Order of Magnitude Revisited

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Extended Abstract

Order of Magnitude Reasoning , as first described by O. Raiman with the formal system FOG [Raiman, 1986], has proven to be an efficient tool to extend reasoning in Qualitative Physics. When there is a lack of quantitative information , it allows nevertheless to solve ambiguities that would remain if only signs of quantities would be considered. FOG has been in particular successfully used for troubleshooting analog circuits in the DEDALE project [Dague, 1987].

Some problems have yet arisen with FOG during this study, the main of which are:

- the difficulty to incorporate partial quantitative information, if available
- the lack of means to control the inference process, this control being necessary because some rules of FOG, true at the formal level, can lead to errors in real cases if applied without precaution
- the trouble that arises, even when remaining at the formal level, when changing from an order of magnitude to another one (or when the question occurs whether an inconsistency exists or not) without the ability of a "smooth" change thanks to overlapping orders of magnitude.

Those problems were pointed out in [Mavrovouniotis, 1987], without receiving a complete satisfactory solution in our opinion.

This work is an attempt to give an answer to those problems, based on sound principles.

- The third problem is solved by introducing a fourth basic operator in addition to the three operators of FOG, in order to represent the intuitive notion: "to be distant from". A minimal set of 15 axioms is given and about 45 properties having an evident and interesting intuitive interpretation in qualitative terms are demonstrated (essentially those of FOG plus about 20 ones using the new operator).

A completely symmetric framework is thus obtained with two operators respectively defined in terms of the two (not totally independent) other ones. These two degrees of freedom (with a constraint) are found again in the mathematical model which is given and which is weaker than the classical model of Non Standard Analysis (where the datum of infinitesimals is just needed: this is equivalent to add the property that the relation "distant from" is nothing more than the negation of the relation "close to").

Along with sign and identity, 15 primitive overlapping relations are thus logically obtained from those operators permitting to smoothly describe the progressively larger and larger qualitative gap between two quantities.

- The two first problems are solved by defining four "quantitative" (in the field of real numbers) operators analogous to the four operators of the formal system, each one being parameterized by a symbolic parameter which represents the scale associated with the corresponding operator. Quantitative counterparts of all axioms and properties of the formal system are then demonstrated, highlighting the symbolic expression of the associated scale of the concluded relation in terms of the scales of relations in premises.

In addition with the ability of directly using quantitative information, it allows, if coupled with a computer algebra system, to automatically control the reasoning inferences. It becomes in particular possible to switch during the inference process from an order of magnitude operator to another one because of a numerical change of scale, or to compute how accurate initial data would have to be to guarantee a correct (in a real world meaning) answer when using a given path of inferences.

As previously, two degrees of freedom are obtained, in terms of two primitive scales with one constraint (one scale is less than the other). The exact counterpart of classical N.S.A. is obtained by adding the property that the two scales are the same, which suppresses one degree of freedom and by the same the ability of expressing smoothly changes in order of magnitude.

In our mind, what has been developed in this work could be a basis for implementing a module which, coupled with a computer algebra system and classical numerical functionalities, would constitute an engineering tool for both quantitative and qualitative, numerical and symbolic algebra. Such a tool could possibly be integrated in the future in a high level programming language such as constraint logic programming.

References

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[Mavrovouniotis, 1987] M. L. Mavrovouniotis, and G. Stephanopoulos. Reasoning with orders of magnitude and approximate relations. In *Proceedings AAAI-87*, pages 626-630, Seattle, PA, American Association for Artificial Intelligence, July 1987.

The Formal System

The formal system is described by the axioms (Ai) and we give some logically derived properties (Pi). Quantities are taken in a totally ordered commutative field, $[A]$ stands for the sign of A (induced by the order) and $|A|$ for the absolute value of A. The logical connectives are written as: "$\mapsto$" for the implication, "$\leftrightarrow$" for the equivalence, "," for the and, "or" for the or. Formulas are assumed to be universally quantified.

$\approx$ ("close") is an equivalence relation:

(A1) $A \approx A$

(A2) $A \approx B \mapsto B \approx A$

(A3) $A \approx B,\ B \approx C \mapsto A \approx C$

$\sim$ ("comparable") is an equivalence relation, which is coarser than $\approx$:

(A4) $A \sim B \mapsto B \sim A$

(A5) $A \sim B,\ B \sim C \mapsto A \sim C$

(A6) $A \approx B \mapsto A \sim B$

$\approx$ and $\sim$ are stable by homotethy:

(A7) $A \approx B \mapsto C.A \approx C.B$

(A8) $A \sim B \mapsto C.A \sim C.B$

The two relations are thus entirely determined by the class of 1 for $\approx$ and the class of 1 for $\sim$.

Any element in the class of 1 for $\sim$ is positive:

(A9) $A \sim 1 \mapsto [A] = +$

In fact, we deduce from (A8) and (A9) that two elements in the same class for $\sim$ (and a fortiori in the same class for $\approx$ by (A6)) have the same sign:

(P1) $A \sim B \mapsto [A] = [B]$

In particular, the class of 0 is reduced to the unique element 0.

$\ll$ ("negligible") is defined in terms of $\approx$ by:

(A10) $A \ll B \leftrightarrow B \approx (B + A)$

With this definition, (A1) can be expressed as:

(P2) $0 \ll A$

From (A10) and (A7), $\ll$ is also stable by homotethy:

(P3) $A \ll B \mapsto C.A \ll C.B$

We assume the following relationship between $\ll$ and $\sim$:

(A11) $A \ll B,\ B \sim C \mapsto A \ll C$

It can be deduced from (A11), by using (A8), (P3), (A4), (P2), (P1), (A10), (A6), that:

(P4) $A \ll B,\ A \sim C \mapsto C \ll B$

(A11) and (P4) express that $\ll$ is compatible with $\sim$ and thus defines a relation between equivalence classes for $\sim$.

With definition (A10), one can express (A2) and (A3) in terms of $\ll$. Using (A6) and (A11) one obtains:

(P5) $A \ll B \mapsto -A \ll B$

(P6) $A \ll C,\ B \ll C \mapsto (A + B) \ll C$

With (A10), (P5) can be expressed by:

(P7) $A \approx (A + B) \mapsto A \approx (A - B)$

From (P5) and (P3), we have:

(P8) $A \ll B \mapsto A \ll -B$

Using (A10), (A6), (A11) and (P5), one has:

(P9) $A \ll B,\ C \ll B \mapsto A \ll (B + C),\ A \ll (B - C)$

thus, in particular:

(P10) $A \ll B \mapsto A \ll (B + A),\ A \ll (B - A)$

Using (A10), (A6), (P4), (P5) and (P6), one proves the transitivity of $\ll$:

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(P11) $A \ll B,\ B \ll C \mapsto A \ll C$

We can conclude from all this that the relation ($A \sim B$ or $A \ll B$) is a partial preorder, that $\sim$ is the associated equivalence relation and that $\ll$ is the induced strict order between equivalence classes for $\sim$ (other than the class reduced to 0).

Using (A6), (A4), (P2), (P4), (P5), (A10), (A2), (A1), (P1), we have:

(P12) $A \approx 0 \leftrightarrow A \sim 0 \leftrightarrow A \ll 0 \leftrightarrow A \ll A \leftrightarrow A = 0$

An other consequence of (P1), using (A10), (A6) and (P5), is:

(P13) $A \ll B \mapsto |A| < |B|$ or $(A = B = 0)$

From (P1) and (P12) and using (A4), (A11), (P11), (P8) we deduce:

(P14) $(A \sim B,\ A \sim -B,\ A \ll B,\ B \ll A)$ are two by two exclusive, except when $(A = B = 0)$

Notice that we do not assume that, given any A and B, one of these relations always holds. This would be equivalent, using (A4), (A8), (P5), (P3), (P8) and (P1), to assuming that ($A \sim B$ or $A \ll B$) is a total preorder on positive elements.

An other consequence of (P1), using (P4), (P5), (A10), (A6) is:

(P15) $C \sim (A + B),\ C \ll A \mapsto [A] = -[B]$

The relations $\approx$ and $\sim$ between two elements are preserved by translation of any quantity of the same sign:

(A12) $A \approx B,\ [C] = [A] \mapsto (A + C) \approx (B + C)$

(A13) $A \sim B,\ [C] = [A] \mapsto (A + C) \sim (B + C)$

Using (A10), we see that (A12) is equivalent to:

(P16) $A \ll B,\ [C] = [B] \mapsto A \ll (B + C)$

(A12) implies more generally, using (A6), (P1), (A3), that:

(P17) $A \approx B,\ C \approx D,\ [C] = [A] \mapsto (A + C) \approx (B + D)$

(A13) implies more generally, using (P1), (A5), that:

(P18) $A \sim B,\ C \sim D,\ [C] = [A] \mapsto (A + C) \sim (B + D)$

The following generalization of (P16) is derived from (P17) and (A10):

(P19) $A \ll B,\ C \ll D,\ [B] = [D] \mapsto (A + C) \ll (B + D)$

Using (P3), (P13) and (P16) one demonstrates the following property of $\ll$ with respect to translation (to compare with (A12) or (A13)):

$$\text{(P20)} \quad (A + C) \ll (B + C),\ [C] = [A] \mapsto A \ll B$$

By the same way, one shows, using (A7) and (A12), that:

$$\text{(P21)} \quad A \approx (A + B + C),\ [B] = [C] \mapsto A \approx (A + B)$$

and, using (A8) and (A13), that:

$$\text{(P22)} \quad A \sim (A + B + C),\ [B] = [C] \mapsto A \sim (A + B)$$

These two properties say that if two quantities are in the same class for $\approx$ (resp. for $\sim$), any quantity between them (for the usual order defined by the sign) is also in the same class for $\approx$ (resp. for $\sim$).

With (A10), (P21) is equivalent to:

$$\text{(P23)} \quad (B + C) \ll A,\ [B] = [C] \mapsto B \ll A$$

We have thus, putting together (P16) and (P23), and using (P5) and (P8):

$$\text{(P24)} \quad A \ll B,\ |C| \le |A|,\ |D| \ge |B| \mapsto C \ll D$$

Using above properties about $\approx$ and $\ll$, it is easy to show that if the relation $\approx$ is not trivial, i.e. does not coincide with the equality (which is equivalent to say that there exists $I \neq 0$ such that $I \ll 1$), then the field is non archimedean.

Recall that we have not assumed that the relation ($A \sim B$ or $A \ll B$) is a total preorder on positive elements. We want nevertheless to be able to compare by this relation "sufficiently" elements. One shows in fact, using (A8), (P3), (A10), (A4), (A6), (A13), (A5) and (P22), that simply assuming that there exist two different positive rational numbers a and b such that ($a \sim b$ or $a \ll b$) is equivalent to the following axiom:

$$\text{(A14)} \quad A \sim (A + A)$$

It follows from (A13), (A14) and (A5) that:

$$\text{(P25)} \quad A \sim B \mapsto A \sim (A + B)$$

Thus, any positive rational number and more generally any element between two positive rational numbers is in the class of 1 for $\sim$ (and any positive element I such that $I \ll 1$ is smaller than all positive rational number).

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$\not\asymp$ ("distant") is defined in term of $\sim$ by:

$$\text{(A15)}\quad A \not\asymp B \leftrightarrow (A - B) \sim A \text{ or } (B - A) \sim B$$

It is clear from this definition that $\not\asymp$ is symmetric:

$$\text{(P26)}\quad A \not\asymp B \mapsto B \not\asymp A$$

The following properties are straightforward:

$$\text{(P27)}\quad 0 \not\asymp A$$

follows from (A15), (A1), (A6)

$$\text{(P28)}\quad A \not\asymp A \leftrightarrow A = 0$$

follows from (A15), (A4), (P12)

$$\text{(P29)}\quad A \not\asymp (A + A)$$

follows from (A15), (A14)

$$\text{(P30)}\quad A \not\asymp -A$$

follows from (A15), (A14), (A4)

The relations $\ll$, $\sim$ and $\not\asymp$ have the following dependences:

$$\text{(P31)}\quad A \ll B \text{ or } B \ll A \text{ or } A \sim -B \mapsto A \not\asymp B$$

results from (A15), (P5), (A10), (A6), (A4), (P25)

$$\text{(P32)}\quad A \ll B \text{ or } A \sim B \text{ or } A \sim -B \mapsto A \not\asymp (A + B),\ A \not\asymp (A - B)$$

results from (A15), (P5), (A10), (A6), (A8), (A4), (P25)

$$\text{(P33)}\quad A \not\asymp (A - B),\ B \not\asymp (B - A) \leftrightarrow A \sim B \text{ or } A \sim -B$$

results from (A15), (A4), (A8), (A5), (P25)

The two relations $\approx$ and $\not\asymp$ are exclusive, except for $A = B = 0$, as it can be shown by using (A15), (A10), (P5), (P4), (P12), (A2), (A6), (A1), (P27):

$$\text{(P34)}\quad A \approx B,\ A \not\asymp B \leftrightarrow A = B = 0$$

But we do not assume that for any given A and B we have ($A \approx B$ or $A \not\asymp B$), which would mean, using (P34), that $\not\asymp$ is exactly the same, on non zero elements, that the negation of $\approx$. In fact, one shows, using (P32), (P5), (A10), (P33), that it would be equivalent to assuming that for any given A and B we have ($A \sim B$ or $A \sim -B$ or $A \ll B$ or $B \ll A$).

It is clear from (A15) and (A8) that $\not\asymp$ is stable by homotethy:

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(P35) $A \not\approx B \mapsto C.A \not\approx C.B$

We have the following three key properties that highlight the relationships between $\approx$ and $\not\approx$:

(P36) $A \not\approx B,\ C \lll A \mapsto C \lll (A - B)$

that follows from (A15), (A4), (A11), (P16), (P12), (P2), (A13), (A5), (P8)

(P37) $A \approx B,\ C \not\approx A \mapsto (C - A) \approx (C - B)$

that is a direct consequence of (P36), using (A10), (P26), (A7)

(P38) $A \not\approx B,\ C \approx A,\ D \approx B \mapsto C \not\approx D$

that follows from (P37) by using (A2), (P26), (A6), (A15), (A4), (A5), (A8). This property shows that $\not\approx$ defines a relation between equivalence classes for $\approx$.

From the definition (A15) it follows immediatly by using (A4) and (A5) the following result concerning translation by an element:

(P39) $A \not\approx B,\ (A + C) \sim A,\ (B + C) \sim B \mapsto (A + C) \not\approx (B + C)$

The conditions about C are in particular satisfied when both ($C \lll A$ or $C \sim A$) and ($C \lll B$ or $C \sim B$) are true (by (A10), (A6), (A4), (P25)).

We have the following property (compare with (P21) and (P22)):

(P40) $A \not\approx (A + B),\ [B] = [C] \mapsto A \not\approx (A + B + C)$

which follows from (A15), (A13), (P1), (A8), (P25), (A4), (P22), (A5).

It can be reformulated by using (P26) as (compare with (P24)):

(P41) $C \leq A < B \leq D,\ A \not\approx B \mapsto C \not\approx D$

By using (P35) and (P40) we have an other result concerning translation by an element (compare with (P20)):

(P42) $(A + C) \not\approx (B + C),\ [C] = [A] \mapsto A \not\approx B$

From (P27), (P26), (P30) and (P40), one demonstrates that two elements that have different signs are always related by $\not\approx$:

(P43) $[A] \neq [B] \mapsto A \not\approx B$

The analog of (P7) for $\not\approx$ follows from (A15), (P25), (P27), (A13), (A8), (A4), (P26), (P42):

(P44) $A \not\approx (A + B) \mapsto A \not\approx (A - B)$

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Finally, using (P43), (A1), (A6), (A8), (P35), (A4), (P26), (A14), (P22), (P29) and (P40), it can be proved that any given two elements are always related by $\sim$ or by $\not\asymp$ (not exclusively cf. (A14) and (P29)):

(P45) $A \sim B$ or $A \not\asymp B$

How using relations

See Fig 1. which shows the different relations which can hold between two positive quantities A and B.

With signs and identity, we obtain 15 primitive relations (see Fig 2.):

$\equiv$ for identity

$\approx^-$ for $\approx, <$

$\approx^+$ for $\approx, >$

$\simeq^-$ for $\neg\not\asymp, <$

$\simeq^+$ for $\neg\not\asymp, >$

$\not\lessapprox$ for $\neg\approx, \sim, <$

$\not\gtrapprox$ for $\neg\approx, \sim, >$

$\not\lesssim$ for $\not\asymp, \sim, <$

$\not\gtrsim$ for $\not\asymp, \sim, >$

$\prec$ for $\not\asymp, \neg\preccurlyeq, <$

$\succ$ for $\not\asymp, \neg\succcurlyeq, >$

$\twoheadleftarrow$ for $\neg\sim, <$

$\twoheadrightarrow$ for $\neg\sim, >$

$\preccurlyeq$

$\succcurlyeq$

By adding relations obtained by disjunction of successive primitive relations, we obtain a total of 62 relations.

The Quantitative Symbolic System

For K a positive real number, we define the following operators acting in the field of real numbers:

$$A \overset{K}{\sim} B \;\leftrightarrow\; |A - B| \le K \times \mathrm{Max}(|A|,|B|)$$
$$A \overset{K}{\not\sim} B \;\leftrightarrow\; |A - B| > K \times \mathrm{Max}(|A|,|B|)$$
$$A \overset{K}{\ll} B \;\leftrightarrow\; |A| < K \times |B|$$

We then demonstrate the counterpart of all axioms and properties of the formal system. We just give here the correspondance of the definitions (see Fig 2):

$$A \approx B \;\leftrightarrow\; A \overset{K_1}{\sim} B$$
$$A \sim B \;\leftrightarrow\; A \overset{1-K_2}{\sim} B$$
$$A \ll B \;\leftrightarrow\; A \overset{K_1}{\ll} B$$
$$A \not\approx B \;\leftrightarrow\; A \overset{K_2}{\not\sim} B$$

with:

$$0 < K_1 < K_2 < \frac{1}{2} < 1 - K_2 < 1$$

$\approx$	$\not\approx$	$\sim$	$\ll$ or $\gg$
$\boxed{A \approx B}$	$\neg(A \not\approx B)$	$A \sim B$	$\neg(A \ll B)$ $\neg(A \gg B)$
?	$\boxed{\neg(A \not\approx B)}$	$A \sim B$	$\neg(A \ll B)$ $\neg(A \gg B)$
$\boxed{\neg(A \approx B)}$	?	$\boxed{A \sim B}$	$\neg(A \ll B)$ $\neg(A \gg B)$
$\neg(A \approx B)$	$\boxed{A \not\approx B}$	$\boxed{A \sim B}$	$\neg(A \ll B)$ $\neg(A \gg B)$
$\neg(A \approx B)$	$\boxed{A \not\approx B}$	?	$\boxed{\neg(A \ll B) \quad \neg(A \gg B)}$
$\neg(A \approx B)$	$A \not\approx B$	$\boxed{\neg(A \sim B)}$	?
$\neg(A \approx B)$	$A \not\approx B$	$\neg(A \sim B)$	$\boxed{A \ll B}$ or $\boxed{A \gg B}$

Fig. 1

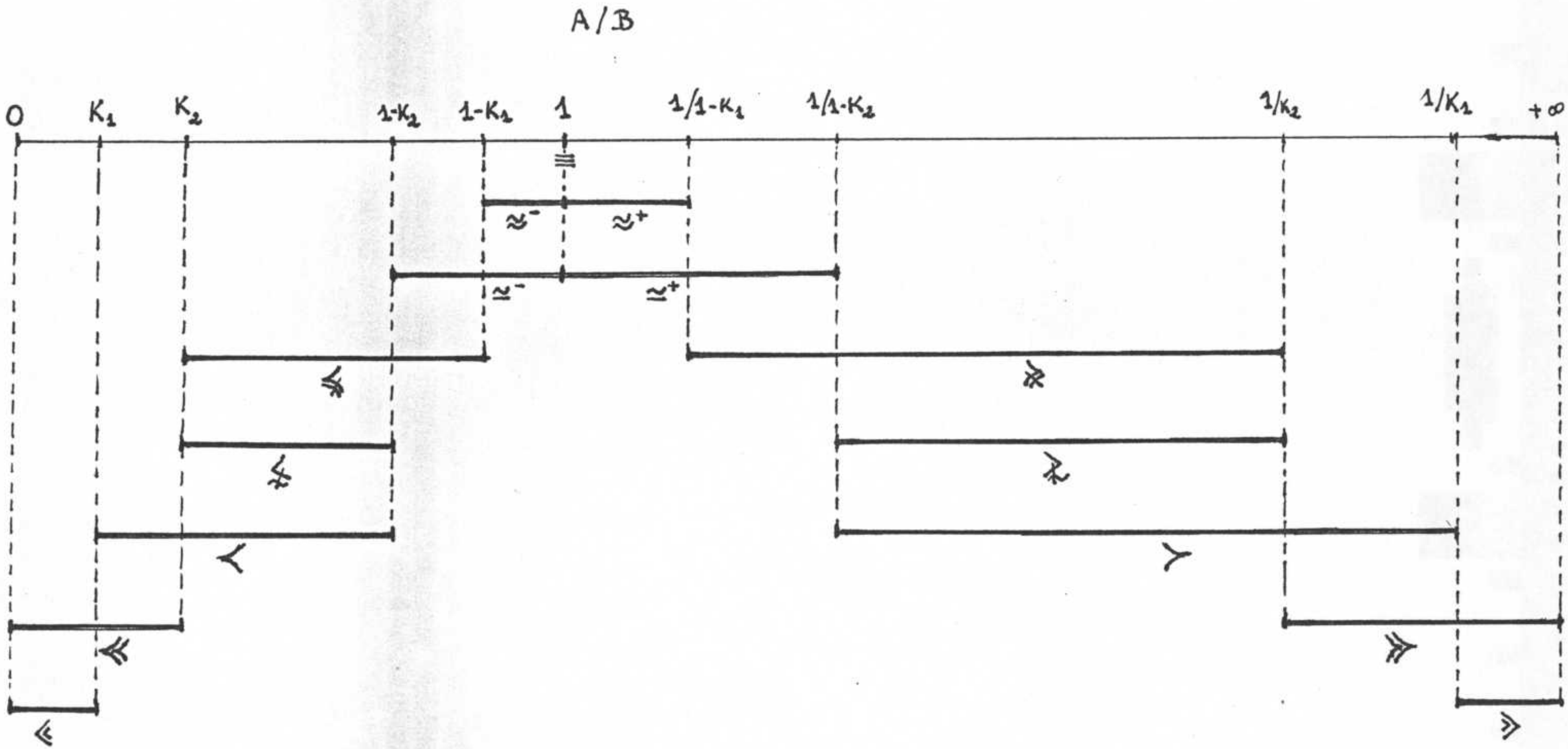

Fig 2.