Piecewise Linear Abstraction

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I find PLA more promising.

PLA performs local and global analysis. Using local analysis, it derives the behavior of a parameterized piecewise linear system over a linear portion of its domain. It solves the system symbolically with the standard algorithm from the theory of linear systems—Laplace transform, partial fractions expansion, and inverse Laplace transform—and invokes the QMR mathematical reasoner to analyze the solution. QMR handles a large class of functions on the reals, including *extended elementary functions:* polynomials and compositions of exponentials, logarithms, trigonometric functions, inverse trigonometric functions, absolute values, maxima, and minima. It infers their *qualitative properties:* signs of the first and second derivatives, discontinuities, singularities, and asymptotes, and records them in data structures called *Q-behaviors*. QMR can answer a wide range of questions about symbolic expressions. It can also sketch them, as demonstrated in Figure 1.

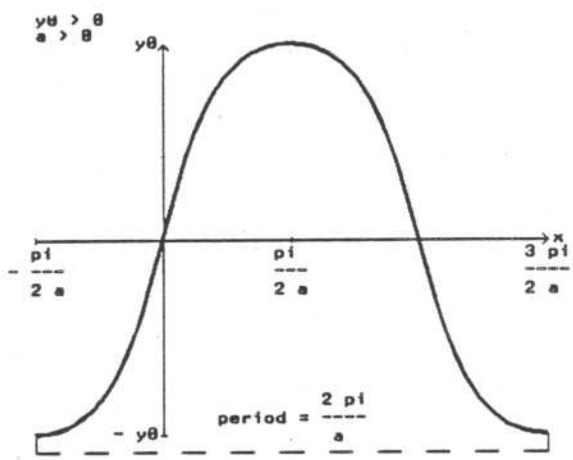

Figure 1: QMR's sketch of $y(t) = y_0 \sin ax$ for y_0 and a positive.

QMR can either analyze simple functions from first principles of calculus or match them against stored patterns. It analyzes complicated functions by analyzing their constituents recursively and combining the results. This process may produce several alternative Q-behaviors, depending on algebraic relations between a capacitor, an inductor, and a nonlinear resistor connected in series. By Kirchoff's laws, the current through the circuit, I, obeys the van der Pol equation

$$I'' + \frac{k}{L}(3I^2 - 1)I' + \frac{1}{LC}I = 0 \tag{1}$$

with C the capacitance, L the inductance, and k a positive scaling factor. Intuitively, the system oscillates because the nonlinear resistor adds energy to the circuit at low currents and drains energy at high currents. One obtains a piecewise linear approximation of equation (1) by replacing the nonlinear resistor model with a piecewise linear one, as illustrated in Figure 3. PLA must consider two cases in analyzing the resulting equations

$$I'' - \frac{2k}{3L}I' + \frac{1}{LC}I = 0 \quad \text{for} \quad |I| \leq \frac{1}{\sqrt{3}} \approx .58 \tag{2}$$

$$I'' + \frac{2k}{3L}I' + \frac{1}{LC}I = 0 \quad \text{for} \quad |I| > \frac{1}{\sqrt{3}} \tag{3}$$

depending on whether their characteristic roots are real or complex. I will only discuss the real case; the complex case is similar.

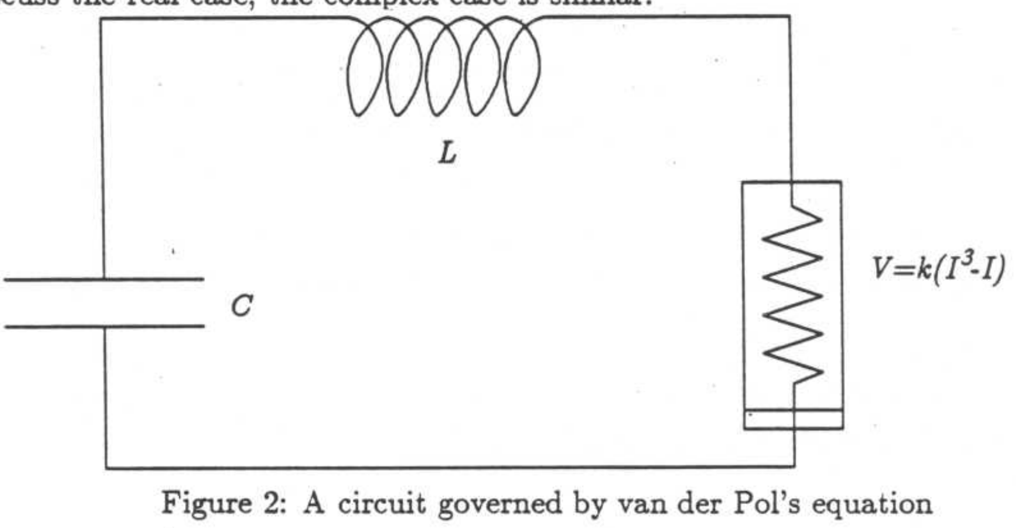

Figure 2: A circuit governed by van der Pol's equation

In the real case, the characteristic roots of equation (2) are positive, while those of equation (3) are negative. Figure 4 depicts the phase diagrams for both equations. The joint phase diagram (Figure 5) contains three significant regions: F, G, and H.

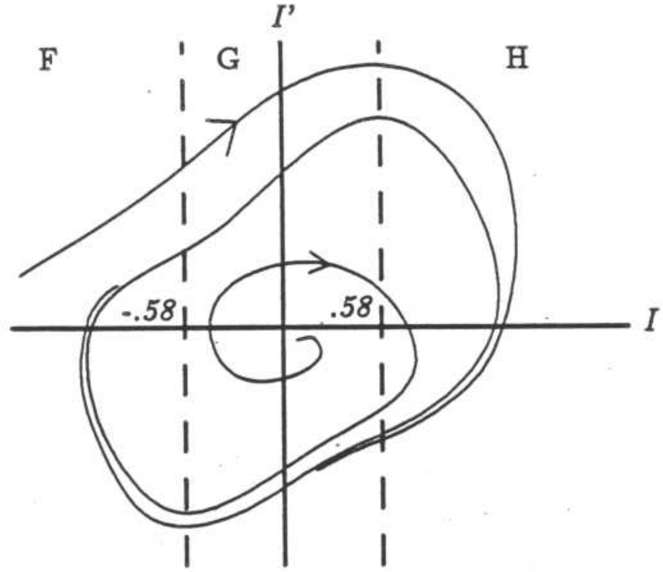

Figure 5: Joint phase diagram for the piecewise van der Pol equations

of detail for engineering applications. The fact that engineers normally find piecewise linear equations adequate for modeling devices provides empirical support for the PLA paradigm. As more concrete evidence, I have demonstrated its analysis of the van der Pol equation. Although mathematicians can handle this equation analytically, the prospect of encoding the prerequisite creativity, sophistication, and knowledge is daunting. PLA provides an algorithmic alternative. It can also handle nonlinear systems that defy known analytic techniques.

The system described here is only partially implemented. The local analysis component, mathematical reasoner, and inequality prover are complete, but the global analysis component is under development. Once completed, PLA must prove its worth empirically by solving useful problems.