Qualitative Vector Algebra

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Abstract

A significant aspect of reasoning about physical situations involves analysis of the interaction of physical parameters that have both magnitude and direction. There have been some attempts to model motion and rotation in two dimensions, but none of these approaches have been extended to general vector analysis. Humans, however, are exceptionally good in reasoning about direction and motion. In this paper, we define a framework called Qualitative Vector Algebra (QVA) for qualitative reasoning about vector interactions and projection onto prespecified axes. A set of lemmas are developed that relate changes in vector magnitude and direction to relative changes in the resultant vector. We demonstrate how QVA can be applied to reasoning about problems in mechanics. The problem solver developed can form the basis for an Intelligent Tutoring System directed toward high school students and college freshmen.

1 Introduction

Consider a person traveling in a motor-powered boat on a river that has a swift current. He is currently at point A in the river and wishes to go across to point B (see Fig. 1). There are numerous ways in which this could be achieved, but consider that he wishes to do it in an efficient manner (i.e., try and minimize the distance he has to travel). To simplify the problem, we assume that he can vary the magnitude and direction of force imparted by the boat motor. The person is not likely to know precise magnitudes of the forces (river current and that imparted by the motor) or the angles Θ_1 and Θ_2, and, therefore, reasons with qualitative estimates. In reality, the person probably chooses a reasonable direction and magnitude for the direct force imparted by the boat motor. Then he makes iterative adjustments to both quantities based on periodic observations until he achieves the desired goal. Thus the operator goes through a series of adjustments, but each one involves a line of reasoning similar to the one discussed above. It is not difficult to conjecture a number of other situations, e.g., sailing, wind surfing, and determining the trajectory of a load dropped from a moving aircraft, where the aggregate of a number of vectors in two- or three-dimensional space needs to be computed in order to derive an overall estimate of the result.

The above example lends support to a Forbus conjecture that "people exhibit fluency in reasoning about motion through space mainly from using their visual apparatus" [1]. An example that illustrates this point is the use of diagrams which reflect spatial relationships among parameters of interest (e.g., Fig. 1). These diagrams allow visual interpretation of spatial information and reasoning about their consequences. The concept of *vectors*, quantities that possess both magnitude and direction, forms a key role in defining physical parameters and developing problem solving techniques in all branches of physics. In this paper, we extend previous work [2,3] dealing with vector quantities, and introduce quantity space descriptions for directly manipulating and reasoning about vector quantities in a qualitative reasoning framework.

Traditional quantity space descriptions adopted in the qualitative physics literature [4,5,6] deal with quantities and parameter values that are scalar in nature, e.g., pressure, temperature, and height. Other parameters (such as fluid flow) which are in fact two- or three-dimensional vectors are represented as scalars, but this restricts the range of behaviors that can be analyzed. Furthermore, to analyze and reason about motion in general (e.g., a bouncing ball), Forbus [5] defines the symbols ± 1 along an axis *dir*. Motion of a quantity Q along that direction is expressed by the predicate $Direction\text{-}Of(dir, Q)$. This framework does not establish a general notion of computation of the resultant of a number of vector quantities (e.g., force) that act along different directions on an object. Nielsen introduced the notion of qualitative reasoning of vector quantities by conceptualizing the representation of qualitative translational directions [2,3]. He adopts the $(+, 0, -)$ representation to define vectors in two- and three-dimensional space, and uses them for analyzing linear and rotational motion of rigid bodies in space. However, his method does not incorporate sufficient detail to reason about change in vector magnitude and direction. Therefore, existing methods are inadequate for visualizing and reasoning about change of vector quantities in two- and three-dimensional space.

This paper defines a Qualitative Vector Algebra (QVA) methodology that can be in-

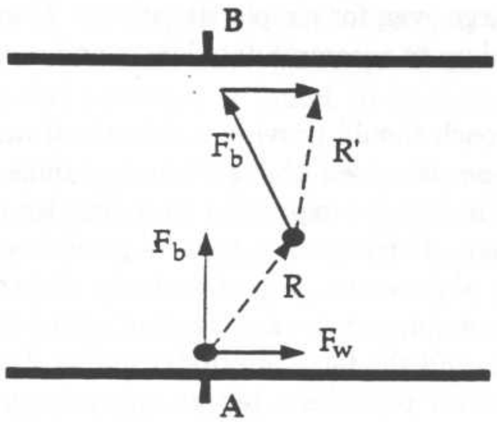


Figure 1. Illustration of qualitative vector manipulation

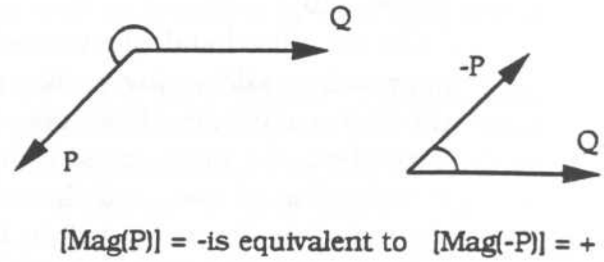


Figure 2. Reversing a vector

corporated into a traditional qualitative reasoning scheme (e.g.. QPE, QSIM) to enable reasoning about vectors in a plane or in a three-dimensional space. We focus on two-dimensional vectors, but this analysis can be extended to manipulate vectors in a spherical coordinate system. This approach can form the basis of a sophisticated problem solver in mechanics, and reason about systems in motion or in static equilibrium. In another application this framework has been used to develop a spatio-temporal model of cardiac electrophysiology [7], the details of which may be obtained from the authors.

Section 2 of the paper reviews previous approaches to modeling qualitative vectors. Section 3 develops the QVA scheme. Section 4 discusses the solution to two different problems in mechanics. Section 5 contains a summary and directions for future work.

2 Previous Approaches

The first attempts to model motion in qualitative physics problem solving appeared in de Kleer's NEWTON[8] and Forbus' FROB systems [1,9]. Qualitative and quantitative methodologies were combined in NEWTON to reason about a ball rolling down a roller coaster track. However, the analysis performed was essentially one-dimensional.

FROB worked in a two-dimensional world to reason about locations and trajectories of balls bouncing off various surfaces. The notion of a *qualitative state vector* was introduced to describe system behavior. This vector included: (i) a quantization of the traditional state variables, i.e., *position* and *velocity*, and (ii) a symbolic description of the type of activity, e.g., fly and collide. Position was described in terms of a *place vocabulary* derived from a *metric diagram* that was based on the geometrical properties of the space under consideration. The direction of velocity was expressed by a conjunction of two axes *up-down* and *right-left* (Each of these represent a *dir* discussed in Section 1).

Using this formalism, the state of the bouncing ball was described in terms of its position (a symbol from the place vocabulary, e.g., *REGION S3*) and velocity (e.g., *left down*). The temporal behavior was derived from a set of highly-specific qualitative simulation rules (in lieu of equations) which described what qualitative states could occur next. However, the overall scheme had two limitations: (i) the simulation rules were too specific to apply in general, and (ii) the representation lacked detail, therefore, the number of possible next states predicted by FROB were very large even for simple situations. Therefore, it is unlikely that this method can easily be scaled up to accommodate larger and more complex models [10].

On the other hand, the vector-based approach should provide a richer and more general approach to addressing problems like the one discussed above. The language offered by the vector notation allows laws and constraints to be expressed in a simpler manner independent of a prespecified coordinate system. Furthermore, vectors provide an intuitive visualization of two- and three-dimensional phenomena (e.g., free body diagrams). The motion of the bouncing ball in the FROB example can be analyzed in terms of two force vectors: an initial force imparted on the ball, and the force of gravity acting downward. A qualitative estimate of the resultant force vector provides a better estimate and explanation of the direction of flight of the ball which can then be used to answer questions, such as "*will the ball fall into the well ?*" [9].

As mentioned breifly in Section 1, Nielsen conceptualizes a representation for qualitative vectors in his development of a qualitative theory for reasoning about the motion and interaction of rigid bodies [2,3]. Qualitative vectors are represented by the symbols $(+, 0, -)$ with respect to their orientation to a Cartesian coordinate system. This representation allows for nine specific landmark values delineating the eight directions about the center of the axis system and one at the center of the axis system (this holds for a two-dimensional system, the method could be as easily extended to a three-dimensional system). For example, $(-, +)$ would represent a vector in the direction of third quadrant between 90° and 180°. Using qualitative addition and qualitative multiplication, the sign of the dot product for two such vectors can be determined, thereby revealing information regarding the angle subtended by the two vectors. This representation lacks the expressiveness to reason about situations involving the relative magnitudes of a set of vectors. Also, this method does not allow for reasoning at greater levels of detail with dynamically generated landmarks as a comparative analysis method may. Nielsen suggests greater levels of detail could be gained by further subdividing the Cartesian coordinate system, but this would necessarily add additional ambiguity. The next section develops our QVA methodology, based on the polar coordinate representation.

3 Qualitative Vector Algebra

As discussed earlier, a vector is a quantity with a magnitude and a direction. For physical quantities, the magnitude maps onto the positive real line (or a subset of it) with appropriate units of measurement. The direction is specified by angles subtended with a prespecified set of coordinate axes. Qualitative physics abstracts parameter values into *quantity spaces*, sets of totally ordered *landmark values* that divide up the real line into alternate sequences of points and open intervals [6]. In this paper, a similar approach is taken to define a qualitative vector representation. A corresponding qualitative algebra is devised to facilitate vector operations, such as addition and projection.

Mathematically, vectors are defined in two ways. A vector may be expressed in terms of components projected on orthogonal coordinate axes (Cartesian coordinate system), such as the x–y axes in two dimensions. Alternatively, the polar notation defines a vector in terms of its magnitude and direction which is usually expressed as an angle subtended with a reference axis. Since qualitative reasoning does not deal with exact magnitudes, the polar notation is found to be more effective as well as intuitive in reasoning about vector operations.

Standard vector algebra defines four major operations: (i) vector addition (subtraction), (ii) multiplication of a vector by a scalar, (iii) dot product of two vectors, and (iv) cross product of two vectors. For the problems we currently deal with, only the first three operations are relevant, so we do not discuss the cross product in this paper. The dot product is also used in a restricted form, i.e., projecting one vector onto another. This is equivalent to computing the dot product of the first vector with a unit vector in the direction of the second.

The QVA framework developed in this paper deals with pairwise vector operations in two-dimensional space. Pairwise operations such as vector addition are commutative and associative, and can be generalized to an arbitrary number of vectors. Throughout the text, vectors will be represented in bold face, e.g., **P** and **Q**. The two quantities associated with a vector are expressed in functional notation, i.e., $Mag(\mathbf{P})$ represents the magnitude of vector **P**, and $\Theta(\mathbf{P},\mathbf{X})$ represents the angle that **P** subtends with a predefined axis **X**. Note that by convention $\Theta(.,.)$ is measured in the counterclockwise direction. $[Mag(\mathbf{P})]$ and $[\Theta(\mathbf{P},\mathbf{X})]$ represent the qualitative values of the magnitude and angle, respectively.

Since our emphasis is on establishing qualitative pairwise operations, the following quantities are of interest:

- $Mag(\mathbf{P})$, $Mag(\mathbf{Q})$: magnitudes of **P** and **Q**, respectively.
- $[Mag(\mathbf{P})]$, $[Mag(\mathbf{Q})]$: qualitative values of corresponding magnitudes.
- $\Theta(\mathbf{P},\mathbf{Q})$: magnitude of the angle between **P** and **Q** measured in the counterclockwise direction.
- $[\Theta(\mathbf{P},\mathbf{Q})]$: qualitative value of the magnitude of the angle between **P** and **Q**.
- $Proj(\mathbf{Q},\mathbf{P})$: projection of vector **Q** on vector **P**. (Note that projection is a magnitude).
- $[Proj(\mathbf{Q},\mathbf{P})]$: qualitative value of the projection of vector **Q** on vector **P**.

Following Kuipers[6], a quantity space description in its most general form may assume three significant landmarks, $\{-\infty, 0, +\infty\}$, and a finite number of additional landmarks (distinguished points) in between. The quantity space for Θ is defined in terms of the following values:

- Aligned: representation of point value 0,
- Acute: representation of interval $(0,\pi/2)$,
- Perpendicular: the point value $\pi/2$,
- Obtuse: the interval $(\pi/2,\pi)$
- Opposite: the point value π.

$[\Theta$ (P,S)]	Proj(P,S)
1. $[\Theta(\mathbf{P},\mathbf{S})]$ = Aligned	Proj(P,S) = Mag(P)
2. $[\Theta(\mathbf{P},\mathbf{S})]$ = Acute	Proj(P,S) < Mag(P); Proj(P,S) > 0
3. $[\Theta(\mathbf{P},\mathbf{S})]$ = Perpendicular	Proj(P,S) = 0
4. $[\Theta(\mathbf{P},\mathbf{S})]$ = Obtuse	Proj(P,S) < 0; |Proj(P,S)| < Mag(P)
5. $[\Theta(\mathbf{P},\mathbf{S})]$ = Opposite	Proj(P,S) < 0; |Proj(P,S)| = Mag(P)

Table 1: Derivation of Vector Projection

Note that the quantity space does not extend beyond π. All angles $\Theta(\mathbf{P},\mathbf{Q})$ that exceed π ($\pi < \Theta(\mathbf{P},\mathbf{Q}) < 2\pi$) are analyzed in the following manner: vector **Q** is reversed (see Fig. 2), and its magnitude is multiplied by -1 to produce an equivalent vector. The angle between **-Q** and **P** is in the range $(0,\pi)$.

3.1 Qualitative Multiplication with a Scalar

Given a vector **P** and a real number a, the product of vector **P** by scalar a denoted by aP defines a new vector **P**′, such that:

$$[Mag(\mathbf{P}')] = [a][Mag(\mathbf{P})].$$

Note that the product defined on the right hand side is a qualitative product. If $a > 0$, the direction of **P** does not change. Scalar multiplication can be used to define the reversal of a vector by setting $a = -1$, so that $Mag(\mathbf{P}') = -Mag(\mathbf{P})$.

3.2 Vector Projection

Given a vector **P** and a direction **S**[1], the projection of **P** onto **S** is defined in Table 1. The use of vector projections is demonstrated in [7].

3.3 Vector Addition

Qualitative vector addition is defined for pairs of vectors **P** and **Q**. The result is a vector $\mathbf{R} = \mathbf{P} + \mathbf{Q}$. Possible magnitudes and directions for **R** are summarized in Table 2. Note that this analysis basically produces constraints or bounds on the possible magnitudes and directions of the resultant.

3.4 Comparative Analysis

A very useful application of QVA is *comparative analysis*[2]. This involves reasoning about the effect of changes made in a given situation. Consider the boat example again, and suppose that the operator decides to increase the force imparted by the boat $\mathbf{F_B}$ while keeping the direction of application unchanged. It is not difficult to establish that the

[1] A direction is defined as a vector with unit magnitude, i.e., $Mag(\mathbf{S}) = 1$.

[2] In a sense, this is similar to Weld's DQ analysis [11] (also Forbus [5]).

[Θ (P, Q)]	angular relationships for R	magnitude of R
1. [Θ(P,Q)] = Aligned	[Θ(P,R)] = [Θ(R,Q)] = Aligned	Mag(R) = Mag(P) + Mag(Q);
2. [Θ(P,Q)] = Acute	[Θ(P,R)] = [Θ(R,Q)] = Acute; (Θ(P,R) < Θ(P,Q)); (Θ(R,Q) < Θ(P,Q));	Mag(R) < (Mag(P) + Mag(Q)); (Mag(R) > Mag(P)); (Mag(R) > Mag(Q))
3. [Θ(P,Q)] = Perpendicular	same as (2)	same as (2)

4. [Θ(P,Q)] = Obtuse	if (Mag(P) ≥ Mag(Q))
	then (Θ(P,R) = Acute);
	
	if ($\vert$(Proj(Q,P))$\vert$ < $\vert$Mag(P)$\vert$)
	then ([Θ(P,R)] = Acute);
	
	if ($\vert$(Proj(Q,P))$\vert$ = $\vert$Mag(P)$\vert$)
	then ([Θ(P,R)] = Perpendicular);
	
	if ($\vert$(Proj(Q,P))$\vert$ > $\vert$Mag(P)$\vert$)
	then ([Θ(P,R)] = Obtuse)
		Mag(R) < (Mag(P) + Mag(Q));

5. [Θ(P,Q)] = Opposite	if (Mag(P) > Mag(Q))
	then ([Θ(P,R)] = Aligned,
	[Θ(R,Q)] = Opposite);
	
	if (Mag(P) < Mag(Q))
	then ([Θ(R,Q)] = Aligned,
	[Θ(P,R)] = Opposite)
		Mag(R) = Mag(P) − Mag(Q);
		if (Mag(P) = Mag(Q))
		then (Mag(R) = 0;

Table 2: Qualitative vector addition

resultant force vector, and the direction of movement of the boat will then swing toward the direction of $\mathbf{F_B}$ and the boat will move more toward the left (Fig. 1). Decreasing $\mathbf{F_B}$ will have the opposite effect in the direction of movement of the boat, i.e., the boat will swing more to the right.

We formalize this reasoning in five lemmas presented below. Their application to reasoning about change in vector directions and magnitudes is illustrated in Section 4.

Lemma 1.1 Given vectors $\mathbf{P}, \mathbf{Q}, \mathbf{Q}'$, $0 < \Theta(\mathbf{P},\mathbf{Q}) < \pi$, s.t. $Mag(\mathbf{Q}) < Mag(\mathbf{Q}')$, $[\Theta(\mathbf{Q},\mathbf{Q}'] =$ Aligned, $\mathbf{R} = \mathbf{P}+\mathbf{Q}$, and $\mathbf{R}' = \mathbf{P}+\mathbf{Q}'$
then the following relations are true:
$\Theta(\mathbf{P},\mathbf{R}) < \Theta(\mathbf{P},\mathbf{R}')$, $\Theta(\mathbf{R},\mathbf{Q}) > \Theta(\mathbf{R}',\mathbf{Q}')$.

Lemma 1.1.1 Given vectors $\mathbf{P}, \mathbf{Q}, \mathbf{Q}'$, $0 < \Theta(\mathbf{P},\mathbf{Q}) < \pi/2$, s.t. $Mag(\mathbf{Q}) < Mag(\mathbf{Q}')$, $[\Theta(\mathbf{Q},\mathbf{Q}'] =$ Aligned, $\mathbf{R} = \mathbf{P}+\mathbf{Q}$, and $\mathbf{R}' = \mathbf{P}+\mathbf{Q}'$
then the following relations are true:

$\Theta(\mathbf{P},\mathbf{R}) < \Theta(\mathbf{P},\mathbf{R}')$, $\Theta(\mathbf{R},\mathbf{Q}) > \Theta(\mathbf{R}',\mathbf{Q}')$, and $Mag(\mathbf{R}') > Mag(\mathbf{R})$.

Lemma 1.2 The same as Lemma 1.1 but $Mag(\mathbf{Q}) > Mag(\mathbf{Q}')$,
then the above relations are reversed, i.e.,
$\Theta(\mathbf{P},\mathbf{R}) > \Theta(\mathbf{P},\mathbf{R}')$, $\Theta(\mathbf{R},\mathbf{Q}) < \Theta(\mathbf{R},\mathbf{Q}')$.

Lemma 1.2.1 The same as Lemma 1.1.1 but $Mag(\mathbf{Q}) > Mag(\mathbf{Q}')$,
then the above relations are reversed, i.e.,
$\Theta(\mathbf{P},\mathbf{R}) > \Theta(\mathbf{P},\mathbf{R}')$, $\Theta(\mathbf{R},\mathbf{Q}) < \Theta(\mathbf{R},\mathbf{Q}')$, and $\mathrm{Mag}(\mathbf{R}') < \mathrm{Mag}(\mathbf{R})$.

Lemma 2.1 Given vectors $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{Q}'$, s.t. $Mag(\mathbf{Q}) = Mag(\mathbf{Q}')$,
$0 < \Theta(\mathbf{P},\mathbf{Q}) < \Theta(\mathbf{P},\mathbf{Q}') < \pi$, $\mathbf{R} = \mathbf{P} + \mathbf{Q}$, and $\mathbf{R}' = \mathbf{P} + \mathbf{Q}'$
then the following relation is true:
$Mag(\mathbf{R}') < Mag(\mathbf{R})$.

Lemma 2.2 The same as above, but $\Theta(\mathbf{P},\mathbf{Q}) > \Theta(\mathbf{P},\mathbf{Q}')$, $\mathbf{R} = \mathbf{P} + \mathbf{Q}$,
then the relations of Lemma 2.1 are reversed, and $Mag(\mathbf{R}') > Mag(\mathbf{R})$.

Lemma 3.1 Given vectors $\mathbf{P}$, $\mathbf{Q}$, s.t. $Mag(\mathbf{P}) = Mag(\mathbf{Q})$, and $\mathbf{R} = \mathbf{P} + \mathbf{Q}$
the following relation is true:
$\Theta(\mathbf{P},\mathbf{R}) = \Theta(\mathbf{R},\mathbf{Q})$.

Lemma 3.2 Given vectors $\mathbf{P}$, $\mathbf{Q}$, $0 < \Theta(\mathbf{P},\mathbf{Q}) < \pi$ s.t. $Mag(\mathbf{P}) > Mag(\mathbf{Q})$, and $\mathbf{R} = \mathbf{P} + \mathbf{Q}$
the following relation is true:
$\Theta(\mathbf{P},\mathbf{R}) < \Theta(\mathbf{R},\mathbf{Q})$.

Lemma 3.3 Given vectors $\mathbf{P}$, $\mathbf{Q}$, $0 < \Theta(\mathbf{P},\mathbf{Q}) < \pi$ s.t. $Mag(\mathbf{P}) < Mag(\mathbf{Q})$, and $\mathbf{R} = \mathbf{P} + \mathbf{Q}$
the following relation is true:
$\Theta(\mathbf{P},\mathbf{R}) > \Theta(\mathbf{R},\mathbf{Q})$.

Lemma 4.1 Given vectors $\mathbf{P}$, $\mathbf{Q}$ and a direction $\mathbf{S}$, s.t. $Mag(\mathbf{P}) > Mag(\mathbf{Q})$ and $\Theta(\mathbf{P},\mathbf{Q}) =$ Aligned
the following relation is true:
$Proj(\mathbf{P},\mathbf{S}) \geq Proj(\mathbf{Q},\mathbf{S})$, if $0 \leq \Theta(\mathbf{P},\mathbf{S}) \leq \pi/2$.
$Proj(\mathbf{P},\mathbf{S}) < Proj(\mathbf{Q},\mathbf{S})$, if $\Theta(\mathbf{P},\mathbf{S}) > \pi/2$.

Lemma 4.2 Given vectors $\mathbf{P}$, $\mathbf{Q}$ and a direction $\mathbf{S}$, s.t. $Mag(\mathbf{P}) = Mag(\mathbf{Q})$ and $\Theta(\mathbf{P},\mathbf{Q}) =$ Aligned,
the following relation is true:
$Proj(\mathbf{P},\mathbf{S}) = Proj(\mathbf{Q},\mathbf{S})$.

Lemma 4.3 Given vectors $\mathbf{P}$, $\mathbf{Q}$ and a direction $\mathbf{S}$, s.t. $Mag(\mathbf{P}) < Mag(\mathbf{Q})$ and $\Theta(\mathbf{P},\mathbf{Q}) =$ Aligned
the following relation is true:
$Proj(\mathbf{P},\mathbf{S}) \leq Proj(\mathbf{Q},\mathbf{S})$, if $0 \leq \Theta(\mathbf{P},\mathbf{S}) \leq \pi/2$.
$Proj(\mathbf{P},\mathbf{S}) > Proj(\mathbf{Q},\mathbf{S})$, if $\Theta(\mathbf{P},\mathbf{S}) > \pi/2$.

Lemma 5.1 Given vectors $\mathbf{P}$, $\mathbf{Q}$ and a direction $\mathbf{S}$, s.t. $Mag(\mathbf{P}) = Mag(\mathbf{Q})$ and $\Theta(\mathbf{S},\mathbf{P}) > \Theta(\mathbf{S},\mathbf{Q})$
the following relation is true:
$Proj(\mathbf{P},\mathbf{S}) < Proj(\mathbf{Q},\mathbf{S})$.

Lemma 5.2 Given vectors **P**, **Q** and a direction **S**, s.t. $Mag(\mathbf{P}) = Mag(\mathbf{Q})$ and $\Theta(\mathbf{S},\mathbf{P}) < \Theta(\mathbf{S},\mathbf{Q})$ the following relation is true:
$Proj(\mathbf{P},\mathbf{S}) > Proj(\mathbf{Q},\mathbf{S})$.

4 Qualitative Reasoning about Vector Quantities

A significant application of the QVA framework in our research has been the development of a spatio-temporal model for cardiac electrophysiology. Basically, this model simulates the conduction of electrical potentials initiated in a pacemaker area of the heart to the heart muscle which, in response, rhythmically contracts and relaxes to pump blood through the entire body. At any given moment during the cardiac cycle, the overall electrical activity, referred to as the *cardiac vector,* is observed as the aggregation of all electric field vectors. A sequence of projections of the cardiac vector on a predefined coordinate axis forms an ECG waveform of the corresponding cardiac electrical activity [12]. A primary purpose of modeling the electrical conduction system of the heart is to explain the relationship between observed ECG waveform patterns and the behavior of cardiac components (i.e. reasoning about cardiac conduction disorders).

Presently, we are able to simulate electrical propagation through the cardiac muscle using a structural model, and derive cardiac vectors with QVA analysis[7]. This extends our previous work which focused on the specialized set of pacemakers and conduction paths which initiate and disseminate electrical impulses to the muscle masses in order to coordinate and synchronize contraction[13,14]. The integrated system derives a robust model of cardiac function which can simulate ECG's qualitatively, and more important, explain ECG changes in terms of functional/structural disorders.

However, QVA is a very general framework that can be extensively used for commonsense reasoning about vectors, and in some way may also relate to how humans process this information visually. For this paper, we apply QVA to problem solving in mechanics, and demonstrate its application to solving two elementary examples in mechanics using qualitative reasoning techniques. The idea is to demonstrate that QVA in conjunction with qualitative reasoning techniques can be used to analyze and interpret the behavior of objects that are either in motion or in static equilibrium when acted upon by multiple forces. We envision this as a first step in developing an Intelligent Tutoring System for teaching elementary mechanics at the high school and college freshmen level. The system will encompass a broad range of concepts that will include: vectors, motion, velocity, acceleration, forces, Newton's laws, and the laws of momentum. Section 4.1 presents the first example, which reasons about objects in static equilibrium (the net force acting on them is zero). Section 4.2 discusses the second example which covers the computation of the motion and trajectory of objects that are acted upon by multiple forces.

4.1 Traffic Light Suspension Problem

Consider a traffic light that is suspended from two wires (Fig. 3) in the middle of a street intersection. Just to make the problem more interesting, consider that the two wires pass over frictionless pulleys that are supported vertically. There are two masses of equal mag-

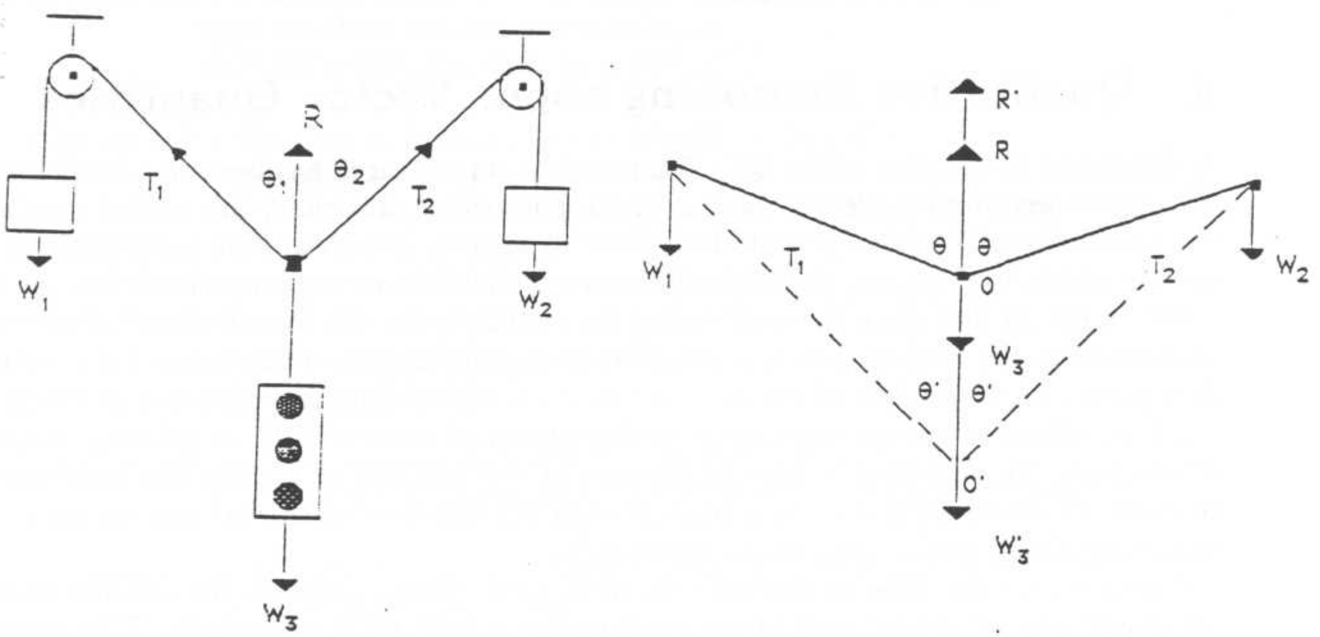


Figure 3: **Traffic Light Suspension Schematic.**

nitude suspended at the end of the two wires.

Analyzing this problem, we determine that there are three forces acting on the traffic light: the tension on the two wires $\mathbf{T_1}$ and $\mathbf{T_2}$, and the weight of the light $\mathbf{W_3}$ acting downward. Since the pulleys are ideal:

$$Mag(\mathbf{T_1}) = Mag(\mathbf{T_2}) = Mag(\mathbf{W_1}) = Mag(\mathbf{W_2}).$$

If the system is in equilibrium, the net force on the traffic light must be 0.

Lemma 3.1 directly applies to this situation, and we derive:

$$\Theta(\mathbf{R}, \mathbf{T_1}) = \Theta(\mathbf{T_2}, \mathbf{R}),$$

where $\mathbf{R}$ is the resultant of $\mathbf{T_1}$ and $\mathbf{T_2}$. Reasoning spatially and using geometry[3], we can then determine that $\mathbf{R}$ will be vertical and opposite of weight $\mathbf{W_3}$, i.e., $\Theta(\mathbf{R}, \mathbf{W_3}) =$ Opposite. Note that if $\mathbf{R}$ is not vertical then there will be a net force on point O in the horizontal direction. Therefore, the system will be in equilibrium if:

$$Mag(\mathbf{R}) = Mag(\mathbf{W_3}).$$

Once we have analyzed the above problem, the same framework can be used to reason about a number of situations that involve comparative analysis.

For example, consider what happens when the current traffic light is replaced by a heavier one. Can equilibrium be restored ?

[3]The spatial reasoning module is not fully operational. An overview of this module will be presented at the workshop.

This implies that $\mathbf{W}_3' > \mathbf{W}_3$. At point O the net force on the traffic light is going to have a non zero downward component. Therefore, point O will move downward. However, reasoning spatially we determine that as point O moves down, the angles $\Theta(\mathbf{T}_2, \mathbf{R})$ and $\Theta(\mathbf{R}, \mathbf{T}_1)$ decrease. Since previous conditions have not changed (i.e., $Mag(\mathbf{T}_1) = Mag(\mathbf{T}_2)$) $\Theta(\mathbf{R}', \mathbf{R}) =$ aligned. Using this fact and the fact that the angles decrease, Lemma 5.2 determines that $Proj(\mathbf{T}_1, \mathbf{R}') > Proj(\mathbf{T}_1, \mathbf{R})$ and $Proj(\mathbf{T}_2, \mathbf{R}') > Proj(\mathbf{T}_2, \mathbf{R})$. Since
$Mag(\mathbf{R}') = Proj(\mathbf{T}_1, \mathbf{R}') + Proj(\mathbf{T}_2, \mathbf{R}') > Proj(\mathbf{T}_1, \mathbf{R}) + Proj(\mathbf{T}_2, \mathbf{R}) = Mag(\mathbf{R})$ the problem solver derives $Mag(\mathbf{R}') > Mag(\mathbf{R})$[4]. This implies that as point O' descends, the resultant $\mathbf{R}'$ will become larger and eventually balance out $\mathbf{W}_{3'}$. Therefore, by qualitative reasoning about vector quantities, we can determine that equilibrium can be restored.

Consider a more complex example, i.e., reasoning how the traffic light could be moved so that it is positioned on the right lane.

Instinctively, to achieve this one would try to shorten the length of wire on the right, and this can be achieved by increasing the mass on that side. Therefore, $\mathbf{W}_2$ is now larger, and $\mathbf{T}_2' = \mathbf{W}_2' > \mathbf{T}_2 = \mathbf{W}_2$.

To simplify the problem, the problem solver reasons about it in a step-by step manner.

Step 1: Consider the impact of the increase in $\mathbf{W}_2$ to $\mathbf{W}_2'$.

Applying Lemma 1.1 we derive:

$$\Theta(\mathbf{R}, \mathbf{T}_1) < \Theta(\mathbf{R}', \mathbf{T}_1),$$
$$\Theta(\mathbf{T}_2, \mathbf{R}) > \Theta(\mathbf{T}_2', \mathbf{R}'), \text{ and}$$
$$Mag(\mathbf{R}') > Mag(\mathbf{R}).$$

This means that the resultant vector $\mathbf{R}'$ swings to the right and equilibrium conditions are no longer satisfied since $\mathbf{R}' \neq \mathbf{W}_3$. We deduce that the forces acting on the traffic light are unbalanced and the resultant force has non zero vertical and horizontal components. Reasoning spatially we deduce that the point **O** moves upward and to the right to say point **O′** in the direction of the resultant force $\mathbf{R}'$.

Step 2: Reason about the change from **O** to **O′**.

Let us analyze the situation at some point **O′** shown in Fig. 4. At this point

$$\Theta(\mathbf{T}_2', \mathbf{R}) > \Theta(\mathbf{T}_2, \mathbf{R}), \text{ and } \Theta(\mathbf{R}, \mathbf{T}_1', \mathbf{R}) > \Theta(\mathbf{R}, \mathbf{T}_1).$$

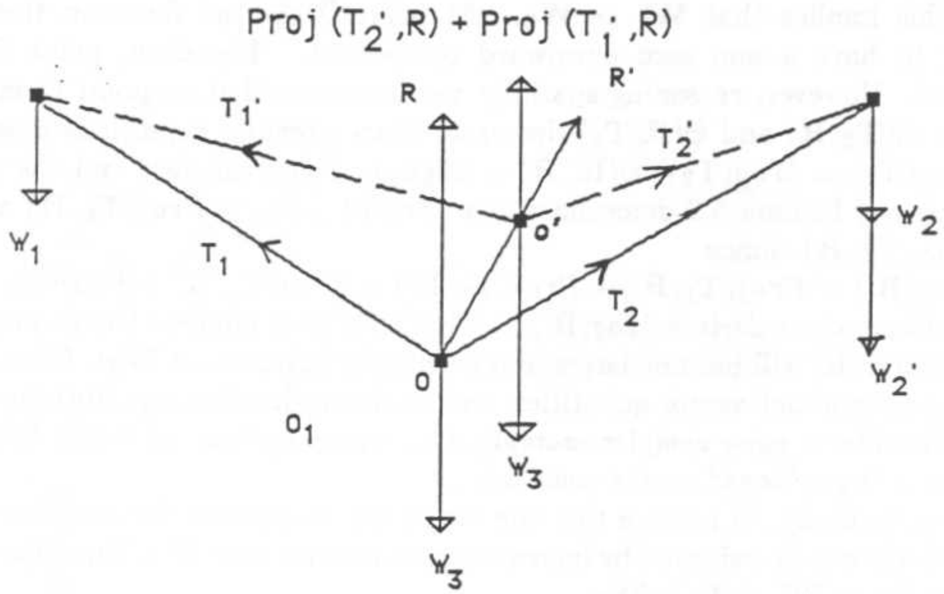
Using Lemma 5.1 we then derive:

$$Proj(\mathbf{T}_2', \mathbf{R}) < Proj(\mathbf{T}_2, \mathbf{R}) \text{ and } Proj(\mathbf{T}_1', \mathbf{R}) < Proj(\mathbf{T}_1, \mathbf{R}).$$

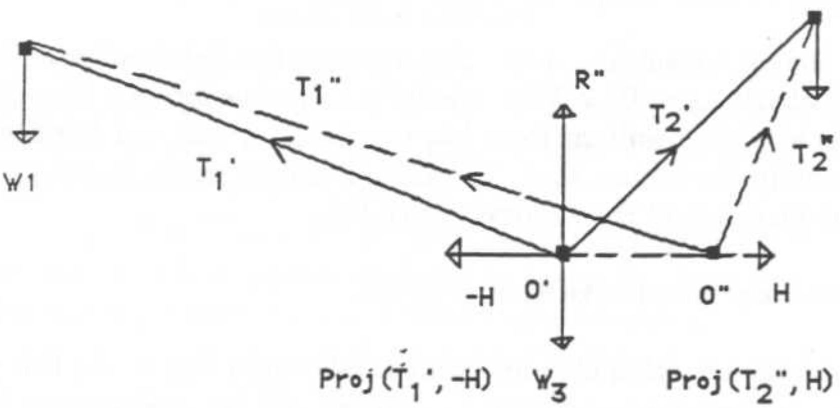
Therefore, the magnitude of the unbalanced force in the vertical direction decreases as the point **O** moves in the direction $\mathbf{R}'$, an should eventually become 0.

Step 3: Let's reason about the unbalanced force in the horizontal direction **H**.

[4]Note that this analysis is not very elegant, but our first attempts have been directed toward making the problem solver work.



Step 1 - Step 2



Step 2 - Step 3

Figure 4: Moving Traffic Light to Right Lane.

Since $\mathbf{W_2}' > \mathbf{W_2}$ initially there will be a net force in the direction **H** as shown in Fig. 4. Let's say the vertical force stabilizes at point $\mathbf{O'}$, but a net force still exists in direction **H**. This will cause $\mathbf{O'}$ to move to the right say to an intermediate point $\mathbf{O''}$. Reasoning spatially we can again establish that between $\mathbf{O'}$ and $\mathbf{O''}$ the following angular relationship holds:

$$\Theta(\mathbf{H}, \mathbf{T}'_2) < \Theta(\mathbf{H}, \mathbf{T}''_2), \text{ and } \Theta(, \mathbf{T}'_1, -\mathbf{H}) > \Theta(\mathbf{T}''_1, -\mathbf{H}).$$

Applying Lemmas 5.1 and 5.2 again enable us to determine that $Proj(\mathbf{T}''_2, \mathbf{H})$ will decrease and $Proj(\mathbf{T}''_1, -\mathbf{H})$ will increase, thus decreasing the unbalanced horizontal force which will eventually reach 0.

Since all the forces are balanced in a new position where the traffic light is now located on the right lane, we have been able to reason how the traffic light achieves equilibrium in a new position.

4.2 Maneuvering a Boat

Let us return to the man in the motor boat problem presented in the introduction. Using QVA and a few observations, we can now analyze in more detail the reasoning steps followed by the operator in going from point A to point B. Fig. 1 illustrates the problem.

Time 0: • Observation: A more knowledgeable person observing the current would probably angle his boat up stream, however, let us assume that our operator takes a more naive approach and sets a course straight across the river toward point B hoping that will get him across the river in the quickest time. Since he is in a hurry, he sets his boat speed fast enough, so that $Mag(\mathbf{F_b}) > Mag(\mathbf{F_w})$.

Time 1: • QVA reasoning: Given that $Mag(\mathbf{F_b}) > Mag(\mathbf{F_w})$ and $[\Theta(\mathbf{F_w}, \mathbf{F_b})] =$ Perpendicular.

Using QVA we can reason $[\Theta(\mathbf{R}, \mathbf{F_b})]$ and $[\Theta(\mathbf{F_w}, \mathbf{R})]$ are Acute. Therefore, the boat really heads downstream from point B.

- Observation: After a short while (say Time 2) the boatman realizes he is not heading in the right direction (it actually has moved the boat downstream from point B), changes the direction of the boat. Not quite having learned his lesson, he angles his boat directly at point B, at the same time increases the boat speed to compensate for his deviation. Now

$$Mag(\mathbf{F}'_b) > Mag(\mathbf{F_b}) > Mag(\mathbf{F_w})$$

Time 2: • QVA reasoning: Again $Mag(\mathbf{F}'_b) > Mag(\mathbf{F_w})$ but we observe that $[\Theta(\mathbf{F_w}, \mathbf{F_b})] =$ Obtuse[5]. Using QVA we reason $[\Theta(\mathbf{R}', \mathbf{F_b})] =$ Acute. This will direct the boat downstream of the objective.

- Observation: Having increased the speed of the boat quite a bit, and being a slow thinker, the boat operator finds he cannot make any changes, and ends up close to the shore downstream from B. However, he has a perfect solution for this dilemma, i.e., direct the boat straight upstream with a force greater than the current.

[5]This is derived by the spatial reasoning module which we have not explicitly discussed

Time 3: • QVA reasoning: Now $Mag(\mathbf{F}''_b) \gg Mag(\mathbf{F_w})$ and $[\Theta(\mathbf{F}''_b, \mathbf{F_w})]$ = Opposite. From Table 2.4, we can reason $[\Theta(\mathbf{F}''_b, \mathbf{R}'')]$ = Aligned. This ultimately puts the naive operator where he wants to be.

We hope that the naive operator, having assimilated QVA and some practical experience will do a better job the next time he operates the boat, such as pointing his boat in a direction to compensate for the current. Observations were chosen to make the example more interesting. Other scenarios are certainly possible, some could have been created which would have led to ambiguous conclusions in the QVA framework, and therefore, created multiple branches in a qualitative reasoning scheme. In general, however, with a few simple qualitative observations we were able to make some fairly powerful deductions in a non-trivial problem setting where quantitative information is not available.

5 Discussion

This paper discusses QVA, a formalism for qualitative manipulation of vectors. An intuitive quantity space description for representing angular directions on a plane is used as the framework for qualitative vector addition and projection onto a prespecified axis. Further, deductions on the relationships of the resultants (lemmas in Section 3) provide a framework for comparative analysis of changes from a given situation. These deductions will prove to be a powerful tool in relating overall behavior to structural changes.

A primary motivation for this work was the qualitative simulation of cardiac electrophysiology as discussed in [7]. In this paper, we have shown how QVA can be applied to more general physics problems of statics and motion. One interesting application would be the development of a basic physics problem solver in the domain of mechanics (both statics and dynamics) that could interact with users using vector and free body diagrams, or deal with phenomena that involves electrical and magnetic fields. Such a system could qualitatively reason about situations involving interacting vectors (e.g., forces), and better explain quantitative equations that model the real world in a qualitative framework.

Future work toward such a goal would need to incorporate geometric and spatial reasoning as well as vector analysis. A problem solver which combines both the vector analysis framework and geometric/spatial reasoning may be related to the way humans tend to visualize and reason about similar phenomena [9]. In summary, the QVA technique seems general enough to be applied to a myriad of physical situations that can be described in terms of vector interactions.

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