarser level. For example, a fifteen minute slot in your calendar might be subsumed by three five minute slots. (For a detailed discussion see [41].) Every partition has minimal cells, i.e., cells that do not have (proper) subcells. For example minimal cells in your calendar may be five minute slots. This, however, does not mean that there do not exist events that are shorter than five minutes. This does only mean that your calendar 'does not care' about those events.

These intuitions about the time-line and its partitions were formalized in the granularity-model proposed in [3]. Moreover, this model takes special kinds of partitions into account, including partitions with holes or gaps, partitions determined by attributes such as working days, holidays, and so on, and partitions with cells of different sizes.

2.2. Occurrents and partitions

In this paper we consider events and processes such as 'Your meeting with your boss on Monday morning from 8 a.m. to 9 a.m. in her office', 'My childhood', 'World War 2', 'AAAI-2000'. Following [38] I call such spatio-temporal objects *occurrents*.

Every event or process, o, is located at a region of time, t, bounded by the beginning and the end of its existence, i.e., $t = \tau(o)$. Occurrents have temporal parts, which are occurrents themselves, and which are located at parts of the temporal regions at which their wholes are located.

Consider now occurrents and their temporal location with respect to partitions of the time-line. This relationship can be described in terms of relations between the exact temporal region of a given occurrent, $\tau(o)$, and the cells of the corresponding partition. For example, we can describe the temporal location of your meeting with your boss, relative to the above-mentioned partition of the time-line into fifteen minute slots, by saying that the temporal location of this occurrent, $\tau(o)$, is identical to the sum of the four consecutive cells between 8 a.m. and 9 a.m.

Consider our three example partitions: calendar-partition, db-partition, and WW2-partition. For each of these partitions we have a number of occurrents whose temporal locations, $\tau(o)$, can be exactly represented with respect to one of these partitions (As in the case of your meeting with your boss). Exact representation in this context means that the boundaries of these events coincide with boundaries of corresponding partition cells. More precisely, we can say that the temporal location, $\tau(o)$, of such occurrents is identical to some mereological sum of partition cells. The majority of occurrents, however, cannot be represented exactly with respect to partitions in this way. This is because their beginnings and endings do not coincide with the beginnings and endings of partition cells. Consider, for example, the location of the beginning of the German carnival season which occurs every year on November 11th at 11 o'clock and 11 minutes. This boundary lies skew to the boundaries of the time-slots of your calendar divided into 15-minute slots that start at each full hour. Consequently, the location of the occurrent 'Carnival season 2001' cannot be represented exactly with respect to the partitions of such calendars. One can easily see that most occurrents and most partitions stand in this kind of relationship. This is because most events and processes occur *independently* of our partitioning activity. (This holds, too, of most meetings with your boss.)

We have two ways of dealing with this issue: (1) Whenever we want to use a partition as a frame of reference we can construct an ad hoc partition of such a sort that the occurrents we refer to are located exactly at some corresponding sums of partition cells. This can be achieved, for example, by choosing partitions which are such that the occurrents of interest are parts of the partition, e.g., 'before the occurrent of interest', 'during the occurrent of interest', 'after the occurrent of interest', etc [2]. Another way

[2] More complex forms of partitions of this sort are often used in Geographic Information Systems. Parti-

is to refine partitions until the occurrents of interest can be represented exactly, e.g., by switching from hours to minutes, from seconds to nano-seconds, and so on. Or (2) we use an approximation theoretic approach and describe temporal location and extension approximately rather than exactly. For example, we say that the German carnival season begins at some time between 11 a.m. and 11.15 a.m., i.e., that it occupies only a part of the corresponding time slot.

Obviously (1) is preferable. Unfortunately it is not always possible to construct or to use partitions in that sense: (a) An important advantage of the use of familiar calendar-like partitions as frames of reference is that such partitions *do not change* and that they can be re-used in different contexts involving in independent events and processes which may need to be synchronized. Frames of reference are, by definition, relatively stable over time. Constructing partitions on the fly as occurrents occur is thus inappropriate. (b) It is often not possible to refine partitions as needed; our measuring instruments have only a finite resolution. (c) Often we do not *know* when certain events exactly occur; for example, I know that my boss came to talk to me during my lunch break, but I do not know exactly when she arrived or when she left.

Assuming the relative stability of the sorts of partitions of the time-line we use as frames of reference, we can distinguish two different classes of occurrents according to the ways they behave with respect to such stable partitions: bona fide occurrents on the one hand, and fiat occurrents, on the other. Consider again your meeting with your boss. It was scheduled for Monday morning and had a corresponding, neat entry in your calendar. The *planned* meeting starts and ends in exactly the way in which it is entered in your calendar. The *actual* meeting, on the other hand, starts at a time which depends on when people actually show up and when the boss decides to end it. It is very important to notice that we have *two distinct* occurrents here: (1) the occurrent 'Planned meeting with the boss' that is at home in the realm of calendars and scheduling applications, and (2) the actual meeting involving actual people and their activities of drinking coffee, standing about, rolling their eyes, etc.

The planned meeting is a *fiat occurrent*. This means that it is defined by its fiat boundaries, which are the result of human conceptualization [39,40]. Human beings have complete control over the temporal location of such fiat occurrents; they can, provided they act early enough, postpone and cancel them at will. In particular such fiat occurrents can be scheduled in such a way that their boundaries coincide with the bound-

tions are refined in stepwise fashion by adding more and more occurrents. If two objects overlap, then their overlap forms a separate cell. This process of constructing a partition is called spatial enforcement [31].

aries of cells of some partition of the time-line [3]: the planned meeting starts exactly at 8 a.m. and ends exactly at 9 a.m.

The actual meeting, in contrast, is a *bona fide occurrent*. This means that it exists to a significant degree independently of human conceptualization [39,40]. This occurrent involves people and their actions. You can get fired during a meeting, you can arrive too late for a meeting, or you may have to leave it early.

Notice that the distinction between bona fide and fiat occurrents does not imply that all fiats coincide with the boundaries of our partitions and that all bona fide occurrents lie skew to them. Often bona fide occurrents are themselves cells of our partitions (for example in case of the WW2-partition). On the other hand there are fiat occurrents that lie skew to the boundaries of our partitions, for example, the fiat occurrent 'the second five minutes of the planned meeting with the boss' lies skew to the boundaries of your calendar partition which consists of fifteen minute slots. The point is that in the fiat domain we often have the freedom to place occurrents nicely since we are in charge of creating and placing them (as in the case of a planned meeting). We often can adjust the fiat occurrents to our reference partitions. On the other hand we often adjust or create our partitions with respect to bona fide occurrents if we are able to do so (as in the case of the WW2-partition). Otherwise we represent bona fide occurrents approximately by describing their relations (e.g., full overlap or partial overlap) to the cells of some reference partition.

Going deeper into the theory of bona fide and fiat occurrents goes beyond the scope of this paper. See [39,40][4] for details. The important point however is that the two kinds of occurrents behave differently with respect to those partitions of the time-line which we human beings construct. Fiat occurrents can be *scheduled* in such a way that they can be represented *exactly*, i.e., their boundaries can be placed in such a way that they coincide with the cell boundaries of the appropriate partition of the time-line. (Think of the way you plan your meetings.) Bona fide occurrents, on the other hand, behave differently, since they are independent of our human conceptualization. They are affected by climate and the mood of the participants, by the punctuality of the transport

[3] Fiat occurrents can of course be scheduled also in such a way that their boundaries do not coincide with regular partitions of the time-line as in the case of the carnival season or in the case of races which begin with the shooting of the starting pistole.

[4] Smith's theory of bona fide and fiat objects shows that the formerly contrary positions of realism and idealism can be combined on the basis of the view that many entities in reality enjoy independent existence but have boundaries which depend on our human demarcations. In this context his work represents a continuation of that of Brentano, Husserl, and Ingarden.

system, and by other, internal and external factors not under our control. That is why they usually do not exactly fit into our partitions.

In order to establish a relationship between bona fide occurrents and calendar partitions we need to deal with *approximations* of the temporal locations of bona fide occurrents, rather than their exact locations (which may in any case not be capable of being exactly determined). This provides the motivation for extending the granularity model proposed by [3] to take approximations into account.

2.3. Approximation

In this section I show that the view of partitions as frames of reference is a special case of approximation. In the remainder of the paper I then concentrate on the notion of approximation.

2.3.1. Rough approximations

The notion of approximation is based on the definition of an indiscernibility relation with respect to a partition of some domain. Rough set theory [35] provides the formal foundations. Rough approximation is based on approximating subsets of a set when the set is equipped with an equivalence relation that partitions the set as a whole into equivalence classes. Given a set X with a partition $G = \{a_i \mid i \in \mathcal{I}\}$, an arbitrary subset $b \subseteq X$ can be approximated by a function $\varphi_b : \mathcal{I} \rightarrow \{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$. The value of $\varphi_b(i)$ is defined to be fo if $a_i \subseteq b$, it is no if $a_i \cap b = \varnothing$, and otherwise the value is po. Intuitively fo, po and no are interpreted as 'full overlap', 'partial overlap' and 'no overlap'.

Using the notion of approximation we then can define an equivalence (indiscernibility) relation in the domain of subsets of X: $a, b \subseteq X : a \sim b \equiv \varphi_a = \varphi_b$. The approximation of a subset b is *exact* if and only if b is identical to a union of elements $a_i \in G$ of partition elements. Formally we can write:

$$\mathit{exact}(\varphi_b) \equiv \forall i : \varphi_b(i) \in \{\mathsf{fo}, \mathsf{no}\}$$

Otherwise the approximation is called *rough*. In the exact case the equivalence class $[b]$ contains a single element.

There is obviously a relation between the granularity (resolution) of a given partition and the number of sets we can represent exactly and the 'roughness' of the approximation of the other sets. This will be discussed in Section 9.

2.3.2. Approximation and temporal granularities

In [9] and [43] the technique of approximation of subsets of a set was applied to regions of space by interpreting the three values fo, po, and no as different degree of spatial overlap. On the spatial interpretation these values measure the extent to which a region b overlaps with the cells of the partition of a spatial region X. In this paper I consider the approximation of regions in the temporal domain with the obvious interpretation of fo, po, and no in a one-dimensional space.

Partitions of the time-line were hithero used mainly to provide a frame of reference with respect to which temporal location and extension were described. From the point of view of approximations the time-granularities used in [3] are approximations that are *exact* in the sense that occurrents have a temporal location that is identical to the mereological sum of corresponding partition cells, i.e., $\forall i : \varphi_b(i) \in \{\mathsf{fo}, \mathsf{no}\}$.

Given the exactness of the approximations, φ_a and φ_b, one can easily derive qualitative relations between the (non-empty) intervals a and b (assuming $\mathsf{fo} > \mathsf{no}$):

$$\begin{aligned}
\forall i : \varphi_a(i) = \varphi_b(i) &\text{ implies } \mathit{equal}(a,b) \\
\forall i : \varphi_b(i) = \mathsf{fo} \rightarrow \varphi_a(i) = \mathsf{fo} \text{ and } \exists i : \varphi_a(i) \neq \varphi_b(i) &\text{ implies } \mathit{contains}(a,b) \\
\forall i : \varphi_b(i) = \mathsf{fo} \rightarrow \varphi_a(i) = \mathsf{fo} \text{ and } \exists i : \varphi_a(i) \neq \varphi_b(i) &\text{ implies } \mathit{containedBy}(b,a) \\
\forall i : (\varphi_a(i) \neq \mathsf{no} \text{ or } \varphi_b(i) \neq \mathsf{no}) \ \rightarrow \varphi_a(i) \neq \varphi_b(i) &\text{ implies } \mathit{disjoint}(a,b) \\
\exists i, j, k : \varphi_a(i) = \varphi_b(i) = \mathsf{fo} \text{ and} & \\
\varphi_a(j) > \varphi_b(j) \text{ and } \varphi_a(k) < \varphi_b(k) &\text{ implies } \mathit{partially\ overlap}(a,b)
\end{aligned}$$

These follow immediately from the definitions. As already mentioned, things become more complicated when multiple frames of reference are involved [3]. In this paper I concentrate on a single frame of reference and I consider the derivation of relations between temporal regions given their *rough* approximation.

3. Qualitative relations between regions

In this section I define qualitative relations between one-dimensional regions. These definitions are based exclusively on the meet operation and provide the basis for the definition of corresponding relations between approximations later on in this paper. The meet operation is interpreted as the overlap of regions. Two regions have a non-empty meet $(x \wedge y \neq \perp)$ if and only if they share parts (or interior points in point-set topological terms). It is important to stress that the same or similar relations have been defined also elsewhere, e.g., [1,21,36,24,15]. I use the notation of RCC from the region connection calculus in order to stress the correspondence between the relations defined in this paper and relations defined by Cohn and his co-workers. Correspondence in this context means that I am talking about regular regions that satisfy the RCC-axioms [36] and that similar relations could be defined or have been defined in terms of RCC, e.g., [36,16,15]. I use sub- and superscripts (e.g., RCC_1^9) where the superscript refers to the number of relations in the denoted set and the subscript refers to the dimension of the regions and the embedding space.

The contribution of this section is the specific style of definition that allows us to generalize relations between temporal regions to relations between approximations of temporal regions. This methodology was originally proposed in [8] for the definition of relations between approximations of spatial regions. The usage of the outcome of the meet operation ($x \wedge y = \bot$, $x \wedge y = x$, $x \wedge y = y$) in order to define relations is somewhat similar to the technique Egenhofer used in his intersection matrices ($x \wedge y = \bot$ and $x \wedge y \neq \bot$), e.g., [21].

3.1. Boundary insensitive relations

3.1.1. RCC5 relations

Given two regions x and y, each boundary insensitive topological relation (RCC5 relation) between them can be determined by considering the following triple of Boolean truth values [8]:

$$(x \wedge y \neq \bot,\ x \wedge y = x,\ x \wedge y = y).$$

The correspondence between such triples and boundary insensitive relations between regions on an undirected line is given in the table in Figure 1 [8].

$x \wedge y \neq \bot$	$x \wedge y = x$	$x \wedge y = y$	RCC5
F	F	F	DR
T	F	F	PO
T	T	F	PP
T	F	T	PPi
T	T	T	EQ

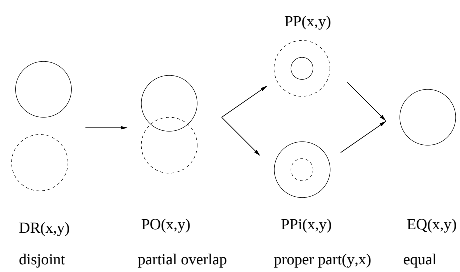


Figure 1. Definition of the RCC5 relations and the corresponding RCC5 lattice. (The bold boundary encloses x and the dashed boundary encloses y.)

As a set the relations defined in the table in Figure 1 are jointly exhaustive and pair-wise disjoint (JEPD). The set of triples is partially ordered by setting $(a_1, a_2, a_3) \leq (b_1, b_2, b_3)$ iff $a_i \leq b_i$ for $i = 1, 2, 3$, where the Boolean values are ordered by $\mathsf{F} < \mathsf{T}$. The resulting ordering is indicated by the arrows in the right part of Figure 1. [8] call this graph the RCC5 lattice in order to distinguish it from the conceptual neighborhood graph given in [26].

3.1.2. RCC_1^9 *relations between intervals*

Intervals are topologically maximally connected one-dimensional regions. The boundary insensitive topological relation between intervals x and y on a *directed* line (RCC_1^9 relations) can be determined by considering the triple:

$$(x \wedge y \not\sim \perp,\ x \wedge y \sim x,\ x \wedge y \sim y)$$

where the evaluation of each component yields a value belonging to the set $\{\mathsf{FLO}, \mathsf{FLI}, \mathsf{T}, \mathsf{FRI}, \mathsf{FRO}\}$ as follows:

$$x \wedge y \not\sim \perp = \begin{cases} \mathsf{FLO} \text{ if } x \wedge y = \perp \text{ and } x \ll y \\ \mathsf{FRO} \text{ if } x \wedge y = \perp \text{ and } x \gg y \\ \mathsf{T} \quad\ \text{ if } x \wedge y \neq \perp \end{cases}$$

and

$$x \wedge y \sim x = \begin{cases} \mathsf{FLO} \text{ if } x \wedge y \neq x \text{ and } x \wedge y \neq y \text{ and } x \ll y \\ \mathsf{FRO} \text{ if } x \wedge y \neq x \text{ and } x \wedge y \neq y \text{ and } x \gg y \\ \mathsf{FLI} \quad \text{if } x \wedge y \neq x \text{ and } x \wedge y = y \text{ and } x \ll y \\ \mathsf{FRI} \quad \text{if } x \wedge y \neq x \text{ and } x \wedge y = y \text{ and } x \gg y \\ \mathsf{T} \quad\ \ \text{if } x \wedge y = x \end{cases}$$

and

$$x \wedge y \sim y = \begin{cases} \mathsf{FLO} \text{ if } x \wedge y \neq y \text{ and } x \wedge y \neq x \text{ and } x \ll y \\ \mathsf{FRO} \text{ if } x \wedge y \neq y \text{ and } x \wedge y \neq x \text{ and } x \gg y \\ \mathsf{FLI} \quad \text{if } x \wedge y \neq y \text{ and } x \wedge y = x \text{ and } y \gg x \\ \mathsf{FRI} \quad \text{if } x \wedge y \neq y \text{ and } x \wedge y = x \text{ and } y \ll x \\ \mathsf{T} \quad\ \ \text{if } x \wedge y = y \end{cases}$$

with

$$x \ll y = \begin{cases} T \text{ if } L(x) \wedge L(y) = L(x) \text{ and } L(x) \wedge L(y) \neq L(y) \\ F \text{ otherwise} \end{cases}$$

$$x \gg y = \begin{cases} T \text{ if } R(x) \wedge R(y) = R(x) \text{ and } R(x) \wedge R(y) \neq R(y) \\ F \text{ otherwise} \end{cases}$$

$L(x)$ ($R(y)$) is the one-dimensional region occupying the whole line to the left (right)[5] of x. The intuition behind $x \wedge y \sim x = \mathsf{FLO}$ ($x \wedge y \sim x = \mathsf{FRO}$) is that $x \wedge y = x$ is false because there are parts of x sticking out to the left (right) of y and because y is not a part of x. The intuition behind $x \wedge y \sim y = \mathsf{FLI}$ ($x \wedge y \sim y = \mathsf{FRI}$) is that $x \wedge y = y$ is false because there are parts of y sticking out to the right (left) x and because x is a part of y.

The triples formally describe jointly exhaustive relations under the assumption that x and y are intervals in a one-dimensional directed space. The correspondence between the triples and the boundary insensitive relations between intervals is given in the table below. Possible geometric interpretations of the defined relations are given in Figure 2.

$x \wedge y \not\sim \bot$	$x \wedge y \sim x$	$x \wedge y \sim y$	RCC_1^9
FLO	FLO	FLO	DRL
FRO	FRO	FRO	DRR
T	FLO	FLO	POL
T	FRO	FRO	POR
T	T	FLI	PPL
T	T	FRI	PPR
T	FLI	T	PPiL
T	FRI	T	PPiR
T	T	T	EQ

For example: the relation $\mathsf{DRL}(x, y)$ holds if and only if x and y do not overlap and x is to the left of y; $\mathsf{POL}(x, y)$ holds if and only if x and y partly overlap and the non-overlapping parts of x are to the left of y; $\mathsf{PPL}(x, y)$ holds if and only if x is contained in y but x does not cover the rightmost parts of y; $\mathsf{PPiL}(x, y)$ holds if and only if y is a part of x and there are parts of x sticking out to the left of y; $\mathsf{PPR}(x, y)$ holds if and only if x is a part of y and x does not cover the leftmost parts of y; $\mathsf{PPiR}(x, y)$ holds if and only if y is a part of x and there are parts of x sticking out to the right of y.

In qualitative reasoning the aim is to define sets of jointly exhaustive and pair-wise disjoint (JEPD) relations. For JEPD sets of relations for arbitrary configurations of objects or regions one and only one relation holds to be true. The set of RCC_1^9 relations is jointly exhaustive for intervals, i.e., arbitrary configurations of intervals are covered. But consider the geometric interpretations of $\mathsf{PPL}(x, y)$ and $\mathsf{PPR}(x, y)$ in Figure 2. *Both* relations hold if x is a part of y and the boundaries of x and y do not intersect. We

[5] I use the spatial metaphor of a line extending from the left to the right rather than the terminology of a time-line extending from the past into the future in order to focus on the aspects of the time-line as a one-dimensional directed space.

have $x \wedge y = x$ and $x \wedge y \neq y$ and $x \ll y$ and $x \gg y$. Consequently the table above does not define pair-wise disjoint relations. Consider the two sets $\mathsf{ppl} = \{(x, y) \mid \mathsf{PPL}(x, y)\}$ and $\mathsf{ppr} = \{(x, y) \mid \mathsf{PPR}(x, y)\}$. Since PPL and PPR are not identical and not pair-wise disjoint we have $\mathsf{ppl} \neq \mathsf{ppr}$ and $\mathsf{ppl} \wedge \mathsf{ppr} \neq \bot$. The same holds for PPiL and PPiR.

It is important to stress that this property of the defined relations is intended and needed in order to generalize those relations to approximations. One could easily make these definitions JEPD by setting $x \wedge y \sim y = \mathsf{FRI}$ iff $x \wedge y \neq y$ and $x \wedge y = x$ and $y \ll x$ and $y \not\gg x$ and by setting $x \wedge y \sim x = \mathsf{FRI}$ iff $x \wedge y \neq x$ and $x \wedge y = y$ and $x \ll y$ and $x \not\gg y$. But this would destroy the symmetry of the relations with respect to EQ. This symmetry is important in our generalization below. The central concern of this paper is not the definition of JEPD relations between regions but rather the discussion of sets of relations that can be generalized to relations between approximations.

Assuming the ordering $\mathsf{FLO} < \mathsf{FLI} < \mathsf{T} < \mathsf{FRI} < \mathsf{FRO}$, and the ordering on triples as defined above, a lattice is formed, which has $(\mathsf{FLO}, \mathsf{FLO}, \mathsf{FLO})$ as minimal element and $(\mathsf{FRO}, \mathsf{FRO}, \mathsf{FRO})$ as maximal element. It is called the RCC_1^9 lattice and the ordering is indicated by the arrows in Figure 2.

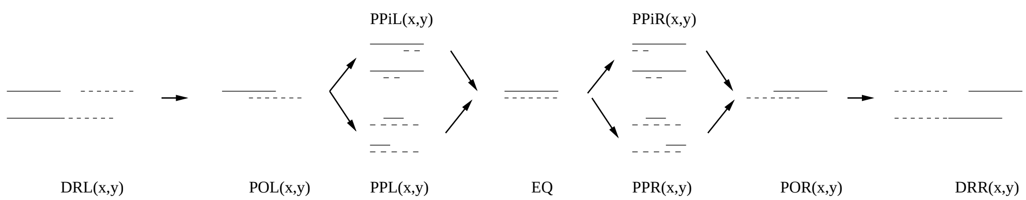


Figure 2. Possible geometric interpretations of the RCC_1^9 relations. For DRL, $\mathsf{PPLPPiL}$, PPR, PPiR, and DRR two example configurations are given. One with and one without boundary intersection. The solid lines signify the interval x and dashed lines signify the interval y.

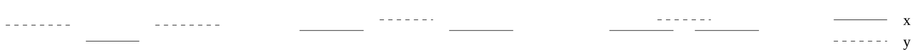
Consider Figure 2. For all relations except POL, POR, and EQ, two distinct geometric interpretations are given. These configurations differ regarding the emptiness or non-emptiness of the intersection of the boundaries of the regions but they cannot be distinguished within the current formal framework. It is the task of further refinement to distinguish those configurations. This will result in definitions that describe the thirteen Allen-relations [1] which are based exclusively on the meet operation between temporal regions (See Section 3.2.2.).

3.1.3. RCC_1^9 *relations and complex regions*

Often temporal regions are not intervals, i.e., they consist of multiple disconnected parts. Imagine, for example, that John grew up in London. When he was 20 years old he moved to New York and returned 10 years later to London where he lived until he died. The temporal region during which John lived in London has two disconnected piece. Imagine Mary moved to London one year after John left and left London for good one year before John came back. There are then parts of 'John's living in London' before (to the left of) and after (to the right of) 'Mary's living in London'. Consider configuration (a) in Figure 3. None of the RCC_1^9 relations defined in the previous section applies to this configuration. Both regions are disjoint but there are no parts of x 'sticking out' to the left and right of y, i.e., $x \not\lll y$ and $x \not\ggg y$. RCC_1^9 relations only incompletely describe configurations (b) and (c) in Figure 3. The RCC_1^9 relations defined above do not generalize to complex one-dimensional regions.

In the domain of complex one-dimensional regions the RCC_1^9 relations defined in the previous section is the specification of relations between the *convex hull* of complex regions. In our example above we would have $\mathsf{PPiL}(JohnInLondon, MaryInLondon)$ and $\mathsf{PPiR}(JohnInLondon, MaryInLondon)$. This interpretation is not necessarily satisfactory if our aim is towards the definition of JEPD relations between arbitrary one-dimensional regions. In order to obtain JEPD relations a further refinement of the RCC_1^9 relations is needed.

The difficult problem of providing a formal theory of complex intervals goes beyond the scope of this paper. For definitions of relations between complex intervals see [29,30] and [32]. These definitions are much more complex than those sketched above. The focus of this paper is the discussion of the generalization of relations between regions to relations between approximations. Consequently I consider relations between temporal regions without explicitly distinguishing between simple and complex intervals and to this end I consider convex hulls of intervals where necessary. However I shall explicitly mark cases that only hold for simple intervals or only hold for complex intervals.



(a) R(x,y) = ? (b) DRL(x,y) and DRR(x,y) (c) POL(x,y) and POR(x,y)

Figure 3. Configurations of complex regions that are not or only incomplete characterized by the RCC_1^9 relations.

3.2. *Boundary sensitive relations*

3.2.1. *RCC8 relations*

In order to describe boundary sensitive relations between regions x and y we use a triple, where the three entries may take one of three truth-values rather than the two Boolean ones [8]. The scheme has the form:

$$(x \wedge y \not\equiv \bot, x \wedge y \equiv x, x \wedge y \equiv y)$$

where

$$x \wedge y \not\equiv \bot = \begin{cases} \mathsf{T} & \text{if the interiors of } x \text{ and } y \text{ overlap, i.e.,} x \wedge y \neq \bot \\ \mathsf{M} & \text{if only the boundaries } x \text{ and } y \text{ overlap, i.e.,} x \wedge y = \bot \text{ and } \delta x \wedge \delta y \neq \bot \\ \mathsf{F} & \text{if there is no overlap between } x \text{ and } y\text{, i.e.,} x \wedge y = \bot \text{ and } \delta x \wedge \delta y = \bot \end{cases}$$

and where[6]

$$x \wedge y \equiv x = \begin{cases} \mathsf{T} & \text{if either } x \text{ and } y \text{ are identical or } x \text{ is contained in the interior of } y\text{, i.e.,} \\ & x = y \text{ or } (x \wedge y = x \text{ and } \delta x \wedge \delta y = \bot) \\ \mathsf{M} & \text{if } x \text{ is contained in } y \text{ and the boundaries overlap} \\ & i.e., x \wedge y = x \text{ and } x \wedge y \neq y \text{ and } \delta x \wedge \delta y \neq \bot \\ \mathsf{F} & \text{if } x \text{ is not contained within } y, i.e., x \wedge y \neq x \end{cases}$$

and where

$$x \wedge y \approx y = \begin{cases} \mathsf{T} & \text{if } x = y \text{ or } (x \wedge y = y \text{ and } \delta x \wedge \delta y = \bot) \\ \mathsf{M} & \text{if } x \wedge y = y \text{ and } x \wedge y \neq x \text{ and } \delta x \wedge \delta y \neq \bot \\ \mathsf{F} & \text{if } x \wedge y \neq y \end{cases}$$

The meaning of $x \wedge y \neq \bot = \mathsf{T}$ is that the intersection of the interior of x and y is non-empty and the meaning of $\delta x \wedge \delta y \neq \bot = \mathsf{T}$ is that the meet of the boundaries of x and y is non-empty. The correspondence between the triples defined above and boundary sensitive topological relations is given in the table of Figure 4 [8]. [8] define $\mathsf{F} < \mathsf{M} < \mathsf{T}$, assume the ordering between triples discussed above, and call the corresponding Hasse diagram (right part of Figure 4) an RCC8 *lattice.*

Consider the definition of the relation $\mathsf{DC}(x, y)$. By Table 4 we have $x \wedge y \not\approx \bot = \mathsf{F}$, $x \wedge y \approx x = \mathsf{F}$, and $x \wedge y \approx y = \mathsf{F}$. Consequently, neither the interiors nor the boundaries of x and y overlap, i.e., $x \wedge y = \bot$ and $\delta x \wedge \delta y = \bot$, and the regions x and y are disconnected. In the case of $\mathsf{EC}(x, y)$ we have $x \wedge y \not\approx \bot = \mathsf{M}$, $x \wedge y \approx x = \mathsf{F}$, and $x \wedge y \approx y = \mathsf{F}$. Consequently, the interiors of x and y do not overlap but the boundaries

[6] In the domain of regions $x = y$ is equivalent to ($x \wedge y = x$ and $x \wedge y = y$) [10].

$x \wedge y \not\approx \perp$	$x \wedge y \approx x$	$x \wedge y \approx y$	RCC8
F	F	F	DC
M	F	F	EC
T	F	F	PO
T	M	F	TPP
T	T	F	NTPP
T	F	M	TPPi
T	F	T	NTPPi
T	T	T	EQ

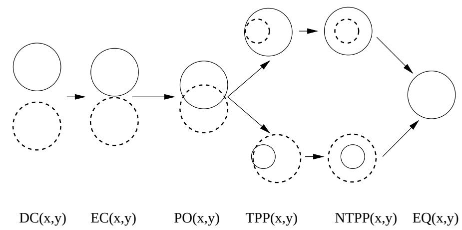


Figure 4. Definition of the RCC8 relations and the corresponding RCC8 lattice

do, i.e., $x \wedge y = \perp$ and $\delta x \wedge \delta y \neq \perp$, and the regions x and y are externally connected. In the case of $\mathsf{NTPP}(x, y)$ we have $x \wedge y \not\approx \perp = \mathsf{T}$, $x \wedge y \approx x = \mathsf{T}$ and $x \wedge y \approx y = \mathsf{F}$. Consequently, x is completely contained in the interior of y: $x \wedge y \neq \perp$, $x \wedge y = x$ and since $x \wedge y \neq y$ we have $\delta x \wedge \delta y = \perp$, i.e., x is a non-tangential proper part of y. In the case of $\mathsf{EQ}(x, y)$ we have $x \wedge y \not\approx \perp = \mathsf{T}$, $x \wedge y \approx x = \mathsf{T}$ and $x \wedge y \approx y = \mathsf{T}$. Both regions are identical, i.e., $x \wedge y = x$ and $x \wedge y = y$.

This paper deals with the regions of a one-dimensional space and with the relations between them. In this context the meaning of $\delta x \wedge \delta y \neq \perp = \mathsf{T}$ is that the boundary points of the one-dimensional regions x and y coincide. Let B_x be the set of boundary points of x and B_y be the set of boundary points of y respectively[7]. We have $\delta x \wedge \delta y \neq \perp = \mathsf{T}$ if and only if $B_x \cap B_y \neq \emptyset$. In this context we assume that if two points coincide then they are identical.

In order to distinguish sets of relations between one-dimensional regions from relations between regions of higher dimension I use the notation RCC_1^8 rather than RCC8. Possible geometric interpretations of the RCC_1^8 relations are given in Figure 5.

For all relations except NTPP, NTPPi, and EQ in this figure, two distinct geometric interpretations are given. Consider the relations $\mathsf{DC}(x, y)$, $EC(x, y)$, and $\mathsf{PO}(x, y)$. These geometric configurations differ according to whether x is to the left of y or vice versa. In the case of $\mathsf{TPP}(x, y)$ and $\mathsf{TPPi}(x, y)$ we cannot distinguish whether the left or the right boundary points of x and y coincide. The configurations cannot be distinguished within the current formal framework. It is the task of further refinement to distinguish them.

[7] Notice that we are allowed to talk about sets of boundary points since in the domain of one-dimensional regions the identity of boundary points is well defined. Moreover the boundary points of a one-dimensional region are countable. In the domain of temporal intervals their number is two.

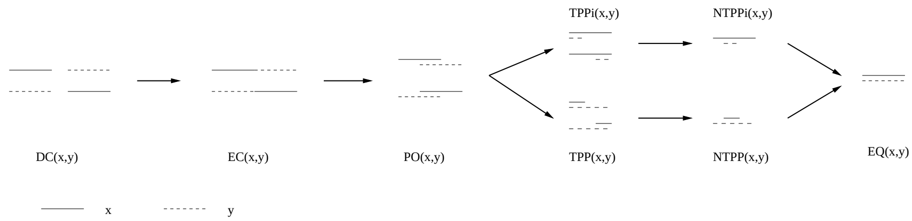


Figure 5. Geometric interpretations of RCC_1^8 relations between one-dimensional regions of a non-directed line.

3.2.2. RCC_1^{15} *relations between intervals*

Boundary insensitive relations between time intervals on a directed time-line do not distinguish emptiness and non-emptiness of intersection at boundary points. Boundary sensitive relations between regions on a non-directed line take boundary intersection into account but do not distinguish left and right. We now define boundary sensitive relations between *intervals* on a *directed* line by combining both approaches.

In order to describe boundary sensitive relations between intervals on a directed line (RCC_1^{15})[8] we define the relationship between intervals x and y by using a triple, where the three entries may take one of nine truth values: FLO, MLO, FLI, MLI, T, MLI, FLI, MLO, FLO. The scheme has the form

$$(x \wedge y \not\approx \bot, x \wedge y \approx x, x \wedge y \approx y)$$

where

$$x \wedge y \not\approx \bot = \begin{cases} \mathsf{T} & x \wedge y \not\equiv \bot = \mathsf{T} \\ \mathsf{MLO} & x \wedge y \not\equiv \bot = \mathsf{M} \text{ and } x \wedge y \not\sim \bot = \mathsf{FLO} \\ \mathsf{MRO} & x \wedge y \not\equiv \bot = \mathsf{M} \text{ and } x \wedge y \not\sim \bot = \mathsf{FRO} \\ \mathsf{FLO} & x \wedge y \not\equiv \bot = \mathsf{F} \text{ and } x \wedge y \not\sim \bot = \mathsf{FLO} \\ \mathsf{FRO} & x \wedge y \not\equiv \bot = \mathsf{F} \text{ and } x \wedge y \not\sim \bot = \mathsf{FRO} \end{cases}$$

[8] To be distinguished from RCC15 relations between concave regions of higher dimension [15].

and where

$$x \wedge y \approx x = \begin{cases} \mathsf{T} & x \wedge y \equiv x = \mathsf{T} \\ \mathsf{MLI} & x \wedge y \equiv x = \mathsf{M} \text{ and } x \wedge y \sim y = \mathsf{FLI} \\ \mathsf{MRI} & x \wedge y \equiv x = \mathsf{M} \text{ and } x \wedge y \sim y = \mathsf{FRI} \\ \mathsf{FLO} & x \wedge y \equiv x = \mathsf{F} \text{ and } x \wedge y \sim x = \mathsf{FLO} \\ \mathsf{FLI} & x \wedge y \equiv x = \mathsf{F} \text{ and } x \wedge y \sim x = \mathsf{FLI} \\ \mathsf{FRO} & x \wedge y \equiv x = \mathsf{F} \text{ and } x \wedge y \sim x = \mathsf{FRO} \\ \mathsf{FRI} & x \wedge y \equiv x = \mathsf{F} \text{ and } x \wedge y \sim x = \mathsf{FRI} \end{cases}$$

and where

$$x \wedge y \approx y = \begin{cases} \mathsf{T} & x \wedge y \equiv y = \mathsf{T} \\ \mathsf{MLI} & x \wedge y \equiv y = \mathsf{M} \text{ and } x \wedge y \sim x = \mathsf{FLI} \\ \mathsf{MRI} & x \wedge y \equiv y = \mathsf{M} \text{ and } x \wedge y \sim x = \mathsf{FRI} \\ \mathsf{FLO} & x \wedge y \equiv y = \mathsf{F} \text{ and } x \wedge y \sim y = \mathsf{FLO} \\ \mathsf{FLI} & x \wedge y \equiv y = \mathsf{F} \text{ and } x \wedge y \sim y = \mathsf{FLI} \\ \mathsf{FRO} & x \wedge y \equiv y = \mathsf{F} \text{ and } x \wedge y \sim y = \mathsf{FRO} \\ \mathsf{FRI} & x \wedge y \equiv y = \mathsf{F} \text{ and } x \wedge y \sim y = \mathsf{FRI} \end{cases}$$

The intuitions behind those definitions are the following[9]: $x \wedge y \not\approx \perp = \mathsf{FLO}$ means that: $x \wedge y \neq \perp$ is false because x is to the left of y and no boundary points of x and y coincide; $x \wedge y \not\approx \perp = \mathsf{MLO}$ means that: $x \wedge y \neq \perp$ is false because x is to the left of y and the boundary points of x and y do coincide; $x \wedge y \approx x = \mathsf{FLO}$ means that: $x \wedge y = x$ is false because of parts of x sticking out to the left of y and because no boundary points of x and y coincide; $x \wedge y \approx y = \mathsf{FLI}$ means that: $x \wedge y = y$ is false because of parts of y sticking out to the right x, because x is a part of y, and because no boundary points of x and y coincide; $x \wedge y \approx y = \mathsf{MLI}$ means that: $x \wedge y = y$ is false because of parts of y sticking out to the right x, and because x is a part of y and the boundary points of x and y do coincide. Definitions for $x \wedge y \not\approx \perp \in \{\mathsf{FLI}, \mathsf{MLI}, \mathsf{FRI}, \mathsf{MRI}\}$, $x \wedge y \approx x \in \{\mathsf{MLO}, \mathsf{MRO}\}$, and $x \wedge y \approx y \in \{\mathsf{MLO}, \mathsf{MRO}\}$ are not meaningful.

The correspondence between such triples, boundary sensitive topological relations between intervals on a directed line, and the 13 relations defined by [1] is given in Table 6. Possible geometric interpretations are given in Figure 7. Consider, for example, the definition of the following relations: $\mathsf{DCL}(x, y)$, i.e., before(x, y), holds if x and y do not overlap and do not share boundary points ($x \wedge y \not\equiv \perp = \mathsf{F}$) and x is to the left of y ($x \wedge y \not\sim \perp = \mathsf{FLO}$), and hence, $x \wedge y \not\approx \perp = \mathsf{FLO}$; $\mathsf{ECL}(x, y)$, i.e., meets(x, y), holds if

[9] I discuss here only the most important cases. Cases not explicitly discussed are similar.

x and y do not overlap but do share boundary points ($x \wedge y \not\equiv \bot = \mathsf{M}$) and x is to the left of y ($x \wedge y \not\sim \bot = \mathsf{FLO}$), and hence, $x \wedge y \not\equiv \bot = \mathsf{MLO}$; $\mathsf{TPPL}(x, y)$, i.e., starts(x, y), holds if x is a proper part of y and the two left boundary points of x and y coincide. We have $x \wedge y = x$, $x \wedge y \neq y$ and $\delta x \wedge \delta y \neq \bot$, and hence $x \wedge y \equiv x = \mathsf{M}$ and $x \wedge y \equiv y = \mathsf{F}$. Furthermore we have $x \wedge y \sim y = \mathsf{FLI}$, i.e., there are parts of y sticking out to the right of x. This is consistent with the coincidence of the left boundary points of x and y. Consequently we have $x \wedge y \not\approx \bot = \mathsf{T}$, $x \wedge y \approx x = \mathsf{MLI}$, and $x \wedge y \approx y = \mathsf{FLI}$, which defines the relation $\mathsf{TPPL}(x, y)$; $\mathsf{TPPiL}(x, y)$, i.e., starts$_i(x, y)$, holds if y is a proper part of x and the two left boundary points of x and y coincide. We have $x \wedge y = y$, $x \wedge y \neq x$ and $\delta x \wedge \delta y \neq \bot$, and hence $x \wedge y \equiv y = \mathsf{M}$ and $x \wedge y \equiv x = \mathsf{F}$. Furthermore we have $x \wedge y \sim x = \mathsf{FLI}$, i.e., there are parts of x sticking out to the right of y. This is consistent with the coincidence of the left boundary points of x and y. Consequently we have $x \wedge y \not\approx \bot = \mathsf{T}$, $x \wedge y \approx y = \mathsf{MLI}$, and $x \wedge y \approx x = \mathsf{FLI}$, which defines the relation $\mathsf{TPPiL}(x, y)$;

Consider the relations $\mathsf{NTPPL}(x, y)$ and $\mathsf{NTPPR}(x, y)$, both corresponding to during(x, y). The intuition is that x is a proper part of y and there is no boundary intersection. This is consistent with parts of y sticking out to the left of x and parts of y sticking out to the right of x. Consequently, as a set the RCC_1^{15} relations are not JEPD. Consider the two sets $\mathsf{ntppl} = \{(x, y) \mid \mathsf{NTPPL}(x, y)\}$ and $\mathsf{ntppr} = \{(x, y) \mid \mathsf{NTPPR}(x, y)\}$. We have $\mathsf{ntppl} = \mathsf{ppr}$. Consequently, the formal distinction between $\mathsf{NTPPL}(x, y)$ and $\mathsf{NTPPR}(x, y)$ does not correspond to distinctions between pairs of one-dimensional intervals on a directed line. The same holds for $\mathsf{NTPPiL}(x, y)$ and $\mathsf{NTPPiR}(x, y)$. We need those distinctions for formal reasons in the generalization to approximations.

We define $\mathsf{FLO} < \mathsf{MLO} < \mathsf{FLI} < \mathsf{MLI} < \mathsf{T} < \mathsf{MRI} < \mathsf{FRI} < \mathsf{MRO} < \mathsf{FRO}$ and call the corresponding Hasse diagram an RCC_1^{15} *lattice* to distinguish it from the conceptual neighborhood graph [24]. The ordering of the lower RCC_1^{15} relations is indicated by the arrows in Figure 7.

4. Approximations

In this section boundary insensitive and boundary sensitive approximations of one-dimensional regions with respect to an underlying partition are formally defined. Boundary sensitive approximations take the relationships of the boundary of the approximated region and the boundaries of partition cells into account. Boundary sensitive approximations are needed if we want to derive boundary sensitive relations from those approx-

$x \wedge y \not\approx \bot$	$x \wedge y \approx x$	$x \wedge y \approx y$	RCC_1^{15}	Allen
FLO	FLO	FLO	DCL	before
FRO	FRO	FRO	DCR	after
MLO	FLO	FLO	ECL	meets
MRO	FRO	FRO	ECR	meets$_i$
T	FLO	FLO	POL	overlaps
T	FRO	FRO	POR	overlaps$_i$
T	MLI	FLI	TPPL	starts
T	MRI	FRI	TPPR	finishes
T	T	FLI	NTPPL	during
T	T	FRI	NTPPR	during
T	FLI	MLI	TPPiL	starts$_i$
T	FRI	MRI	TPPiR	finishes$_i$
T	FLI	T	NTPPiL	during$_i$
T	FRI	T	NTPPiR	during$_i$
T	T	T	EQ	equal

Figure 6. Definition of RCC_1^{15} relations.

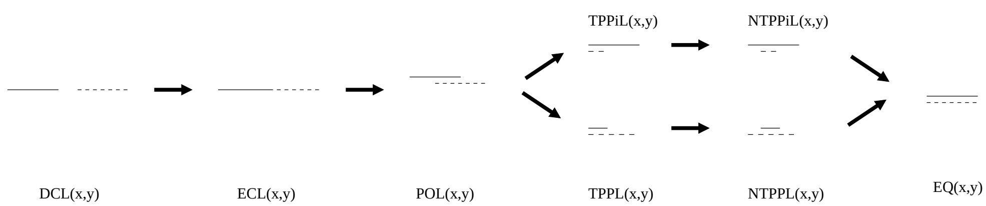


Figure 7. Geometric interpretations of the lower ($\leq$ EQ) RCC_1^{15} relations between connected intervals. The solid lines signify the interval x and dashed lines signify the interval y.

imations.

4.1. Approximating regions

4.1.1. Boundary insensitive approximation

Consider the set of regions, R, of a one-dimensional space. By imposing a partition, G, on R we can approximate elements of R by elements of Ω_3^G [9]. That is, we approximate regions in R by functions from G to the set $\Omega_3 = \{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$. The function which assigns to each region $r \in R$ its approximation will be denoted $\alpha_3 : R \to \Omega_3^G$. The value of $(\alpha_3 r)g$ is fo if r covers all of the cell g, it is po if r covers some but not all of the interior of g, and it is no if there is no overlap between r and g. [9] call the elements of Ω_3^G the overlap and containment sensitive approximations of regions $r \in R$ with respect to the underlying regional partition G.

4.1.2. Boundary sensitive approximation

Consider a one-dimensional non-directed space. We can further refine the approximation of regions R with respect to the partition G by taking boundary points shared by neighboring partition regions into account. That is, we approximate regions in R by functions from $G \times G$ to the set $\Omega_4 = \{\mathsf{fo}, \mathsf{bo}, \mathsf{nbo}, \mathsf{no}\}$. The function which assigns to each region $r \in R$ its boundary sensitive approximation will be denoted $\alpha_4 : R \to \Omega_4^{G \times G}$. The value of $(\alpha_4 r)(g_i, g_j)$ is fo if r covers all of the cell g_i, it is bo if r covers the boundary point, (g_i, g_j), shared by the cell g_i and g_j and some but not all of the interior of g_i, it is nbo if r does not cover the boundary point (g_i, g_j) and covers some but not all of the interior of g_i, and it is no if there is no overlap between r and g_i.

4.1.3. The semantics of approximate regions

Each approximate region $X_G \in \Omega_3^G$ ($X_G \in \Omega_4^{G \times G}$) stands for a set of precise regions, i.e., all those precise regions having the approximation X_G with respect to the partition G. This set will be denoted $[\![X_G]\!]^3$ ($[\![X_G]\!]^4$) and provides a semantics for approximate regions.

$$[\![X_G]\!]^3 = \{r \in R \mid \alpha_3 r = X_G\},\ [\![X_G]\!]^4 = \{r \in R \mid \alpha_4 r = X_G\}$$

Wherever the context is clear the subscripts and superscripts are omitted.

4.2. Approximate operations

The domain of regions is equipped with join and meet operations, $\vee$ and $\wedge$. [9] showed that these operations on regions can be approximated by pairs of greatest minimal and least maximal operations on approximations. In this paper I discuss the operations $\overline{\wedge}$ and $\underline{\wedge}$ on boundary insensitive approximations and boundary sensitive approximations. A detailed discussion can be found in [9].

4.2.1. Boundary insensitive operations

Firstly we define operations $\underline{\wedge}$ and $\overline{\wedge}$ on the set $\Omega_3 = \{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$.

$\underline{\wedge}$	no	po	fo
no	no	no	no
po	no	no	po
fo	no	po	fo

$\overline{\wedge}$	no	po	fo
no	no	no	no
po	no	po	po
fo	no	po	fo

These operations extend to elements of Ω_3^G (i.e. the set of functions from G to Ω_3) by $(X \underline{\wedge} Y)g = (Xg) \underline{\wedge} (Yg)$ and similarly for $\overline{\wedge}$. [9] showed that the outcome of the operations $\underline{\wedge}$ and $\overline{\wedge}$ on approximations X and Y constrains the possible outcome of the operation $x \wedge y$ for $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ in such a way that $X \underline{\wedge} Y \leq (\alpha_3\,(x \wedge y)) \leq X \overline{\wedge} Y$. The symbol $\leq$ designates a partial order between approximations defined by $X \leq Y$ if and only if for all $g \in G$ $(X\,g) \leq (Y\,g)$ with $\mathsf{no} < \mathsf{po} < \mathsf{fo}$.

4.2.2. Boundary sensitive operations

We define the operation $\overline{\wedge}$ on the set $\Omega_4 = \{\mathsf{fo}, \mathsf{bo}, \mathsf{nbo}, \mathsf{no}\}$ as shown in the left table below. This operation extends to elements of $\Omega_4^{G \times G}$ (i.e. the set of functions from $G \times G$ to Ω_4) by $(X \overline{\wedge} Y)(g_i, g_j) = (X(g_i, g_j)) \overline{\wedge} (Y(g_i, g_j))$.

$\overline{\wedge}$	no	nbo	bo	fo
no	no	no	no	no
nbo	no	nbo	nbo	nbo
bo	no	nbo	bo	bo
fo	no	nbo	bo	fo

$\underline{\wedge}^N$	no	nbo	bo	fo
no	no	no	no	no
nbo	no	$\gamma(N)$	$\gamma(N)$	nbo
bo	no	$\gamma(N)$	bo	bo
fo	no	nbo	bo	fo

The definition of the operation $\underline{\wedge}$ is slightly more complicated (right table above). In this case we need to take the approximation values referring to *both* boundary points (g_i, g_{i-1}) and (g_i, g_{i+1}) into account. Let

$$N(g_i) = \{((X\,(g_i, g_{i-1})), (Y\,(g_i, g_{i-1}))), ((X\,(g_i, g_{i+1})), (Y\,(g_i, g_{i+1})))\}$$

be the set of pairs of approximation values of X and Y with respect to g_i. We define the operation $X \underline{\wedge} Y$ as follows:

$$(X \,\underline{\wedge}\, Y)(g_i, g_{i+1}) = (X(g_i, g_{i+1}))\ (\underline{\wedge}^{N(g_i)})\ (Y(g_i, g_{i+1}))$$

where $(\underline{\wedge}^{N(g_i)})$ is defined as shown in the right table above and $\gamma(N)$ is defined as follows:

$$\gamma(N) = \begin{cases} \mathsf{no} & \text{if } (\mathsf{bo}, \mathsf{bo}) \notin N \\ \mathsf{nbo} & \text{if } (\mathsf{bo}, \mathsf{bo}) \in N \end{cases}.$$

This definition corresponds to the definition of operations on boundary sensitive approximations of two-dimensional regions in the plane discussed in [9]. Again, the outcome of the operations $\underline{\wedge}$ and $\overline{\wedge}$ on approximations X and Y constrains the possible outcome of the operation $x \wedge y$ for $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ in such a way that $X \underline{\wedge} Y \leq (\alpha_4 (x \wedge y)) \leq X \overline{\wedge} Y$, with $\leq$ defined as above with $\mathsf{no} < \mathsf{nbo} < \mathsf{bo} < \mathsf{fo}$.

5. Generalization of RCC5 relations

[8] showed that there are two approaches to generalizing RCC relations between precise regions to approximate ones: the semantic and the syntactic. In the semantic case one defines the RCC relationship between approximations X and Y to be the set of relationships which occur between any pair of precise regions approximated by X and Y, i.e., $\mathcal{SEM}(X,Y) = \{RCC(x,y) \mid x \in [\![X]\!]$ and $y \in [\![Y]\!]\}$. In the syntactic case one takes a formal definition of RCC in the precise case and replaces variables ranging over regions by variables ranging over approximations and the meet operation, $\wedge$, between regions by the corresponding operations $\underline{\wedge}$ and $\overline{\wedge}$ between approximations. This syntactic replacement yields pairs of relations between approximations: one relation defined using $\underline{\wedge}$ and another defined using $\overline{\wedge}$. These relations constrain the possible relations, $\rho(x,y)$, that can hold between $x \in [\![X]\!]$ and $y \in [\![Y]\!]$. In the remainder of this section I discuss syntactic and semantic generalizations for RCC5 . A similar approach is taken for RCC_1^8 , RCC_1^9 , and RCC_1^{15} in Sections 6-8.

5.1. Syntactic generalization

If X and Y are approximate regions (i.e. functions from G to Ω_3) we can consider the two triples of Boolean truth-values [8]:

$$(X \underline{\wedge} Y \neq \perp,\ X \underline{\wedge} Y = X,\ X \underline{\wedge} Y = Y), (X \overline{\wedge} Y \neq \perp,\ X \overline{\wedge} Y = X,\ X \overline{\wedge} Y = Y).$$

In the context of approximate regions, the bottom element, $\perp$, is the function from G to Ω_3 which takes the value no for every element of G. Each of the above triples defines an RCC5 relation, so the relation between X and Y can be measured by a pair of RCC5 relations. These relations will be denoted by $\underline{R}(X,Y)$ and $\overline{R}(X,Y)$. Let $\leq$ be the ordering of the RCC5 lattice then the following holds:

Theorem 1 [8]. The pairs $(\underline{R}(X,Y), \overline{R}(X,Y))$ which can occur are all those pairs (a,b) for which $a \leq b$ with the exception of $(\mathsf{PP}, \mathsf{EQ})$ and $(\mathsf{PPi}, \mathsf{EQ})$.

5.2. Correspondence of semantic and syntactic generalization

Let the syntactic generalization of RCC5 be defined by $\mathcal{SYN}(X,Y) = (\underline{R}(X,Y), \overline{R}(X,Y))$, where $\underline{R}$ and $\overline{R}$ are as defined above.

Theorem 2 [8]. For any approximate regions X and Y the syntactic and semantic generalization of RCC5 are equivalent in the sense that

$$\mathcal{SEM}(X,Y) = \{\rho \in \text{RCC5} \mid \underline{R}(X,Y) \leq \rho \leq \overline{R}(X,Y)\},$$

where RCC5 is the set $\{\mathsf{EQ}, \mathsf{PP}, \mathsf{PPi}, \mathsf{PO}, \mathsf{DR}\}$, and $\leq$ is the ordering in the RCC5 lattice.

6. Generalization of RCC_1^8 relations

6.1. Syntactic generalization

Let X and Y be boundary sensitive approximations of regions x and y. The generalized scheme has the form

$$((X \underline{\wedge} Y \not\equiv \bot, X \underline{\wedge} Y \equiv X, X \underline{\wedge} Y \equiv Y), (X \overline{\wedge} Y \not\equiv \bot, X \overline{\wedge} Y \equiv X, X \overline{\wedge} Y \equiv Y))$$

where

$$X \underline{\wedge} Y \not\equiv \bot = \begin{cases} \mathsf{T} & X \underline{\wedge} Y \neq \bot \\ \mathsf{M} & X \underline{\wedge} Y = \bot \text{ and } \delta X \wedge \delta Y \neq \bot \\ \mathsf{F} & X \underline{\wedge} Y = \bot \text{ and } \delta X \wedge \delta Y = \bot \end{cases}$$

and where

$$X \underline{\wedge} Y \equiv X = \begin{cases} \mathsf{T} & X \underline{\simeq} Y \text{ or } (X \underline{\not\simeq} Y \text{ and } X \underline{\wedge} Y = X \text{ and } \delta X \wedge \delta Y = \bot) \\ \mathsf{M} & X \underline{\not\simeq} Y \text{ and } X \underline{\wedge} Y = X \text{ and } \delta X \wedge \delta Y \neq \bot \\ \mathsf{F} & X \underline{\wedge} Y \neq X \end{cases}$$

and similarly for $X \underline{\wedge} Y \equiv Y$ (by commutativity of $\underline{\wedge}$), and for $X \overline{\wedge} Y \not\equiv \bot$, $X \overline{\wedge} Y \equiv X$, and $X \overline{\wedge} Y \equiv Y$ using $\overline{\wedge}$ instead of $\underline{\wedge}$ and $\overline{\simeq}$ instead of $\underline{\simeq}$. In this context the bottom element, $\bot$, is either the value no or the function from $G \times G$ to Ω_4 which takes the value no for every element of $G \times G$. The formula $X \underline{\simeq} Y$ is true if and only if $X \underline{\wedge} Y = X$ and $X \underline{\wedge} Y = Y$. The formula $X \overline{\simeq} Y$ is true if and only if $X \overline{\wedge} Y = X$ and $X \overline{\wedge} Y = Y$. These definitions correspond to the definition $x = y$ if and only if $x \wedge y = x$ and $x \wedge y = y$ in Section 3.2.1.

The formula $\delta X \wedge \delta Y \neq \bot$ is true if and only there are partition cells, $g_i, g_{i+1} \in G$, such that one of the following conditions holds: (1) $(X\,(g_i, g_{i+1})) \geq \mathsf{bo}$ and $(Y\,(g_i, g_{i+1})) \geq \mathsf{bo}$ and $(X\,(g_{i+1}, g_i)) < \mathsf{bo}$ and $(Y\,(g_{i+1}, g_i)) < \mathsf{bo}$; (2) $(X\,(g_i, g_{i+1})) < \mathsf{bo}$ and $(Y\,(g_i, g_{i+1})) < \mathsf{bo}$ and $(X\,(g_{i+1}, g_i)) \geq \mathsf{bo}$ and $(Y\,(g_{i+1}, g_i)) \geq \mathsf{bo}$; (3) $(X\,(g_i, g_{i+1})) \geq \mathsf{bo}$ and $(Y\,(g_i, g_{i+1})) < \mathsf{bo}$ and $(X\,(g_{i+1}, g_i)) < \mathsf{bo}$ and $(Y\,(g_{i+1}, g_i)) \geq \mathsf{bo}$; (4) $(X\,(g_i, g_{i+1})) < \mathsf{bo}$ and $(Y\,(g_i, g_{i+1})) \geq \mathsf{bo}$ and $(X\,(g_{i+1}, g_i)) \geq \mathsf{bo}$ and $(Y\,(g_{i+1}, g_i)) < \mathsf{bo}$.

Each of the above triples defines an RCC_1^8 relation, so the relation between X and Y can be measured by a pair of RCC_1^8 relations. These relations will be denoted by $\underline{R^8}(X, Y)$ and $\overline{R^8}(X, Y)$. Let X and Y be approximations of one-dimensional regions in a one-dimensional space and let $\leq$ be the ordering of the RCC_1^8 lattice. Then the following holds:

Theorem 3. The pairs $(\underline{R^8}(X, Y), \overline{R^8}(X, Y))$ which can occur are all pairs (a, b) where $a \leq b$ with the exception of $(\mathsf{TPP}, \mathsf{EQ})$, $(\mathsf{TPPi}, \mathsf{EQ})$,$(\mathsf{NTPP}, \mathsf{EQ})$, $(\mathsf{NTPPi}, \mathsf{EQ})$, $(\mathsf{EC}, \mathsf{TPP})$, $(\mathsf{EC}, \mathsf{TPPi})$, $(\mathsf{EC}, \mathsf{EQ})$, $(\mathsf{DC}, \mathsf{EC})$, $(\mathsf{DC}, \mathsf{TPP})$, $(\mathsf{DC}, \mathsf{TPPi})$, $\mathsf{EC}, \mathsf{NTPP})$, $(\mathsf{EC}, \mathsf{NTPPi})$, $(\mathsf{TPP}, \mathsf{NTPP})$, $(\mathsf{TPP}, \mathsf{NTPPi})$

Proof. (1) We first show that $\underline{R}(X, Y) \leq \overline{R}(X, Y)$ where $X, Y \in \Omega_4^{G \times G}$ and $\underline{R}(X, Y)$ and $\overline{R}(X, Y)$ are defined as discussed in Section 5.1. The structure of the argument corresponds to the proof of Theorem 1 in [8]. We simply use the boundary sensitive operation tables discussed in Section 4.2.2. Consequently, we have $(a \leq b)$ if a and b are refinements of distinct RCC5 relations[10]. Assume that a and b are refinements of the same RCC5 relation, i.e., refinements of DR, PP, or PPi. Pairs (a, b) where a and b are refinements of the same RCC5 - relation and $a \neq b$ cannot occur, since the refined relations are distinguished by the outcome of $\delta X \wedge \delta Y$ which is independent of the operations $\underline{\wedge}$ and $\overline{\wedge}$.

(2) The cases $(\mathsf{TPP}, \mathsf{EQ})$, $(\mathsf{TPPi}, \mathsf{EQ})$,$(\mathsf{NTPP}, \mathsf{EQ})$, $(\mathsf{NTPPi}, \mathsf{EQ})$ cannot occur, since they are refinements of $(\mathsf{PP}(i), \mathsf{EQ})$, which cannot occur by Theorem 1.

(3) The cases $(\mathsf{DC}, \mathsf{TPP})$, $(\mathsf{DC}, \mathsf{TPPi})$, $(\mathsf{EC}, \mathsf{NTPP})$, and $(\mathsf{EC}, \mathsf{NTPPi})$ cannot occur. In the definition of both elements of these pairs the sub-formula $\delta X \wedge \delta Y$ occurs which result is independent of the choice of $\underline{\wedge}$ and $\overline{\wedge}$. Consequently in the definitions of compatible relations, either both relations have $\delta X \wedge \delta Y = \bot$ or both have $\delta X \wedge \delta Y \neq \bot$. This rules out the occurrence of these cases.

[10] Boundary sensitive relations are refinements of boundary insensitive relations, i.e., DC and EC are refinements of DR and $\mathsf{TPP}(i)$ and $\mathsf{NTPP}(i)$ are refinements of $\mathsf{PP}(i)$.

(4) The cases $(\mathsf{EC}, \mathsf{TPP})$, $(\mathsf{EC}, \mathsf{TPPi})$, and $(\mathsf{EC}, \mathsf{EQ})$ cannot occur. Assume $\overline{R^8}(X, Y) \in \{\mathsf{TPP}, \mathsf{EQ}\}$. Since $\delta X \wedge \delta Y \neq \bot$ and $X \overline{\wedge} Y = X$ there must be $g_i, g_j \in G$ such that $(X(g_i, g_j)) \geq \mathsf{bo}$ and $(Y(g_i, g_j)) \geq \mathsf{bo}$. By definition of $\underline{\triangle}$ we have $\mathsf{bo} \underline{\triangle} \mathsf{bo} = \mathsf{bo}$ and consequently $X \underline{\triangle} Y \neq \bot$. This contradicts $\underline{R^8}(X, Y) = \mathsf{EC}$ which implies $X \underline{\triangle} Y = \bot$. A similar argument applies to $\overline{R^8}(X, Y) = \mathsf{TPPi}$.

A Haskell program generating all remaining cases can be obtained from the author. □

Consider Table 1. The numbers indicate which case discussed in the proof above prevents the particular pair $(\underline{R^8}(X, Y), \overline{R^8}(X, Y))$ from occurring. For pairs of relations that can occur $\mathcal{SYN}(X, Y)$ is given.

$\underline{R^8} \backslash \overline{R^8}$	DC	EC	PO	TPP	NTPP	EQ
DC	$\{\mathsf{DC}\}$	(1)	$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}\}$	(3)	$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}\}$	$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\}$
EC	(1)	$\{\mathsf{EC}\}$	$\underline{\{\mathsf{EC}, \mathsf{PO}\}}$	(4)	(3)	(4)
PO	(1)	(1)	$\{\mathsf{PO}\}$	$\underline{\{\mathsf{PO}, \mathsf{TPP}\}}$	$\underline{\{\mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}\}}$	$\underline{\{\mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\}}$
TPP	(1)	(1)	(1)	$\{\mathsf{TPP}\}$	(1)	(2)
NTPP	(1)	(1)	(1)	(1)	$\{\mathsf{NTPP}\}$	(2)
EQ	(1)	(1)	(1)	(1)	(1)	$\{\mathsf{EQ}\}$

Table 1

Possible pairs of minimal and maximal relations (The relations TPPi and NTPPi are omitted.)

Consider the underlined sets of relations in Table 1. At the syntactic level these relations can occur only if we allow for approximations that describe only complex regions, i.e., approximations X such that all $x \in [\![X]\!]$ are complex regions. This will be discussed in more detail in Section 7.1.

6.2. *Correspondence of syntactic and semantic generalization*

Let R be the set of regular one-dimensional regions in a one-dimensional space. Those regions may be intervals, i.e., maximally connected regions, or complex regions consisting of multiple disconnected parts. Let X and Y be boundary sensitive approximations of regions $x, y \in R$ and let RCC range over RCC5 , RCC_1^8 , RCC_1^9 , RCC_1^{15} . Then the following holds:

Lemma 4. If there are $g_i, g_j \in G$ such that $(X(g_i, g_j)) = (Y(g_i, g_j)) = \mathsf{bo}$ then $\min\{\rho \in \text{RCC} \mid \rho(x, y), x \in [\![X]\!], y \in [\![Y]\!]\} = \mathsf{PO}$.

Proof. Assume that $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ are regular regions. Since there are $g_i, g_j \in G$ such that $(X(g_i, g_j)) = (Y(g_i, g_j)) = \mathsf{bo}$ we have $\delta(x \wedge g_i) \wedge \delta(y \wedge g_i) \neq \bot$ and, by regularity, $x \wedge y \neq \bot$. By definition of bo, we have $(x \wedge g_i) < g_i$[11]. Since x and y are arbitrary, possibly complex one-dimensional regions we have $\min((x \wedge g_i) \wedge (y \wedge g_i) = (x \wedge g_i)) = \mathsf{F}$ and $\min((x \wedge g_i) \wedge (y \wedge g_i) = (y \wedge g_i)) = \mathsf{F}$. By the definition of RCC5 and all its refinements we have $\min\{\rho \in \text{RCC} \mid \rho(x, y), x \in [\![X]\!], y \in [\![Y]\!]\} = \mathsf{PO}$. $\square$

It is important to stress that Lemma 4 presupposes that we allow for complex regions. Consider, for example, configuration (h) in Figure 8. We have $(X(g_i, g_{i-1})) = \mathsf{bo}$ and $(Y(g_i, g_{i-1})) = \mathsf{bo}$ and $x \in [\![X]\!]$ and $\{y_1, y_2, y_3\} \subset [\![Y]\!]$ and $\mathsf{PO}(x, y_2)$, $\mathsf{TPPi}(x, y_1)$, and $\mathsf{TPP}(x, y_3)$). Consequently we have $x \wedge y \neq \bot = \mathsf{T}$, $\min(x \wedge y = x) = \mathsf{F}$, $\min(x \wedge y = y) = \mathsf{F}$, and $\min\{\rho \in \mathsf{RCC}_1^8 \mid \rho(x, y), x \in [\![X]\!], y \in [\![Y]\!]\} = \mathsf{PO}$.

Let $SEM(X, Y)$ be a set of RCC_1^8 relations defined as $SEM(X, Y) = \{\rho \in \mathsf{RCC}_1^8 \mid \rho(x, y), x \in [\![X]\!], y \in [\![Y]\!]\}$ with $[\![X]\!] \subset R$ and $[\![Y]\!] \subset R$. Assume $(X(g_i, g_j)) = \mathsf{bo}$ and $(Y(g_i, g_j)) = \mathsf{bo}$. Since $\mathsf{bo} \,\underline{\wedge}\, \mathsf{bo} = \mathsf{bo}$ we have $X \,\underline{\wedge}\, Y \neq \bot$ and possibly, depending on the outcome of $(X(g_k, g_l)) \,\underline{\wedge}\, (Y(g_k, g_l))$ with $i \neq k$ and $j \neq l$, $X \,\underline{\wedge}\, Y = X$ and/or $X \,\underline{\wedge}\, Y = Y$. This means that $\underline{R^8}(X, Y) \geq \mathsf{PO}$. If $\underline{R^8}(X, Y) > \mathsf{PO}$ then this conflicts with $\min(SEM(X, Y)) = \mathsf{PO}$. That is why we define the semantically corrected syntactic generalization of RCC_1^8 as:

$$\mathcal{SYN}(X, Y) = (\underline{R_c^8}(X, Y), \overline{R^8}(X, Y))$$

where $\underline{R_c^8}(X, Y) = \mathsf{PO}$ if there are $g_i, g_j \in G$ such that $(X(g_i, g_j)) = (Y(g_i, g_j)) = \mathsf{bo}$ and $\underline{R_c^8}(X, Y) = \underline{R^8}(X, Y)$ otherwise. The semantic generalization of RCC_1^8 relations is defined as $\mathcal{SEM}(X, Y) = \{\rho \in \mathsf{RCC}_1^8 \mid \underline{R_c^8}(X, Y) \leq \rho \leq \overline{R^8}(X, Y)\}$.

Theorem 5. For any boundary sensitive approximations X and Y of regular one-dimensional regions, the syntactic and semantic generalizations of RCC_1^8 are equivalent in the sense that $\mathcal{SYN}(X, Y) = \mathcal{SEM}(X, Y) = SEM(X, Y)$.

[11] $x < y$ iff $x \wedge y = x$ and $x \wedge y \neq y$.

Proof. Corresponding to the proof of Theorem 2 in [8] there are three things which must be shown. First, that for all $x \in [\![X]\!]$, and $y \in [\![Y]\!]$, that $\underline{R_c^8}(X, Y) \leq \rho(x, y)$ with $\rho \in \mathsf{RCC}_1^8$. Secondly, for all x and y as before, that $\rho(x, y) \leq \overline{R^8}(X, Y)$, and thirdly that if ρ is any RCC8 relation such that $\underline{R_c^8}(X, Y) \leq \rho \leq \overline{R^8}(X, Y)$ then there exist particular x and y which stand in the relation ρ to each other.

First, we need to consider two cases: (i) There are $g_i, g_j \in G$ such that $(X(g_i, g_j)) = \mathsf{bo}$ and $(Y(g_i, g_j)) = \mathsf{bo}$. In this case we have $\underline{R_c^8}(X, Y) = \mathsf{PO} \leq \rho(x, y)$ by Lemma 4. (ii) Otherwise: In this case it is necessary to consider each of the three components $X \underline{\wedge} Y \not\equiv \perp$, $X \underline{\wedge} Y \equiv X$, and $X \underline{\wedge} Y \equiv Y$.

If $X \underline{\wedge} Y \not\equiv \perp = \mathsf{M}$ then we have to show that for all $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ that $x \wedge y \not\equiv \perp \geq \mathsf{M}$. If $X \underline{\wedge} Y \not\equiv \perp = \mathsf{M}$ then $\delta X \wedge \delta Y \neq \perp$. By definition of $\delta X \wedge \delta Y \neq \perp$ there are four possible cases. We discuss the case $(X\,(g_i, g_{i+1})) \geq \mathsf{bo}$ and $(Y\,(g_i, g_{i+1})) < \mathsf{bo}$ and $(X\,(g_{i+1}, g_i)) < \mathsf{bo}$ and $(Y\,(g_{i+1}, g_i)) \geq \mathsf{bo}$. The other cases are similar. By definition of bo and no at least one boundary point of all $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ coincide with the boundary point, $g_i \cap g_{i+1}$, shared by the cells g_i and g_{i+1}. If B_x is the set of boundary points of x and B_y is the set of boundary points of y then we have $g_i \cap g_{i+1} \in B_x$ and $g_i \cap g_{i+1} \in B_y$ and, hence, $\delta x \wedge \delta y \neq \perp$ and $x \equiv y \neq \perp = \mathsf{M}$.

If $X \underline{\wedge} Y \not\equiv \perp = \mathsf{T}$ then for all $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ we show that $x \wedge y \not\equiv \perp = \mathsf{T}$. If $X \underline{\wedge} Y \not\equiv \perp = \mathsf{T}$ then there are $g_i, g_j \in G$ such that (a) $(X(g_i, g_j)) = \mathsf{bo}$ and $(Y(g_i, g_j)) = \mathsf{bo}$ or (b) $(X(g_i, g_j)) = \mathsf{fo}$ and $(Y(g_i, g_j)) \neq \mathsf{no}$ or (c) $(X(g_i, g_j)) \neq \mathsf{no}$ and $(Y(g_i, g_j)) = fo$. In case (a) we have $x \wedge y \neq \perp$ by Lemma 4. In case (b) we have cells g_i with $x \wedge g_i = g_i$, $y \wedge g_i \neq \perp$ and, hence, $x \wedge y \neq \perp$ and similarly for (c). Consequently we have $x \wedge y \not\equiv \perp = \mathsf{T}$.

If $X \underline{\wedge} Y \equiv X = \mathsf{M}$ then we have to show for all $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ that $x \wedge y \equiv x \geq \mathsf{M}$. If $X \underline{\wedge} Y \equiv X \neq \mathsf{F}$ then $X \underline{\wedge} Y = X$. If $X \underline{\wedge} Y = X$ then for all $g_i, g_j \in G$ if $(X(g_i, g_j)) \neq \mathsf{no}$ then $(Y(g_i, g_j)) = \mathsf{fo}$[12]. By definition of nbo, bo, and fo we have if $x \wedge g_i \neq \perp$ then $y \wedge g_i = g_i$ and, hence, $x \wedge y = x$. Consequently we have $x \wedge y \equiv x \geq \mathsf{M}$.

If $X \underline{\wedge} Y \equiv X = \mathsf{T}$ then we have $X \underline{\simeq} Y$ or $(X \underline{\not\simeq} Y$ and $X \underline{\wedge} Y = X$ and $\delta X \wedge \delta Y = \perp)$. We discuss the case $X \underline{\not\simeq} Y$ and omit $X \underline{\simeq} Y$. We have to show for all $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ that $x \wedge y \equiv x = \mathsf{T}$, i.e., $x \wedge y = x$ and $\delta x \wedge \delta y = \perp$. We have already shown above that if $X \underline{\wedge} Y = X$ then $x \wedge y = x$ and that if $\delta X \wedge \delta Y \neq \perp$ then $\delta x \wedge \delta y \neq \perp$, and hence, if $\delta X \wedge \delta Y = \perp$ then $\delta x \wedge \delta y = \perp$. If $X \underline{\wedge} Y \equiv X = \mathsf{T}$ then $X \underline{\wedge} Y = X$ and $\delta X \wedge \delta Y = \perp$. Consequently we have $x \wedge y \equiv x = \mathsf{T}$. The proof for $X \underline{\wedge} Y \equiv Y$

[12] The case $(X(g_i, g_j)) = (Y(g_i, g_j)) = \mathsf{fo}$ was already dealt with in (i).

is similar.

Second: The proof $\rho(x,y) \leq \overline{R^8}(X,Y)$ is similar and is omitted here.

Third: we have to show that if ρ is any RCC8 relation such that $\underline{R_c^8}(X,Y) \leq \rho \leq \overline{R^8}(X,Y)$ then there exist particular x and y which stand in the relation ρ to each other, i.e., we have to show that all sets of relations in Table 1 do actually occur. We discuss the case $(\mathsf{DC}, \mathsf{PO})$ and give examples for the others. Due to $\underline{R} = \mathsf{DC}$ we have $X \underline{\wedge} Y = \bot$ and due to $\overline{R} = \mathsf{PO}$ we have $X \overline{\wedge} Y \neq \bot$, $X \overline{\wedge} Y \neq X$ and $X \overline{\wedge} Y \neq Y$. There cannot be $g_i, g_j \in G$ with $(X(g_i,g_j)) = \mathsf{bo}$ and $(Y(g_i,g_j)) = \mathsf{bo}$, $(X(g_i,g_j)) = \mathsf{fo}$ and $(Y(g_i,g_j)) \neq \mathsf{no}$, and $(X(g_i,g_j)) \neq \mathsf{no}$ and $(Y(g_i,g_j)) = \mathsf{fo}$. There must be $g_{i-1}, g_i, g_{i+1} \in G$ such that $(X(g_i,g_{i-1})) = \mathsf{bo}$, $(Y(g_i,g_{i-1})) = \mathsf{nbo}$, $(X(g_i,g_{i+1})) = \mathsf{nbo}$ and $(Y(g_i,g_{i+1})) = \mathsf{bo}$. Consequently there are $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ such that the relation ρ that holds between x and y is anywhere between $\mathsf{DC} \leq \rho \leq \mathsf{PO}$. This holds in particular for configuration (a) in Figure 8. We have $\mathsf{DC}(x,y_1)$, $\mathsf{EC}(x,y_2)$, and $\mathsf{PO}(x,y_3)$.

Examples for the remaining cases are given in configurations (b-g) in Figure 8. Configuration (b) is an example for $(\mathsf{DC}, \mathsf{NTPPi})$, i.e., $\mathsf{DC}(x,y_5)$, $\mathsf{EC}(x,y_3)$, $\mathsf{PO}(x,y_4)$, $\mathsf{TPPi}(x,y_2)$ and $\mathsf{NTPPi}(x,y_1)$; (c) is an example for $(\mathsf{EC}, \mathsf{PO})$ (This case can only occur if x and y are complex regions), i.e., $\mathsf{EC}(x,y_1)$ and $\mathsf{PO}(x,y_2)$; (d) is an example for $(\mathsf{PO}, \mathsf{NTPPi})$, i.e., $\mathsf{PO}(x,y_3)$, $\mathsf{TPPi}(x,y_2)$, and $\mathsf{NTPPi}(x,y_1)$; (e) is an example for $(\mathsf{PO}, \mathsf{TPPi})$, i.e., $\mathsf{PO}(x,y_2)$ and $\mathsf{TPPi}(x,y_1)$ (This case can only occur if x or y is a complex region); (f) is an example for $(\mathsf{PO}, \mathsf{EQ})$; (g) is an example for $(\mathsf{DC}, \mathsf{EQ})$. $\square$

7. Generalization of RCC_1^9 relations

7.1. Convex hull operation

Let X and Y be boundary sensitive approximations of regions x and y. Since RCC_1^9 relations are defined for one-dimensional intervals and convex hulls, $\hat{x}$, of complex one-dimensional regions, x, we need to define a corresponding operation, $\hat{X}$, in the approximation domain. We start by defining operations $l, r: \ \Omega^{G \times G} \to \Omega \times \Omega$ that return: $(l(X))$ the leftmost pair $(X\,(g_i,g_{i-1}), X\,(g_i,g_{i+1}))$ with $(X\,(g_i,g_{i-1})) \neq \mathsf{no}$; $(r(X))$ the rightmost pair $(X\,(g_j,g_{j-1}), X\,(g_j,g_{j+1}))$ with $(X\,(g_j,g_{j+1})) \neq \mathsf{no}$. If $i \neq j$ then the convex hull operation replaces these values in $\hat{X}$ as indicated in the table below and sets $(\hat{X}\,(g_k,g_{k-1})) = \mathsf{fo}$ and $(\hat{X}\,(g_k,g_{k+1})) = \mathsf{fo}$ for all $i+1 < k < j$.

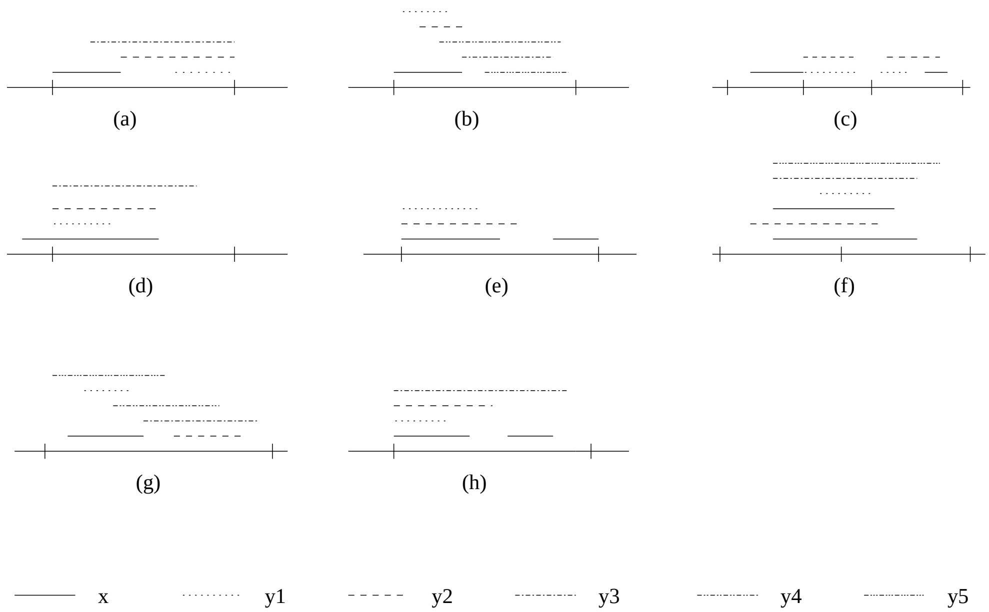


Figure 8. Each configuration depicts cells of a partitioned line (cell boundaries are indicated by small vertical lines). One-dimensional regions x and y_1 - y_4 are located on the line. Depictions of these regions are drawn above the line. The actual regions are located **on** the line and can be identified by orthogonal projections of their depictions onto the line. The legend in the bottom of the figure shows which line-style signifies which region.

$l(X)$,$r(X)$	$(\mathsf{nbo},\mathsf{nbo})$	$(\mathsf{nbo},\mathsf{bo})$	$(\mathsf{bo},\mathsf{nbo})$	$(\mathsf{bo},\mathsf{bo})$	$(\mathsf{fo},\mathsf{fo})$
$l(\hat{X})$	$(\mathsf{nbo},\mathsf{bo})$	$(\mathsf{nbo},\mathsf{bo})$	$(\mathsf{fo},\mathsf{fo})$	$(\mathsf{fo},\mathsf{fo})$	$(\mathsf{fo},\mathsf{fo})$
$r(\hat{X})$	$(\mathsf{bo},\mathsf{nbo})$	$(\mathsf{fo},\mathsf{fo})$	$(\mathsf{bo},\mathsf{nbo})$	$(\mathsf{fo},\mathsf{fo})$	$(\mathsf{fo},\mathsf{fo})$

If $i = j$ then we have $l(X) = r(X)$ and $r(\hat{X}) = l(\hat{X}) = l(X)$ except for pairs $l(X) = (\mathsf{bo},\mathsf{bo})$ where in $l(\hat{X})$ pairs $(\mathsf{bo},\mathsf{bo})$ are replaced by pairs $(\mathsf{fo},\mathsf{fo})$, i.e., if $l(X) = (\mathsf{bo},\mathsf{bo})$ then $l(\hat{X}) = (\mathsf{fo},\mathsf{fo})$.

Let X be a boundary sensitive approximation. The corresponding set of regions $[\![X]\!]$ certainly contains complex regions and may possibly contain intervals. The convex-hull-set of X, denoted by $CH(X)$, is the set of the convex hulls of all $x \in [\![X]\!]$: $CH(X) = \{\hat{x} \mid x \in [\![X]\!]\}$. Lemma 6 tells us that $CH(X)$ is a proper subset of the regions, $[\![\hat{X}]\!]$, that are approximated by $\hat{X}$, i.e., that there are regions in $[\![\hat{X}]\!]$ that are not convex hulls of regions in $[\![X]\!]$. There are some complex regions in $[\![\hat{X}]\!]$.

Lemma 6. $CH(X) \subset [\![\hat{X}]\!]$.

	$int(r(x), g_j)$	$r(x) = r(g_j)$
$l(x) = l(g_i)$	$l(X) = (\mathsf{bo}, \in \{\mathsf{nbo}, \mathsf{bo}\})$ or $l(X) = (\mathsf{fo}, \mathsf{fo})$ $\Rightarrow l(\hat{X}) = (\mathsf{fo}, \mathsf{fo})$ $r(X) = (\in \{\mathsf{nbo}, \mathsf{bo}\}, \mathsf{nbo})$ $\Rightarrow r(\hat{X}) = (\mathsf{bo}, \mathsf{nbo})$	$l(X) = (\mathsf{bo}, \in \{\mathsf{nbo}, \mathsf{bo}\})$ or $l(X) = (\mathsf{fo}, \mathsf{fo})$ $\Rightarrow l(\hat{X}) = (\mathsf{fo}, \mathsf{fo})$ $r(X) = (\in \{\mathsf{nbo}, \mathsf{bo}\}, \mathsf{bo})$ or $r(X) = (\mathsf{fo}, \mathsf{fo})$ $\Rightarrow r(\hat{X}) = (\mathsf{fo}, \mathsf{fo})$
$int(l(x), g_i)$	$l(X) = (\mathsf{nbo}, \in \{\mathsf{nbo}, \mathsf{bo}\})$ $\Rightarrow l(\hat{X}) = (\mathsf{nbo}, \mathsf{bo})$ $r(X) = (\in \{\mathsf{nbo}, \mathsf{bo}\}, \mathsf{nbo})$ $\Rightarrow r(\hat{X}) = (\mathsf{bo}, \mathsf{nbo})$	$l(X) = (\mathsf{nbo}, \in \{\mathsf{nbo}, \mathsf{bo}\})$ $\Rightarrow l(\hat{X}) = (\mathsf{nbo}, \mathsf{bo})$ $r(X) = (\in \{\mathsf{nbo}, \mathsf{bo}\}, \mathsf{bo})$ or $r(X) = (\mathsf{fo}, \mathsf{fo})$ $\Rightarrow r(\hat{X}) = (\mathsf{fo}, \mathsf{fo})$

Table 2

Proof. Let $l(z)$ return the leftmost boundary point of z and let $r(z)$ return the rightmost boundary point of z for arbitrary one-dimensional regions z in a directed space. By the definition of the convex hull for one-dimensional regions we have $l(z) = l(\hat{z})$ and $r(z) = r(\hat{z})$ and $\hat{z}$ is the interval bounded by $l(\hat{z})$ and $l(\hat{z})$. Assume further a predicate $int(y, z)$ that returns True if y is an interior point of the region z and false otherwise.

$CH(X) \subseteq [\![\hat{X}]\!]$. Let x be an arbitrary element of $CH(X)$, i.e., $x \in CH(X)$. Consequently, there is an x' in $[\![X]\!]$ such that $x = \hat{x}'$ with $l(x) = l(x')$ and $r(x) = r(x')$. Consider the partition cells g_i and g_j with $i < j$[13]. Table 2 lists the cases can occur. We discuss the case $l(x) = l(g_i)$ and $int(r(x), g_j)$ the others are similar. In this case we have $l(X) = (\mathsf{bo}, \mathsf{nbo})$, $l(X) = (\mathsf{bo}, \mathsf{bo})$, or $l(X) = (\mathsf{fo}, \mathsf{fo})$ and, $r(X) = (\mathsf{nbo}, \mathsf{nbo})$ or $r(X) = (\mathsf{bo}, \mathsf{nbo})$. By definition of $\hat{X}$ we get $l(\hat{X}) = (\mathsf{fo}, \mathsf{fo})$ and $r(\hat{X}) = (\mathsf{bo}, \mathsf{nbo})$. Consequently we have for all $x'' \in [\![\hat{X}]\!] : l(x'') = l(g_i)$ and $in(r(x''), g_j)$. By definition of $\hat{X}$ (we 'fill' everything between $l(X)$ and $r(X)$ with fo) all intervals with $l(x'') = l(g_i)$ and $in(r(x''), g_j)$ are elements of $[\![\hat{X}]\!]$ and in particular x, i.e., $x \in [\![\hat{X}]\!]$.

$[\![\hat{X}]\!] \not\subseteq CH(X)$. Assume $R(\hat{X}) = (\mathsf{bo}, \mathsf{nbo})$, i.e., there is a partition cell such that $(\hat{X}\,(g_j, g_{j-1}) = \mathsf{bo}$. By definition of bo all regions in $[\![\hat{X}]\!]$ partially overlap the left part of the cell g_j. In particular there is a region $x \in [\![\hat{X}]\!]$ such that $x \wedge g_j$ is a complex region. By definition of $CH(X)$ we have $x \notin CH(X)$. □

By Lemma 6 $[\![\hat{X}]\!]$ contains the convex hulls, $\hat{x}$, of all $x \in [\![X]\!]$ but if X is such that the left most and/or the right most parts of the $x \in [\![X]\!]$ partially overlap partition cells g_i and/or g_j then the intersection of these x with g_i and/or g_j may be a complex

[13] The special case $i = j$ is omitted here but is similar to the cases discussed in what follows.

region, and hence, these x are complex regions. The definition of $\hat{X}$ ensures that there are no such approximations that semantic interpretations contain only complex regions. Furthermore it ensures that if there are complex regions in $[\![\hat{X}]\!]$ then the complex parts of those regions are only at the very left and the very right (relative to the underlying partition).

7.2. *Syntactic generalization*

Let X and Y be boundary sensitive approximations of regions x and y. Since RCC_1^9 relations are only defined for intervals we need to apply the convex hull operator, i.e., we consider $\hat{X}$ and $\hat{Y}$. At the level of approximations we cannot completely exclude complex regions. We can only exclude approximations that contain only complex regions. We can consider the two triples of truth values[14]:

$$(\hat{X} \,\underline{\triangle}\, \hat{Y} \not\sim \bot,\ \hat{X} \,\underline{\triangle}\, \hat{Y} \sim \hat{X},\ \hat{X} \,\underline{\triangle}\, \hat{Y} \sim \hat{Y}), (\hat{X} \,\overline{\wedge}\, \hat{Y} \not\sim \bot,\ \hat{X} \,\overline{\wedge}\, \hat{Y} \sim \hat{X},\ \hat{X} \,\overline{\wedge}\, \hat{Y} \sim \hat{Y}).$$

with

$$\hat{X} \,\underline{\triangle}\, \hat{Y} \not\sim \bot = \begin{cases} \mathsf{FLO} \text{ if } \hat{X} \,\underline{\triangle}\, \hat{Y} = \bot \text{ and } (\hat{X} \triangleleft \hat{Y}) \\ \mathsf{FRO} \text{ if } \hat{X} \,\underline{\triangle}\, \hat{Y} = \bot \text{ and } (\hat{X} \triangleright \hat{Y}) \\ \mathsf{T} \quad\ \text{ if } x \wedge y \neq \bot \end{cases}$$

and with

$$\hat{X} \,\underline{\triangle}\, \hat{Y} \sim \hat{X} = \begin{cases} \mathsf{FLO} \text{ if } \hat{X} \,\underline{\triangle}\, \hat{Y} \neq \hat{X} \text{ and } \hat{X} \,\underline{\triangle}\, \hat{Y} \neq \hat{Y} \text{ and } (\hat{X} \triangleleft \hat{Y}) \\ \mathsf{FRO} \text{ if } \hat{X} \,\underline{\triangle}\, \hat{Y} \neq \hat{X} \text{ and } \hat{X} \,\underline{\triangle}\, \hat{Y} \neq \hat{Y} \text{ and } (\hat{X} \triangleright \hat{Y}) \\ \mathsf{FLI} \ \text{ if } \hat{X} \,\underline{\triangle}\, \hat{Y} \neq \hat{X} \text{ and } \hat{X} \,\underline{\triangle}\, \hat{Y} = \hat{Y} \text{ and} (\hat{X} \triangleleft \hat{Y}) \\ \mathsf{FRI} \ \text{ if } \hat{X} \,\underline{\triangle}\, \hat{Y} \neq \hat{X} \text{ and } \hat{X} \,\underline{\triangle}\, \hat{Y} = \hat{Y} \text{ and } (\hat{X} \triangleright \hat{Y}) \\ \mathsf{T} \quad\ \text{ if } \hat{X} \,\underline{\triangle}\, \hat{Y} = \hat{X} \end{cases}$$

and with

$$\hat{X} \,\overline{\wedge}\, \hat{Y} \sim \hat{X} = \begin{cases} \mathsf{FLO} \text{ if } \hat{X} \,\overline{\wedge}\, \hat{Y} \neq \hat{X} \text{ and } \hat{X} \,\overline{\wedge}\, \hat{Y} \neq \hat{Y} \text{ and } (\hat{X} \triangleleft \hat{Y}) \\ \mathsf{FRO} \text{ if } \hat{X} \,\overline{\wedge}\, \hat{Y} \neq \hat{X} \text{ and } \hat{X} \,\overline{\wedge}\, \hat{Y} \neq \hat{Y} \text{ and } (\hat{X} \triangleright \hat{Y}) \\ \mathsf{FLI} \ \text{ if } \hat{X} \,\overline{\wedge}\, \hat{Y} \neq \hat{X} \text{ and } \hat{X} \,\overline{\wedge}\, \hat{Y} = \hat{Y} \text{ and} (\hat{Y} \triangleright \hat{X}) \\ \mathsf{FRI} \ \text{ if } \hat{X} \,\overline{\wedge}\, \hat{Y} \neq \hat{X} \text{ and } \hat{X} \,\overline{\wedge}\, \hat{Y} = \hat{Y} \text{ and } (\hat{Y} \triangleleft \hat{X}) \\ \mathsf{T} \quad\ \text{ if } \hat{X} \,\overline{\wedge}\, \hat{Y} = \hat{X} \end{cases}$$

and similarly for $\hat{X} \,\underline{\triangle}\, \hat{Y} \not\sim \bot$, $\hat{X} \,\overline{\wedge}\, \hat{Y} \not\sim \bot$, $\hat{X} \,\underline{\triangle}\, \hat{Y} \sim \hat{Y}$, and $\hat{X} \,\overline{\wedge}\, \hat{Y} \sim \hat{Y}$.

The notation $X \triangleleft Y$ is an abbreviation for $X \ll Y \neq \mathsf{F}$ and $X \ll Y \geq X \gg Y$ and the notation $X \triangleright Y$ is an abbreviation for $X \gg Y \neq \mathsf{F}$ and $X \gg Y > X \ll Y$.

[14] Similar relations were defined in [4] using boundary *in*sensitive approximations.

We define $X \ll Y$ as shown in the table below, where $X = Y$ is an abbreviation for $X \,\underline{\triangle}\, Y = X$ and $X \,\underline{\triangle}\, Y = Y$, $X < Y$ is an abbreviation for $X \,\underline{\triangle}\, Y = X$ and $X \neq Y$, and $X > Y$ is an abbreviation for $X \,\underline{\triangle}\, Y = Y$ and $X \neq Y$.

$X \ll Y$	$L(X) \barwedge X = \bot$ or $L(Y) \barwedge Y = \bot$	$L(X) \barwedge X \neq \bot$ and $L(Y) \barwedge Y \neq \bot$
$L(X) < L(Y)$	T	T
$L(X) = L(Y)$	F	M
$L(X) > L(Y)$	F	F

We define $X \gg Y$ following the same pattern but using $R(X)$ and $R(Y)$ instead of $L(X)$ and $L(Y)$. L and R are defined assuming that partition cells g_i are numbered in increasing order in the direction of the underlying space. Let $\min, \max : \Omega^{G \times G} \rightarrow N$ be functions returning the index of the corresponding leftmost (rightmost) partition cell with $(X\ (g_i, g_j)) \neq \mathsf{no}$, i.e., $\min(X) = \min\{k \mid (X\ (g_k, g_l)) \neq \mathsf{no}\}$ and $\max(X) = \max\{k \mid (X\ (g_k, g_l)) \neq \mathsf{no}\}$. We define $L(X)$

$$(L(X)\,(g_i, g_j)) = \begin{cases} \mathsf{fo} \text{ if } i < \min(X) \\ \mathsf{bo} \text{ if } i = \min(X) \text{ and } j = i - 1 \text{ and } (X\ (g_i, g_j)) = \mathsf{nbo} \\ \mathsf{no} \text{ otherwise} \end{cases}$$

and $R(X)$

$$(R(X)\,(g_i, g_j)) = \begin{cases} \mathsf{fo} \text{ if } i > \max(X) \\ \mathsf{bo} \text{ if } i = \max(X) \text{ and } j = i + 1 \text{ and } (X\ (g_i, g_j)) = \mathsf{nbo} \\ \mathsf{no} \text{ otherwise} \end{cases}.$$

Each of the above triples defines an RCC_1^9 relation, so the relation between $\hat{X}$ and $\hat{Y}$ can be measured by a pair of RCC_1^9 relations. These relations will be denoted by $\underline{R^9}(\hat{X}, \hat{Y})$ and $\overline{R^9}(\hat{X}, \hat{Y})$.

Theorem 7. The pairs

$$(\min(\underline{R^9}(\hat{X}, \hat{Y}), \overline{R^9}(\hat{X}, \hat{Y})), \max(\underline{R^9}(\hat{X}, \hat{Y}), \overline{R^9}(\hat{X}, \hat{Y})))$$

that can occur are all pairs (a, b) where $a \leq b \leq \mathsf{EQ}$ and $\mathsf{EQ} \leq a \leq b$ with the exception of $(\mathsf{PPL}, \mathsf{EQ})$, $(\mathsf{PPR}, \mathsf{EQ})$, $(\mathsf{PPiL}, \mathsf{EQ})$, $(\mathsf{POL}, \mathsf{PPL})$, $(\mathsf{PPL}, \mathsf{PPiL})$, $(\mathsf{POR}, \mathsf{PPR})$, $(\mathsf{PPR}, \mathsf{PPiR})$, $(\mathsf{PPiR}, \mathsf{EQ})$, and $(\mathsf{EQ}, \mathsf{DRR})$.

Proof. (1) We start by showing that there are no $\hat{X}$ and $\hat{Y}$ such that $\underline{R^9}(\hat{X}, \hat{Y}) < \mathsf{EQ} < \overline{R^9}(\hat{X}, \hat{Y})$ or $\overline{R^9}(\hat{X}, \hat{Y}) < \mathsf{EQ} < \underline{R^9}(\hat{X}, \hat{Y})$. Assume a deterministic procedure that always checks FLO first and then checks FRO, FLI, FRI, T in this order for all entries in the definition table.[15] Whether or not $R(\hat{X},\hat{Y}) < \mathsf{EQ}$ or $R(\hat{X},\hat{Y}) > \mathsf{EQ}$ depends only on the truth-value of the formulas $\hat{X} \ll \hat{Y}$, $\hat{X} \gg \hat{Y}$, $\hat{Y} \ll \hat{X}$, and $\hat{Y} \gg \hat{X}$. In any case the truth-value of the formulas is the *same* for $\underline{R^9}$ and for $\overline{R^9}$. Consequently the cases $\underline{R^9}(\hat{X},\hat{Y}) < \mathsf{EQ} < \overline{R^9}(\hat{X},\hat{Y})$ and $\overline{R^9}(\hat{X},\hat{Y}) < \mathsf{EQ} < \underline{R^9}(\hat{X},\hat{Y})$ cannot occur. Without loss of generality we need only consider pairs (a,b) where $a \leq \mathsf{EQ}$ and $b \leq \mathsf{EQ}$. Since the outcome of $\ll$ and $\gg$ is the same for a and b the ordering depends only on the outcome of the operations $\underline{\wedge}$ and $\overline{\wedge}$. This leaves only the pairs $(\hat{X} \underline{\wedge} \hat{Y} \neq \bot, \hat{X} \underline{\wedge} \hat{Y} = \hat{X}, \hat{X} \underline{\wedge} \hat{Y} = \hat{Y})$ and $(\hat{X} \overline{\wedge} \hat{Y} \neq \bot, \hat{X} \overline{\wedge} \hat{Y} = \hat{X}, \hat{X} \overline{\wedge} \hat{Y} = \hat{Y})$ that need to be considered, i.e., the RCC5 case. By Theorem 1 we have $a \leq b$, i.e., $\underline{R^9}(\hat{X},\hat{Y}) \leq \overline{R^9}(\hat{X},\hat{Y}) \leq \mathsf{EQ}$. A similar argument applies to pairs (a,b) where $a \geq \mathsf{EQ}$ and $b > \mathsf{EQ}$. Due to the reverse ordering $\mathsf{T} < \mathsf{FRI} < \mathsf{FRO}$ we have $\mathsf{EQ} \leq \overline{R^9}(\hat{X},\hat{Y}) \leq \underline{R^9}(\hat{X},\hat{Y})$.

(2) The pairs $(\mathsf{PPL},\mathsf{EQ})$, $(\mathsf{PPR},\mathsf{EQ})$, $(\mathsf{PPiL},\mathsf{EQ})$, $(\mathsf{PPiR},\mathsf{EQ})$ cannot occur, since the corresponding relations $(\mathsf{PP},\mathsf{EQ})$, $(\mathsf{PPi},\mathsf{EQ})$ cannot occur in the RCC5 case.

(3) The pair $(\mathsf{EQ},\mathsf{DRR})$ cannot occur. Consider approximations $\hat{X}$ and $\hat{Y}$. For $(\mathsf{EQ},\mathsf{DRR})$ (and $(\mathsf{DRL},\mathsf{EQ})$) to hold there must be a single $g_i \in G$ such that $(\hat{X}\,(g_i, g_{i-1})) = \mathsf{nbo}$, $(\hat{X}\,(g_i, g_{i+1})) = \mathsf{nbo}$, $(\hat{Y}\,(g_i, g_{i-1})) = \mathsf{nbo}$, and $(\hat{Y}\,(g_i, g_{i+1})) = \mathsf{nbo}$. Due to the non-symmetric definition of FLO and FRO in $\hat{X} \underline{\wedge} \hat{Y} \not\sim \bot$ and $\hat{X} \overline{\wedge} \hat{Y} \not\sim \bot$ the case $\overline{R^9}(\hat{X},\hat{Y}) = \mathsf{DRR}$ cannot occur.

(4) The pairs $(\mathsf{POL},\mathsf{PPL})$, $(\mathsf{PPL},\mathsf{PPiL})$, $(\mathsf{POR},\mathsf{PPR})$, $(\mathsf{PPR},\mathsf{PPiR})$, $(\mathsf{POL},\mathsf{EQ})$, $(\mathsf{POR},\mathsf{EQ})$ cannot occur since since these cases can only occur for approximations X and Y where either $[\![X]\!]$ or $[\![Y]\!]$ or both contain only complex regions. These cases are excluded by using $\hat{X}$ and $\hat{Y}$ rather than X and Y in the definitions above. A Haskell program generating all remaining cases can be obtained from the author. $\square$

Consider Table 3. The numbers indicate which case of the proof discussed above prevents the particular pair from occurring. The meaning of the row $\underline{R^9_c} = \mathsf{POL}$ will be discussed below.

7.3. *Correspondence of semantic and syntactic generalization*

At the syntactic level the pair $(\mathsf{DRL},\mathsf{EQ})$ represents the most indeterminate case. It occurs if there is a single $g_i \in G$ such that $(\hat{X}\,(g_i, g_{i-1})) = \mathsf{nbo}$, $(\hat{X}\,(g_i, g_{i+1})) = \mathsf{nbo}$, $(\hat{Y}\,(g_i, g_{i-1})) = \mathsf{nbo}$, and $(\hat{Y}\,(g_i, g_{i+1})) = \mathsf{nbo}$. Since $(\mathsf{DRL},\mathsf{EQ})$ is consistent with $(\mathsf{EQ},\mathsf{DRR})$ and $(\mathsf{DRL},\mathsf{EQ})$ was chosen arbitrarily (the non-symmetry in the definitions

[15] This assumption is needed since the RCC^9_1 relations are not JEPD.

$\underline{R^9}\backslash\overline{R^9}$	DRL	POL	PPL	PPiL	EQ
DRL	{DRL}	{DRL,POL}	{DRL,POL,PPL}	{DRL,POL,PPiL}	{DRL,POL,PPL, PPiL,EQ}
$\underline{R^9} = \mathsf{POL}$	(1)	{POL}	(4)	(4)	(4)
$\underline{R^9_c} = \mathsf{POL}$	(1)	{POL}	{POL,PPL}	{POL,PPiL}	{POL,PPL, PPiL,EQ}
PPL	(1)	(1)	{PPL}	(1)	(2)
PPiL	(1)	(1)	(1)	{PPiL}	(2)
EQ	(1)	(1)	(1)	(1)	{EQ}

Table 3
Syntactic generalization of RCC_1^9

could be the other way around as well), $(\mathsf{DRL}, \mathsf{EQ})$ is corrected for semantic reasons to $(\mathsf{DRL}, \mathsf{DRR})$. The corrected relation will be denoted by $\overline{R^9_c}$.

Consider Lemma 4. It tells us that, if there are $g_i, g_j \in G$ such that $(X(g_i, g_j)) = (Y(g_i, g_j)) = \mathsf{bo}$, then there are $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ with $\mathsf{PO}(x, y)$. Notice that Lemma 4 is only true if we allow complex regions. By Lemma 6 there may be complex regions in $[\![\hat{X}]\!]$. Due to these regions Lemma 4 can be applied. Consequently we need to distinguish two cases: (a) We have $\underline{R^9_c}(\hat{X}, \hat{Y}) = \mathsf{POL}$ if there are $g_i, g_{i+1} \in G$ such that $(\hat{X}(g_i, g_{i+1})) = (\hat{Y}(g_i, g_{i+1})) = \mathsf{bo}$ and $i = min(\hat{X}) = min(\hat{Y})$[16], and $\underline{R^9_c}(\hat{X}, \hat{Y}) = \underline{R^9}(\hat{X}, \hat{Y})$ otherwise. (b) We have $\overline{R^9_c}(\hat{X}, \hat{Y}) = \mathsf{POR}$ if there are g_j, g_{j-1} such that $(\hat{X}(g_j, g_{j-1})) = (\hat{Y}(g_j, g_{ji1})) = \mathsf{bo}$ and $j = max(\hat{X}) = max(\hat{Y})$. Otherwise we have $\overline{R^9_c} = \overline{R^9}$ or $\overline{R^9_c} = \mathsf{DRR}$.

Let the syntactic generalization of RCC_1^9 relations, $\mathcal{SYN}$, be defined by

$$\mathcal{SYN}(\hat{X}, \hat{Y}) = (\min(\underline{R^9_c}(\hat{X}, \hat{Y}), \overline{R^9_c}(\hat{X}, \hat{Y})), \max(\underline{R^9_c}(\hat{X}, \hat{Y}), \overline{R^9_c}(\hat{X}, \hat{Y}))),$$

where $\underline{R^9_c}$ and $\overline{R^9_c}$ are defined as discussed above and let the semantic generalization of RCC_1^9 relations, $\mathcal{SEM}$, be defined as

$$\mathcal{SEM}(\hat{X}, \hat{Y}) = \{\rho \in \mathsf{RCC}_1^9 \mid \min(\underline{R^9_c}(\hat{X}, \hat{Y}), \overline{R^9_c}(\hat{X}, \hat{Y})) \leq \rho \leq \max(\underline{R^9_c}(\hat{X}, \hat{Y}), \overline{R^9_c}(\hat{X}, \hat{Y}))\},$$

where $\leq$ is the ordering in the RCC_1^9 lattice. Consider Table 3 and ignore the row $\underline{R^9} = \mathsf{POL}$. It shows $\mathcal{SYN}(\hat{X}, \hat{Y})$ for $\underline{R^9_c} \leq \overline{R^9_c} \leq \mathsf{EQ}$. The pairs $(\mathsf{POL}, \mathsf{PPL})$, $(\mathsf{PO}, \mathsf{PPiL})$, and $(\mathsf{PO}, \mathsf{EQ})$ do occur due to the semantic correction of the pairs

[16] $min(X)$ and $max(X)$ are defined as discussed in Section 7.2.

$(\mathsf{PPL}, \mathsf{PPL})$, $(\mathsf{PPiL}, \mathsf{PPiL})$, and $(\mathsf{EQ}, \mathsf{EQ})$ in cases where Lemma 4 applies. Theorem 8 tells us that the semantic generalization produces the same table.

Theorem 8. For approximations $\hat{X}$ and $\hat{Y}$ the syntactic and semantic generalizations of RCC_1^9 relations are equivalent in the sense that $\mathcal{SYN}(\hat{X}, \hat{Y}) = \mathcal{SEM}(\hat{X}, \hat{Y}) = SEM(X, Y)$.

Proof. We consider three cases: $\underline{R_c^9} \leq \rho \leq \overline{R_c^9}$ with $\underline{R_c^9} < \mathsf{EQ}$ and $\overline{R_c^9} > \mathsf{EQ}$, $\underline{R_c^9} \leq \rho \leq \overline{R_c^9} \leq \mathsf{EQ}$, and $\mathsf{EQ} \leq \overline{R_c^9} \leq \rho < \underline{R_c^9}$. For each case there are three things to demonstrate: (i) for all $x \in [\![\hat{X}]\!]$, and $y \in [\![\hat{Y}]\!]$, that $\underline{R_c^9}(\hat{X}, \hat{Y}) \leq \rho(x, y)$ with $\rho \in \mathsf{RCC}_1^9$; (ii) for all x and y as before, that $\rho(x, y) \leq \overline{R_c^9}(\hat{X}, \hat{Y})$; and (iii) if ρ is any RCC_1^9 relation such that $\underline{R_c^9}(\hat{X}, \hat{Y}) \leq \rho \leq \overline{R_c^9}(\hat{X}, \hat{Y})$ then there exist particular x and y which stand in the relation ρ to each other.

Firstly, there are two cases where there are $\rho_1, \rho_2 \in SEM$ which are such that $\rho_1 < \mathsf{EQ} < \rho_2$: $\mathsf{DRL} \leq \rho \leq \mathsf{DRR}$ and $\mathsf{POL} \leq \rho \leq \mathsf{POR}$. In the case of $\mathsf{DRL}(\hat{X}, \hat{Y}) \leq \rho(x, y) \leq \mathsf{DRR}(\hat{X}, \hat{Y})$ there are only $g_i \in G$ such that $(\hat{X}\,(g_i, g_{i-1})) = \mathsf{nbo}$, $(\hat{X}\,(g_i, g_{i+1})) = \mathsf{nbo}$, $(\hat{Y}\,(g_i, g_{i-1})) = \mathsf{nbo}$, and $(\hat{Y}\,(g_i, g_{i+1})) = \mathsf{nbo}$. By definition of nbo all $x \in [\![\hat{X}]\!]$ and all $y \in [\![\hat{Y}]\!]$ are proper parts of g_i that do not intersect the boundary of g_i. One can see (Figure 9 (a)) that there are enough non-boundary-parts, x and y, of g_i which are such that all relations $\rho(x, y)$ with $\mathsf{DRL}(x, y) \leq \rho(x, y) \leq \mathsf{DRR}(x, y)$ do actually occur.

Consider the case $\mathsf{PPL}(\hat{X}, \hat{Y}) \leq \rho \leq \mathsf{PPR}(\hat{X}, \hat{Y})$; it occurs if $\underline{R^9} = \overline{R^9} = \mathsf{EQ}$ and there are $g_i, g_{i+1} \in G$ such that $(\hat{X}(g_i, g_{i+1})) = (\hat{Y}(g_i, g_{i+1})) = \mathsf{bo}$ and $i = min(\hat{X}) = min(\hat{Y})$ and there are $g_j, g_{j-1} \in G$ such that $(\hat{X}(g_j, g_{j-1})) = (\hat{Y}(g_j, g_{j-1})) = \mathsf{bo}$ and $j = max(\hat{X}) = max(\hat{Y})$. By definition of bo all $x \in [\![\hat{X}]\!]$ and all $y \in [\![\hat{Y}]\!]$ do overlap. Consequently, the relations $\mathsf{DRL}(x, y)$ and $\mathsf{DRR}(x, y)$ cannot occur. Since $\underline{R^9} = \overline{R^9} = \mathsf{EQ}$ there are $x \in [\![\hat{X}]\!]$ and $y \in [\![\hat{Y}]\!]$ such that $\mathsf{EQ}(x, y)$. Consider Figure 9 (b) there is enough freedom for $x \in [\![\hat{X}]\!]$ and $y \in [\![\hat{Y}]\!]$ such that $\mathsf{PPL}(\hat{X}, \hat{Y}) \leq \rho(x, y) \leq \mathsf{PPR}(\hat{X}, \hat{Y})$ e.g., $\mathsf{POL}(x, y_4)$, $\mathsf{POR}(x, y_1)$, $\mathsf{PPL}(x, y_4)$, $\mathsf{POR}(x, y_3$, and $\mathsf{EQ}(x, y_2)$.

Secondly, in order to show (i) for $\underline{R_c^9} \leq \rho \leq \overline{R_c^9} \leq \mathsf{EQ}$ two cases need to be considered: (a) There are $g_i, g_{i-1} \in G$ such that $(\hat{X}(g_i, g_{i-1})) = \mathsf{bo}$ and $(\hat{Y}(g_i, g_{i-1})) = \mathsf{bo}$. In this case we have $\underline{R_c^9}(\hat{X}, \hat{Y}) = \mathsf{POL} \leq \rho(x, y)$ by Lemma 4 and Lemma 6. (b) Otherwise: In this case it is necessary to consider each of the three components $\hat{X} \,\underline{\triangle}\, \hat{Y} \not\equiv \perp$, $\hat{X} \,\underline{\triangle}\, \hat{Y} \equiv \hat{X}$, and $\hat{X} \,\underline{\triangle}\, \hat{Y} \equiv \hat{Y}$. Since we have $\underline{R_c^9} \leq \rho \leq \overline{R_c^9} \leq \mathsf{EQ}$ it is sufficient to show that if $\hat{X} \,\underline{\triangle}\, \hat{Y} \neq \perp$ then for all $x \in [\![\hat{X}]\!]$ and all $y \in [\![\hat{Y}]\!]$ $x \wedge y \neq \perp$ and similar for $\hat{X} \,\underline{\triangle}\, \hat{Y} \equiv \hat{X}$, and $\hat{X} \,\underline{\triangle}\, \hat{Y} \equiv \hat{Y}$.

If $\hat{X} \,\underline{\triangle}\, \hat{Y} \neq \bot$ then there are $g_i, g_j \in G$ with $\hat{X}\,(g_i, g_j) = \mathsf{fo}$ and $\hat{Y}\,(g_i, g_j) \neq \mathsf{no}$ or $\hat{X}\,(g_i, g_j) \neq \mathsf{no}$ and $\hat{Y}\,(g_i, g_j) = \mathsf{fo}$. By definition of nbo and fo we have for all $x \in [\![\hat{X}]\!]$ and all $y \in [\![\hat{Y}]\!]$ $x \wedge y \neq \bot$.

If $\hat{X} \,\underline{\triangle}\, \hat{Y} = \hat{X}$ then if $\hat{X}\,(g_i, g_j) \neq \mathsf{no}$ then $\hat{Y}\,(g_i, g_j) = \mathsf{fo}$. By definition of nbo and fo we have for all $x \in [\![\hat{X}]\!]$ and all $y \in [\![\hat{Y}]\!]$ $x \wedge y = x$. Similarly if $\hat{X} \,\underline{\triangle}\, \hat{Y} = \hat{Y}$ then $x \wedge y = y$. To show (ii) for $\underline{R_c^9} \leq \rho \leq \overline{R_c^9} \leq \mathsf{EQ}$ is accomplished by a similar analysis and is omitted here.

Consider Table 3. In order to show to show (iii) for $\underline{R_c^9} \leq \rho \leq \overline{R_c^9} \leq \mathsf{EQ}$ we need to show that all sets of relations in this table can actually occur. We limit ourself to examples for $\{\mathsf{DRL}, \mathsf{PPL}\}$ and $\{\mathsf{POL}, \mathsf{PPiL}\}$. Consider Figure 9. We have $\mathsf{DRL}(x, y_2)$ and $\mathsf{POL}(x, y_1)$ in configuration (c) and we have $\mathsf{POL}(x, y_1)$ and $\mathsf{PPiL}(x, y_2)$ in configuration (d).

Thirdly, to show (i), (ii), and (iii) for $\mathsf{EQ} \leq \underline{R_c^9} \leq \rho \leq \overline{R_c^9}$ is similar and the proof is omitted here. $\square$

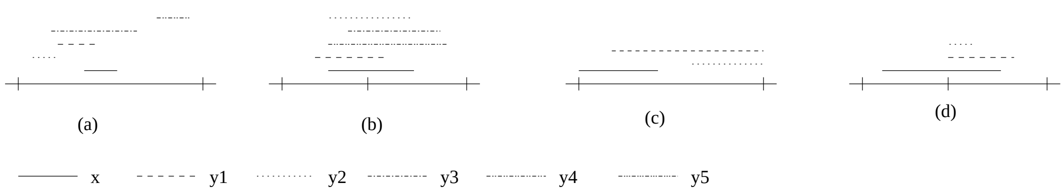


Figure 9.

8. Generalization of RCC_1^{15} relations

8.1. Syntactic generalization

Let X and Y be boundary sensitive approximations of regions x and y in a directed one-dimensional space. Since RCC_1^9 relations are defined for one-dimensional intervals and convex hulls, $\hat{x}$, of complex one-dimensional regions, x, we need to apply the convex hull operator $\hat{X}$ in the approximation domain. We consider the following pair of triples of truth values:

$$(\hat{X} \,\underline{\triangle}\, \hat{Y} \not\approx \bot,\ \hat{X} \,\underline{\triangle}\, \hat{Y} \approx \hat{X},\ \hat{X} \,\underline{\triangle}\, \hat{Y} \approx \hat{Y}), (\hat{X} \,\overline{\wedge}\, \hat{Y} \not\approx \bot,\ \hat{X} \,\overline{\wedge}\, \hat{Y} \approx \hat{X},\ \hat{X} \,\overline{\wedge}\, \hat{Y} \approx \hat{Y}).$$

where

$$\hat{X} \wedge \hat{Y} \not\approx \bot = \begin{cases} \mathsf{T} & \hat{X} \,\underline{\wedge}\, \hat{Y} \not\equiv \bot = \mathsf{T} \\ \mathsf{MLO} & \hat{X} \,\underline{\wedge}\, \hat{Y} \not\equiv \bot = \mathsf{M} \text{ and } \hat{X} \,\underline{\wedge}\, \hat{Y} \not\sim \bot = \mathsf{FLO} \\ \mathsf{MRO} & \hat{X} \,\underline{\wedge}\, \hat{Y} \not\equiv \bot = \mathsf{M} \text{ and } \hat{X} \,\underline{\wedge}\, \hat{Y} \not\sim \bot = \mathsf{FRO} \\ \mathsf{FLO} & \hat{X} \,\underline{\wedge}\, \hat{Y} \not\equiv \bot = \mathsf{F} \text{ and } \hat{X} \,\underline{\wedge}\, \hat{Y} \not\sim \bot = \mathsf{FLO} \\ \mathsf{FRO} & \hat{X} \,\underline{\wedge}\, \hat{Y} \not\equiv \bot = \mathsf{F} \text{ and } \hat{X} \,\underline{\wedge}\, \hat{Y} \not\sim \bot = \mathsf{FRO} \end{cases}$$

and where

$$\hat{X} \,\underline{\wedge}\, \hat{Y} \approx x = \begin{cases} \mathsf{T} & \hat{X} \,\underline{\wedge}\, \hat{Y} \equiv \hat{X} = \mathsf{T} \\ \mathsf{MLI} & \hat{X} \,\underline{\wedge}\, \hat{Y} \equiv \hat{X} = \mathsf{M} \text{ and } \hat{X} \,\underline{\wedge}\, \hat{Y} \sim \hat{Y} = \mathsf{FLI} \\ \mathsf{MRI} & \hat{X} \,\underline{\wedge}\, \hat{Y} \equiv \hat{X} = \mathsf{M} \text{ and } \hat{X} \,\underline{\wedge}\, \hat{Y} \sim \hat{Y} = \mathsf{FRI} \\ \mathsf{FLO} & \hat{X} \,\underline{\wedge}\, \hat{Y} \equiv \hat{X} = \mathsf{F} \text{ and } \hat{X} \,\underline{\wedge}\, \hat{Y} \sim \hat{X} = \mathsf{FLO} \\ \mathsf{FLI} & \hat{X} \,\underline{\wedge}\, \hat{Y} \equiv \hat{X} = \mathsf{F} \text{ and } \hat{X} \,\underline{\wedge}\, \hat{Y} \sim \hat{X} = \mathsf{FLI} \\ \mathsf{FRO} & \hat{X} \,\underline{\wedge}\, \hat{Y} \equiv \hat{X} = \mathsf{F} \text{ and } \hat{X} \,\underline{\wedge}\, \hat{Y} \sim \hat{X} = \mathsf{FRO} \\ \mathsf{FRI} & \hat{X} \,\underline{\wedge}\, \hat{Y} \equiv \hat{X} = \mathsf{F} \text{ and } \hat{X} \,\underline{\wedge}\, \hat{Y} \sim \hat{X} = \mathsf{FRI} \end{cases}$$

and similarly for $\hat{X} \,\overline{\wedge}\, \hat{Y} \not\approx \bot$, $\hat{X} \,\overline{\wedge}\, \hat{Y} \approx \hat{X}$, $\hat{X} \,\underline{\wedge}\, \hat{Y} \approx \hat{Y}$, and $\hat{X} \,\overline{\wedge}\, \hat{Y} \approx \hat{Y}$.

Each of the above triples provides a RCC_1^{15} relation, so the relation between $\hat{X}$ and $\hat{Y}$ can be measured by a pair of RCC_1^{15} relations. These relations will be denoted by $\underline{R^{15}}$ and $\overline{R^{15}}(\hat{X}, \hat{Y})$.

Theorem 9. The pairs of relations

$$(\min(\underline{R^{15}}(\hat{X}, \hat{Y}), \overline{R^{15}}(\hat{X}, \hat{Y})), \max(\underline{R^{15}}(\hat{X}, \hat{Y}), \overline{R^{15}}(\hat{X}, \hat{Y})))$$

that can occur are all pairs (a, b) where $a \leq b \leq \mathsf{EQ}$ and $\mathsf{EQ} \leq a \leq b$ with the exception $(\mathsf{DCL}, \mathsf{ECL})$, $(\mathsf{DCL}, \mathsf{TPPL})$, $(\mathsf{ECL}, \mathsf{POL})$, $(\mathsf{ECL}, \mathsf{TPPL})$, $(\mathsf{ECL}, \mathsf{TPPiL})$, $(\mathsf{ECL}, \mathsf{NTPPL})$, $(\mathsf{ECL}, \mathsf{NTPPiL})$, $(\mathsf{ECL}, \mathsf{EQ})$, $(\mathsf{POL}, \mathsf{TPPL})$, $(\mathsf{POL}, \mathsf{TPPiL})$, $(\mathsf{POL}, \mathsf{NTPPL})$, $(\mathsf{POL}, \mathsf{NTPPiL})$, $(\mathsf{POL}, \mathsf{EQ})$, $(\mathsf{TPPL}, \mathsf{NTPPL})$, $(\mathsf{TPPiL}, \mathsf{NTPPiL})$, $(\mathsf{TPPL}, \mathsf{EQ})$, $(\mathsf{TPPiL}, \mathsf{EQ})$, $(\mathsf{NTPPL}, \mathsf{EQ})$, $(\mathsf{NTPPiL}, \mathsf{EQ})$, and the corresponding pairs with both components greater than or equal to EQ.

Proof. RCC_1^{15} relations are refinements of RCC_1^{8} and refinements of RCC_1^{9} relations. Consequently, if a pair of relations cannot occur in the RCC_1^{8} case or in the RCC_1^{9} case then the corresponding refinements cannot occur in the RCC_1^{15} case. By Theorem 7 there are no (a, b) such that $\underline{R^{15}}(\hat{X}, \hat{Y}) < \mathsf{EQ} < \overline{R^{15}}(\hat{X}, \hat{Y})$ or $\overline{R^{15}}(\hat{X}, \hat{Y}) < \mathsf{EQ} < \underline{R^{15}}(\hat{X}, \hat{Y})$. Pairs (a, b) with $a \leq b \leq \mathsf{EQ}$ and pairs (a, b) with $\mathsf{EQ} \leq a < b$ are governed by Theorem 3 with the additional constraint that RCC_1^{15} relations are only defined for intervals, i.e., for $\hat{X}$ rather than for approximations X in general. Consequently the pairs $(\mathsf{POL}, \mathsf{TPPL})$, $(\mathsf{POL}, \mathsf{TPPiL})$, $(\mathsf{POL}, \mathsf{NTPPL})$, $(\mathsf{POL}, \mathsf{NTPPiL})$, $(\mathsf{POL}, \mathsf{EQ})$ and the corresponding pairs with both components greater than EQ cannot occur, in correspondence to Theorem 7.4. A Haskell program generating all remaining cases can be obtained from the author. ☐

Consider Table 4. It lists the pairs (a, b) with $a \leq b \leq \mathsf{EQ}$ and $a, b \not\in \{\mathsf{TPPiL}, \mathsf{NTPPiL}\}$ and points to the relevant cases in Theorems 7 and 3 that apply.

$\underline{R^{15}} \backslash \overline{R^{15}}$	DCL	ECL	POL	TPPL	NTPPL	EQ
DCL	{DCL}	3.1	{DCL, …, POL}	3.3	{DCL, …, NTPPL}	{DCL, …, EQ}
ECL	3.1	{ECL}	7.4	3.4	3.3	3.4
$\underline{R^{15}}$ = POL	3.1	3.1	{POL}	7.4	7.4	7.4
$\underline{R^{15}_c}$ = POL	3.1	3.1	{POL}	7.4	{POL, …, NTPPL}	{POL, …, EQ}
TPPL	3.1	3.1	3.1	{TPPL}	3.1	3.2
NTPPL	3.1	3.1	3.1	3.1	{NTPPL}	3.2
EQ	3.1	3.1	3.1	3.1	3.1	{EQ}

Table 4

Syntactically possible pairs (a, b) of minimal and maximal RCC_1^{15} relations with with $a \leq b \leq \mathsf{EQ}$ and $a, b \not\in \{\mathsf{TPPiL}, \mathsf{NTPPiL}\}$.

8.2. Correspondence of semantic and syntactic generalization

Corresponding to the generalization of the RCC_1^8 and the RCC_1^9 relations semantic corrections are needed in order to generalize RCC_1^{15} relations between intervals, x and y, to pairs of RCC_1^{15} relations between approximations $\hat{X}$ and $\hat{Y}$.

At the syntactic level the pair $(\mathsf{DCL}, \mathsf{EQ})$ represents the most indeterminate case. As in the RCC_1^9 case it occurs if there is a single $g_i \in G$ such that $(\hat{X}\,(g_i, g_{i-1})) = \mathsf{nbo}$, $(\hat{X}\,(g_i, g_{i+1})) = \mathsf{nbo}$, $(\hat{Y}\,(g_i, g_{i-1})) = \mathsf{nbo}$, and $(\hat{Y}\,(g_i, g_{i+1})) = \mathsf{nbo}$. Since $(\mathsf{DCL}, \mathsf{EQ})$ is consistent with $(\mathsf{EQ}, \mathsf{DCR})$ and $(\mathsf{DCL}, \mathsf{EQ})$ was chosen arbitrarily, $(\mathsf{DCL}, \mathsf{EQ})$ is corrected for semantic reasons to $(\mathsf{DCL}, \mathsf{DCR})$. The corrected relation will be denoted by $\overline{R^{15}_c}$.

Lemma 4 and Lemma 6 tell us that if there are $g_i, g_j \in G$ such that $(\hat{X}(g_i, g_j)) = (\hat{Y}(g_i, g_j)) = \mathsf{bo}$ then there are $x \in [\![\hat{X}]\!]$ and $y \in [\![\hat{Y}]\!]$ with $\mathsf{PO}(x, y)$. As in the RCC_1^9 case we need to distinguish two cases: (a) We have $\underline{R_c^{15}}(\hat{X}, \hat{Y}) = \mathsf{POL}$ if there are $g_i, g_{i+1} \in G$ such that $(\hat{X}(g_i, g_{i+1})) = (\hat{Y}(g_i, g_{i+1})) = \mathsf{bo}$ and $i = min(\hat{X}) = min(\hat{Y})$, and $\underline{R_c^{15}}(\hat{X}, \hat{Y}) = \underline{R^{15}}(\hat{X}, \hat{Y})$ otherwise. (b) We have $\overline{R_c^{15}}(\hat{X}, \hat{Y}) = \mathsf{POR}$ if there are $g_j, g_{j-1} \in G$ $(\hat{X}(g_j, g_{j-1})) = (\hat{Y}(g_j, g_{ji1})) = \mathsf{bo}$ and $j = max(\hat{X}) = max(\hat{Y})$. Otherwise we have $\overline{R_c^{15}} = \overline{R^{15}}$ or $\overline{R_c^{15}} = \mathsf{DCR}$.

Consider Table 4 and ignore the row $\underline{R^{15}} = \mathsf{POL}$. It shows $\mathcal{SYN}(\hat{X}, \hat{Y})$ for $\underline{R_c^{15}} \leq \overline{R_c^{15}} \leq \mathsf{EQ}$. The pairs $(\mathsf{POL}, \mathsf{NTPPL})$, $(\mathsf{POL}, \mathsf{NTPPiL})$, and $(\mathsf{POL}, \mathsf{EQ})$ do occur due to the semantic correction of the pairs $(\mathsf{NTPPL}, \mathsf{NTPPL})$, $(\mathsf{NTPPiL}, \mathsf{NTPPiL})$, and $(\mathsf{EQ}, \mathsf{EQ})$ in cases where Lemma 4 and Lemma 6 apply. Theorem 10 tells us that the semantic generalization produce the same table.

Let the syntactic generalization of RCC_1^{15} be defined as

$$\mathcal{SYN}(\hat{X}, \hat{Y}) = (\min\{\underline{R_c^{15}}(\hat{X}, \hat{Y}), \overline{R_c^{15}}(\hat{X}, \hat{Y})\}, \max\{\underline{R_c^{15}}(\hat{X}, \hat{Y}), \overline{R_c^{15}}(\hat{X}, \hat{Y})\}),$$

where $\underline{R_c^{15}}$ and $\overline{R_c^{15}}$ are defined as discussed above.

Theorem 10. For approximations $\hat{X}$ and $\hat{Y}$ syntactic and semantic generalization of RCC_1^{15} relations are equivalent in the sense that

$$\begin{aligned} \mathcal{SEM}(\hat{X}, \hat{Y}) = \{\rho \in \mathsf{RCC}_1^{15} \ \mid & \min\{\underline{R_c^{15}}(\hat{X}, \hat{Y}), \overline{R_c^{15}}(\hat{X}, \hat{Y})\} \\ & \leq \rho \leq \max\{\underline{R_c^{15}}(\hat{X}, \hat{Y}), \overline{R_c^{15}}(\hat{X}, \hat{Y})\}\}, \end{aligned}$$

where RCC_1^{15} is the set of RCC_1^{15} relations and $\leq$ is the ordering in the RCC_1^{15} lattice.

Proof. We consider three cases: $\underline{R_c^{15}} \leq \rho \leq \overline{R_c^{15}}$ with $\underline{R_c^{15}} < \mathsf{EQ}$ and $\overline{R_c^{15}} > \mathsf{EQ}$, $\underline{R_c^{15}} \leq \rho \leq \overline{R_c^{15}} \leq \mathsf{EQ}$, and $\mathsf{EQ} \leq \overline{R_c^{15}} \leq \rho \leq \underline{R_c^{15}}$. For each case there are three things to demonstrate: (i) for all $x \in [\![\hat{X}]\!]$, and $y \in [\![\hat{Y}]\!]$, that $\underline{R_c^{15}}(\hat{X}, \hat{Y}) \leq \rho(x, y)$ with $\rho \in \mathsf{RCC}_1^{15}$; (ii) for all x and y as before, that $\rho(x, y) \leq \overline{R_c^{15}}(\hat{X}, \hat{Y})$; and (iii) if ρ is any RCC_1^{15} relation such that $\underline{R_c^{15}}(\hat{X}, \hat{Y}) \leq \rho \leq \overline{R_c^{15}}(\hat{X}, \hat{Y})$ then there exist particular x and y which stand in the relation ρ to each other.

Firstly, there are two cases where are $\rho_1, \rho_2 \in \mathcal{SEM}$ such that $\rho_1 < \mathsf{EQ} < \rho_2$: $\mathsf{DCL} \leq \rho \leq \mathsf{DCR}$ and $\mathsf{POL} \leq \rho \leq \mathsf{POR}$. To show that (i), (ii), and (iii) hold corresponds to Theorem 8 and is omitted here.

Secondly, (i) and (ii) for $\underline{R_c^{15}} \leq \rho \leq \overline{R_c^{15}} \leq \mathsf{EQ}$ are a consequence of Theorem 5. (iii) is a consequence of Theorem 5 except for the pairs $(\mathsf{ECL}, \mathsf{POL})$, $(\mathsf{POL}, \mathsf{TPPL})$, $(\mathsf{POL}, \mathsf{TPPiL})$, $(\mathsf{POL}, \mathsf{NTPPL})$, $(\mathsf{POL}, \mathsf{NTPPiL})$, $(\mathsf{POL}, \mathsf{EQ})$ since

RCC_1^{15} relations are only defined for intervals. The cases $(\mathsf{ECL}, \mathsf{POL})$, $(\mathsf{POL}, \mathsf{TPPL})$, and $(\mathsf{POL}, \mathsf{TPPiL})$ cannot occur since for these cases to occur all $x \in [\![\hat{X}]\!]$ and all $y \in [\![\hat{Y}]\!]$ would need to be complex regions and this is excluded by the definition of $\hat{X}$ and $\hat{Y}$. For the remaining cases it remains to show that if $\underline{R_c^{15}}(\hat{X}, \hat{Y}) \leq \rho \leq \overline{R_c^{15}}(\hat{X}, \hat{Y})$ then there exist particular x and y which stand in the relation ρ to each other. This is a consequence of Lemma 4 and 6. □

9. Discussion

In this section I discuss the results presented in this paper (1) with respect to the contributions presented in earlier literature, (2) possible generalizations, and (3) possible areas of application. I start by discussing the relationships between granularity and approximations.

9.1. Granularity

Granularity has been an important research issue for many years [28,24,15,22,13, 3,41]. In these efforts different views on granularity have been taken. In the context of this paper three views are relevant and will be discussed in this section:

1. granularity of theory, e.g. [28];
2. granularity of approximation (of some spatio-temporal domain), e.g. [3];
3. granularity of sets of qualitative relations, e.g. [24], [15].

In all cases the notion of granularity is closely related to the notion of indiscernibility, which refers to the fact that objects, properties or relations are beneath a certain level of resolution (not necessarily related to size) indiscernible, i.e., they are not capable of being distinguished. The approaches differ however in the way indiscernibility is defined and in what aspects of the matters in hand are considered indescernible. The most general view is taken in [28], which defines as indiscernible what cannot be distinguished by a given formal theory. The other approaches are more specific to certain domains, such as indiscernibility of regions of space or time [3], or indiscernibility of relations between such regions [24,15].

9.1.1. Granularity of theory

Hobbs [28] argued that granularity is based on more or less detailed formal theories, where the degree of detail refers to different kinds and numbers of predicates within a given formalized theory: $x \sim y \equiv \forall p : p(x) \leftrightarrow p(y)$ (where p ranges over predicates of the theory and $\sim$ symbolizes indiscernibility). Following this view of granularity I considered four theories and the granularity they dictate: (i) Mereology, (ii) Mereotopology, (iii) Mereology + order, and (iv) Mereotopology + order. This resulted in four sets of qualitative relations between temporal regions that are summarized in Table 5.

Mereology [38] is a formal theory of parts of wholes. Its basic relation is the relation $a \leq b$ denoting that x is a part of y. The relationship between mereology and the meet-based formalization in this paper is given by the well-known equivalence: $x \leq y \equiv x \wedge y = x$. At the mereological level we were able to distinguish the five RCC5 relations DR, PO, $\mathsf{PP}(i)$, and EQ. In terms of mereotopology [42] we also took the topological distinction between the interior and the boundary of an object or region into account (e.g., $x \wedge y \neq \bot$ vs. $\delta x \wedge \delta y \neq \bot$). This resulted in the definition of the eight RCC_1^8 relations, where the relation DR was refined into DC and EC and $\mathsf{PP}(i)$ was refined into $\mathsf{TPP}(i)$ and $\mathsf{NTPP}(i)$. At this level of granularity we were able to specify that the invasion of Poland is a part of World War 2 and that both share a boundary using the relation TPP. Furthermore we are able to distinguish the invasion of Poland from the German attempt to occupy Leningrad, since the latter is a non-tangential proper part of World War 2.

In terms of mereotopology we were able to formalize the distinction between interior and boundary but not that between beginning and ending. We then enriched mereology by primitives describing ordering relations such as *before* and *after* (i.e., left and right in a one-dimensional directed space) and defined the RCC_1^9 relations accordingly. Every RCC5 relation except EQ was refined into two relations, e.g., DR was refined into DRL and DRR. This enabled us to say that World War 1 was before World War 2 rather than that they were only disjoint. The theory with the finest granularity in Hobbs sense was attained by enriching mereotopology by means of such ordering relations. This resulted in the RCC_1^{15} relations that are similar to the well known Allen-relations and that allow us, for example, to say that the beginning of the invasion of Poland coincides with the beginning of World War 2.

The granularity of the underlying theory also dictates different kinds of approximations: the boundary sensitive and the boundary insensitive. Boundary *in*sensitive approximations are based on an underlying mereological theory. They are defined using approximation functions of signature $\alpha_3 : R \rightarrow (G \rightarrow \Omega_3)$ (Section 4). Every temporal region $r \in R$ is approximated by a function that measures the degree of overlap between r and each partition cell $g \in G$ based on an underlying mereological theory. In Ω_3 we distinguished: full overlap, that is ($g \mapsto_r \mathsf{fo} \equiv \mathsf{PP}(g, r)$ or $\mathsf{EQ}(g, r)$) in

RCC5	boundary and order insensitive qualitative relations, based on mereology only;
RCC_1^8	boundary sensitive and order insensitive qualitative relations, based on mereotopology;
RCC_1^9	boundary insensitive and order sensitive qualitative relations, based on mereology and order;
RCC_1^{15}	boundary sensitive and order sensitive qualitative relations, based on mereotopology and order.

Table 5

Qualitative relations between temporal regions based on theories on different levels of granularity.

terms of RCC5 ; partial overlap ($g \mapsto_r \mathsf{po} \equiv \mathsf{PO}(g, r)$ or $\mathsf{PP}(r, g)$); and non-overlap ($g \mapsto_r \mathsf{no} \equiv \mathsf{DR}(g, r)$).

Boundary sensitive approximation are based on an underlying mereotopological theory reflected by approximation functions of signature $\alpha_3 : R \to (G \times G \to \Omega_4)$. The degree of overlap, measured by Ω_4, takes the relationship between the boundary-point shared by the adjacent partition cells g_i and g_j and the approximated region r into account. We distinguished: no overlap (no), partial overlap without coverage of the boundary-point (nbo), partial overlap with coverage of the boundary-point (bo), and full overlap (fo). These definitions can be represented easily in terms of the mereotopological relations RCC_1^8 .

Assuming identical underlying partitions boundary sensitive approximations have a finer granularity than boundary insensitive in the sense that more distinctions can be made. If n is the number of partition cells, then there are 3^n distinctions possible in terms of boundary insensitive approximations and $4^{2(n+1)}$ distinctions possible in terms of boundary sensitive approximations (assuming we include complex intervals).

9.1.2. Granularity of approximation

The most obvious form of granularity occurs at the level of the underlying regional partition, for example a partition into nano-seconds vs. a partition into days. Granularity in this context refers to the size of the minimal cells of the partition. The corresponding indiscernibility relation is based on identity of approximation. We used the notation $[\![X]\!]$ in order to refer to the set of regions indescernible in terms of the approximation X (with X a function from G to Ω_3 in the case of boundary insensitive approximation and with $x \sim y \equiv X = Y$). Given a partition G, with equal cell size and a finite set of regions R, whose elements are evenly distributed with respect to G, we can say intuitively that the finer the partition (the more cells) the more distinctions we can make and the fewer the number of elements of R which will be indiscernible, i.e., the fewer the number of elements of R which will be in every $[\![X]\!]$.

This kind of granularity relates to the indiscernibility of size and location differences below a minimal level of resolution (below the size of the minimal cells). Partitions have different granularities and partitions of different granularity can be hierarchically organized in the sense that cells of the finer partition subsume cells at coarser levels.

It is important to see that the existence of minimal cells in partitions does not imply that there are atoms on the side of the objects (or intervals) which we approximate. This may or may not be the case. If there are no atoms in the domain we are approximating then, of course, we can define finer and finer partitions with smaller and smaller minimal cells as well. Examples for domains which do have temporal atoms can be found in the digital world where atomic temporal units are defined by the tact of the CPU. In this case (minimal) partition cells coincide with the atoms in the domain. With respect to the cells in those partitions the beginnings and endings of all events coincide with boundaries of atomic units and the partial overlap of events and partition cells cannot occur. Approximations with respect to those partitions are exact in the sense defined above.

9.1.3. Granularity of sets of qualitative relations

If we have a set of jointly exhaustive and pair-wise disjoint (JEPD) relations then we can form relations of lower granularity as disjunctions of relations on the base level. For example, the relation of overlap, $O(x, y)$, can be defined as the disjunction of RCC5 base relations: $O(x, y) \equiv \mathsf{PO}(x, y)$ or $\mathsf{PP}(x, y)$ or $\mathsf{PPi}(x, y)$ or $\mathsf{EQ}(x, y)$. If we define $(a, b) \sim (c, d) \equiv O(a, b) \leftrightarrow O(c, d)$ then the relations $\mathsf{PO}, \mathsf{PP}(i)$ and EQ are indescernible with respect to $\sim$. The composition of the base relation often yields such relations of coarser level of resolution [24,15,37]. A similar effect occurred when reasoning about approximations.

Consider the RCC5 relations. As a set these base-relations are jointly exhaustive and pair-wise disjoint and form the RCC5 lattice as depicted in Figure 1. Theorems 1 and 2 show that subsets of these relations form relations of coarser levels of granularity. We obtained these relations by performing the syntactic generalization (Theorem 1). This produced pairs $(\underline{R}, \overline{R}) \subset RCC5 \times RCC5$ with $\underline{R} \leq \overline{R}$ and $(\underline{R}, \overline{R}) \neq (\mathsf{PP}(i), \mathsf{EQ})$ which contain the base relations, $\underline{R} = \overline{R}$, as special cases. The semantic generalization (Theorem 2) showed that each pair $(\underline{R}, \overline{R})$ represents a set of relation that are indistinguishable at the level of approximations X and Y with $R(X, Y) = (\underline{R}, \overline{R})$. Indistinguishable means that for any of the base-relations, $\rho \in RCC5$, constrained by the pairs, $\underline{R} \leq \rho \leq \overline{R}$, there are temporal regions $x \in [\![X]\!]$ and $y \in [\![Y]\!]$, which are such that $\rho(x, y)$ holds. Since the regions $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ are indistinguishable, the relations ρ constrained by $(\underline{R}, \overline{R})$ are indistinguishable as well. Notice, however, that the relations $(\underline{R}, \overline{R})$ are at coarser levels of granularity not necessarily jointly exhaustive and pair-wise disjoint. As a set these pairs form a lattice if we define $(a, b) \preceq (c, d) \equiv c \leq a$ and $b \leq d$, where $\leq$ is the ordering of the RCC5 lattice.

The generalizations of the base relations RCC_1^8 , RCC_1^9 , and RCC_1^{15} generate sets of relations of coarser granularity in the same way as discussed for the RCC5 relations. This was demonstrated in Theorems 3, 5, 7, 8, 9 , and 10.

9.2. *Limitations*

In this paper I focussed on the temporal domain. In related work it has been shown that the ideas presented here can be generalized to two-dimensional space [8] (boundary sensitive and boundary insensitive approximations) and to arbitrary domains [34], [41], [5]. The higher the dimensionality of the space, the more distinctions are possible in terms of mereotopological relations between regions and boundaries shared by neighboring partition cells. It is the author's belief that the generalization of boundary sensitive approximations to spaces of dimension higher than three is not useful. Another limitation is that all regions need to satisfy the axioms of the RCC-theory. This excludes discrete spaces. Consequently, the techniques presented in this paper cannot be applied to approximations themselves, i.e., we cannot approximate approximations.

Another major limitation of the present approach is that it is limited to approximations within a single partition. For realistic applications approximations with respect to multiple partitions need to be considered. In this context the formalization of the meet operation between approximations in distinct but hierarchically organized partitions are of particular interest. An example would be to derive possible relations between an event or process approximated with respect to a partitions of the time-line into hours and another event or process approximated within a partition into fifteen minute slots. In this context it is necessary to bring together the results that have been achieved in the domain of time granularities (approximations that are exact, in the language of this paper) and to generalize those results to approximations using the methodology of syntactic and semantic generalization as discussed above. Considering approximations with respect to two partitions with cells that lie completely skew to each other is much harder. In this context the incorporation of the GIS-technique of spatial enforcement [31] seems to be promising.

A third point of limitation is that, as already mentioned above, the treatment of complex intervals in the presented formalism is not satisfactory yet. To apply the techniques presented in this paper to generalized intervals in Ligozat's sense [32] is an important open question.

9.3. Potential fields of application

The contribution of this paper is mainly theoretical in nature and concrete applications are the subject of future work. As already sketched in Section 2, it is the conviction of the author that approximate representations of events or processes are needed whenever the boundaries of the latter lie skew to the boundaries of our partitions of the time-line. This was demonstrated in particular in the context of bona fide occurrents.

I see an important field of application of the formalism presented in this paper in (spatio-)temporal databases. The explicitly approximate representation of the temporal location of events and processes is able more naturally and correctly to represent the nature of the relationships between human partitions and events and processes. This also reflects more explicitly the limits of human knowledge as concerns temporal location. Measurement has only limited resolution and observations of temporal change are often not made continuously but only at scattered intervals. Based on the proposed formalism queries about temporal relations would then yield results at a coarser level of granularity. These results, however, would be more accurate in the sense that this granularity represents the actual extent of our knowledge rather than an artificial crisping introduced to compensate for the limitations of the underlying representation.

The need for approximate reasoning about temporal location becomes even clearer in relation to do reasoning about the relations between the actual temporal location of an event or process, the partition of the time-line with respect to which we represent this temporal location, and the partition of the time-line created by the successive update operations of our database system. Reasoning of this kind is needed in order to improve the robustness and the quality of the results generated by (spatio-)temporal query engines.

Another important area of application is the treatment of vagueness. There are large classes of events and processes that are inherently vague in the sense that there is no determinate way to measure their exact beginning and ending. For example: When did the last ice-age end or the last rainstorm? When does a political or economical crisis begin and end? When did your flu start and when were you once again healthy? Often we can only approximate the temporal location of such events with respect to some appropriate partition of the time-line. For example, I felt well on Saturday. When I measured my temperature I had a fever on Monday and on Tuesday; and I felt healthy again on Thursday morning. Even if our knowledge of the beginning and ending of an event or process is vague, it is possible using the methods sketched above to specify the relevant approximate locations *exactly*. For further discussion of the presented or similar formalisms in the context of vagueness see [14], [7], [6], [17], [11].

10. Conclusions

In this paper I defined methods of approximate qualitative temporal reasoning. An approximation represents a set of regions that are indescernible in their relations to a partition of the time-line (assuming a fixed set of relations to be considered). Approximate temporal reasoning is performed by deriving possible relations between two temporal regions given only their approximations. At the formal level I proposed a methodology that allows us to define relations between temporal regions in such a way that we can generalize these definitions to approximations by relatively simple syntactic operations.

The methodology is based on three major components: (1) Sets of qualitative relations between regions, which are defined in terms of the meet operation over the domain of regions. As a set these relations must form a lattice with a bottom and top element. (2) Approximations of regions with respect to a regional partition of the underlying space. (3) Pairs of meet operations on those approximations, which constrain the meet operation on regions.

Based on these components syntactic and semantic generalizations of qualitative relations between one-dimensional regions were defined. Generalized relations hold between approximations of regions rather than between the regions themselves. Syntactic generalization is based on replacing the meet operation in the definitions of relations between regions by its minimal and maximal counterparts on approximations. Semantically, syntactic generalizations yield upper and lower bounds (within the underlying lattice structure) on relations that can hold between the corresponding approximated regions.

In the temporal domain I defined four sets of topological relations between one-dimensional regions:

RCC5 Boundary *in*sensitive binary topological relations between regions in a *non*-directed one-dimensional space.

RCC_1^9 Boundary *in*sensitive binary topological relations between maximally connected regions (intervals) in a directed one-dimensional space.

RCC_1^8 Boundary sensitive binary topological relations between regions in a *non*-directed one-dimensional space.

RCC_1^{15} Boundary sensitive binary topological relations between maximally connected regions (intervals) in a directed one-dimensional space.

For each of these sets of relations between regions I discussed the syntactic and semantic generalization for the corresponding approximations and showed the equivalence of both approaches. This provides the formal basis for qualitative temporal reasoning about approximate location in time.

Approximate representation and reasoning was shown to be an generalization of time-granularities in the sense of [3]. Based on the proposed framework we are able to represent the fact that events and processes often lie skew to our partitions of the time-line (granularities) and we are able to model the resulting limits of our knowledge about the exact relations between those events and processes explicitly.

Acknowledgements

Thanks are due to Barry Smith and the anonymous reviewers for their helpful comments. The support by DARPA under the Command Post of the Future program and the National Science Foundation under the Research on Learning and Education program is gratefully acknowledged.

References

[1] J. F. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843, 1983.

[2] C. Bettini, S. Jajodia, and X. S. Wang. *Time Granularities in Databases, Data Mining, and Temporal Reasoning*. Springer-Verlag, 2000.

[3] C. Bettini, X. Wang, and S. Jajodia. A general framework for time granularity and its application to temporal reasoning. *Annals of Mathematics and Artificial Intelligence*, 22:29–58, 1998.

[4] T. Bittner. Rough sets in spatio-temporal data mining. In K. Hornsby and J. Roddick, editors, *International Workshop on Temporal, Spatial, and Spatio-Temporal Data Mining*, volume 2007 of *Lecture Notes in Artificial Intelligence*. Springer Verlag, 2001.

[5] T. Bittner and B. Smith. A taxonomy of granular partitions. In *Proceedings of the Conference on Spatial Information Theory (COSIT 2001)*, Lecture Notes in Computer Science (LNCS). Springer Verlag, 2001.

[6] T. Bittner and B. Smith. Vagueness and granular partitions. In *Proceedings of the Second International Conference on Formal Ontology in Information Systems (FOIS 2001)*, Sheridan Printing Company, Inc., 2001.

[7] T. Bittner and J. G. Stell. Rough location of vague objects. *geoinformatica*, 2001.

[8] T. Bittner and J. G. Stell. Rough sets in approximate spatial reasoning. In *Proceedings of the Second International Conference on Rough Sets and Current Trends in Computing (RSCTC'2000)*, Lecture Notes in Computer Science (LNCS). Springer Verlag, 2000.

[9] T. Bittner and J. G. Stell. A boundary-sensitive approach to qualitative location. *Annals of Mathematics and Artificial Intelligence*, 24:93–114, 1998.

[10] R. Casati and A. Varzi. The structure of spatial localization. *Philosophical Studies*, 82(2):205–239, 1995.

[11] L. Chittaro and C. Combi. Extending the event calculus with temporal granularity and indeterminacy. In *Proceedings of the AAAI 2000 Workshop on Spatial and Temporal Granularity*, Austin, TX, USA, 2000. AAAI Press, Menlo Park, CA, USA,.

[12] L. Chittaro and A. Montanari. Temporal representation and reasoning in artificial intelligence: Issues and approaches. *Annals of Mathematics and Artificial Intelligence*, 28, 2000.

[13] Emanuele Ciapessoni, Edoardo Corsetti, Angelo Montanari, and Pierluigi San Pietro. Embedding time granularity in a logical specification language for synchronous real-time systems, 1993.

[14] A. G. Cohn and N. M. Gotts. The 'egg-yolk' representation of regions with indeterminate boundaries. In P. Burrough and A. U. Frank, editors, *Geographic Objects with Indeterminate Boundaries*, GISDATA Series II. Taylor and Francis, London, 1996.

[15] A. G. Cohn, B. Bennett, J. Goodday, and N. Gotts. Qualitative spatial representation and reasoning with the region connection calculus. *geoinformatica*, 1(3):1–44, 1997.

[16] A. G. Cohn, J. M. Gooday, and B. Bennett. A comparison of structures in spatial and temporal logics. In R. Casti, B. Smith, and G. White, editors, *Philosophy and the Cognitive Science: Proceedings of the 16th International Wittgenstein Symposium*. Hölder-Pichler-Tempsky, 1994.

[17] C. Combi. Representing absolute time expressions with vagueness, indeterminacy, and different granularities. In *Proceedings of the AAAI 2000 Workshop on Spatial and Temporal Granularity*, Austin, TX, USA, 2000. AAAI Press, Menlo Park, CA, USA,.

[18] C. Combi, F. Pinciroli, and G. Pozzi. Managing different time granularities of clinical information by an interval-based temporal data model. *Methods of Information in Medicine*, 34(5):458–474, 1995.

[19] E. Corsetti, A. Montanari, and E. Ratto. Dealing with different granularities in formal specifications of real-time systems. *Journal of Real-Time Systems*, 3(2), 1991.

[20] C. Dyreson and R. Snodgrass. Temporal granularity. In R. T. Snodgrass, editor, *The TSQL2 Temporal Query Language*, pages 347–385. Kluwer Academic Press, 1995.

[21] Max J. Egenhofer and Robert D. Franzosa. Point-set topological spatial relations. *International Journal of Geographical Information Systems*, 5(2):161–174, 1991.

[22] Jerome Euzenat. An algebraic approach for granularity in qualitative space and time representation. In *Proceedings of the IJCAI-95*, pages 894–900, 1995.

[23] A. U. Frank. Qualitative temporal reasoning in GIS - ordered time scales. In T.C. Waugh and R.G. Healey, editors, *SDH*, volume 1, Edinburgh, Scotland, UK, 1994. Taylor & Francis.

[24] C. Freksa. Temporal reasoning based on semi-intervals. *Artificial Intelligence*, 54:199–227, 1992.

[25] P. Geach. Some problems about time. *Proceedings of the British Academy*, 11, 1966.

[26] J. M. Goodday and A. G. Cohn. Conceptual neighborhoods in temporal and spatial reasoning. In *ECAI-94 Spatial and Temporal Reasoning Workshop*, 1994.

[27] I. A. Goralwalla, Y. Leontiev, M. T. Özsu, D. Szafron, and C. Combi. Temporal granularity for unanchored temporal data. In G. Gardarin, J. C. French, N. Pissinou, K. Makki, and L. Bouganim, editors, *Proceedings of the 1998 ACM CIKM International Conference on Information and Knowledge Management, Bethesda, Maryland, USA, November 3-7, 1998*, pages 414–423. ACM, 1998.

[28] J. Hobbs. Granularity. In *Proceedings of the IJCAI 85*, pages 432–435, 1985.

[29] P. Ladkin. Time representation: A taxonomy of interval relations. In *Proceedings of AAAI-86, Philadelphia*, pages 360–366, 1986.

[30] P. Ladkin. *The Logic of Time Representation*. PhD thesis, University of California, 1987.

[31] R. Laurini and D. Thompson. *Fundamentals of Spatial Information Systems.* The APIC series. Academic Press, 1994.

[32] G. Ligozat. Generalized intervals: A guided tour. In *Proceedings of the ECAI-98 Workshop on Spatial and Temporal Reasoning*, Brighton, UK, 1998.

[33] J. McTaggart. *The Nature of Existence*, volume II. Cambridge University Press, 1927.

[34] Z. Pawlak. Rough sets. *Internat. J. Comput. Inform*, 11:341–356, 1982.

[35] Z. Pawlak. *Rough sets : theoretical aspects of reasoning about data*. Theory and decision library. Series D, System theory, knowledge engineering, and problem solving ; v. 9. Kluwer Academic Publishers, Dordrecht ; Boston, 1991.

[36] D. A. Randell, Z. Cui, and A. G. Cohn. A spatial logic based on regions and connection. In *3rd Int. Conference on Knowledge Representation and Reasoning.* Boston, 1992.

[37] J. Renz and B. Nebel. On the complexity of qualitative spatial reasoning: A maximal tractable fragment of the region connection calculus. *Artificial Intelligence*, 108(1-2):69–123, 1999.

[38] P. Simons. *Parts, A Study in Ontology*. Clarendon Press, Oxford, 1987.

[39] B. Smith. On drawing lines on a map. In A.U. Frank and W. Kuhn, editors, *Conference on Spatial Information Theory, COSIT*, volume 988. Springer, Semmering, Austria, 1995.

[40] B. Smith. Fiat objects. *Topoi*, 20(2), 2001.

[41] B. Smith and B. Brogaard. Quantum mereotopology. *Annals of Mathematics and Artificial Intelligence*, to appear.

[42] A. C. Varzi. On the boundary between mereology and topology. In R. Casati, B. Smith, and G. White, editors, *Philosophy and the Cognitive Science*, pages 423–442. Hölder-Pichler-Tempsky, 1994.

[43] M. F. Worboys. Imprecision in finite resolution spatial data. *Geoinformatica*, 1998.