Extracting Duration Facts in Qualitative Simulation using Comparison Calculus

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Abstract

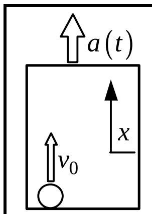
In a previous work, we described the *duration consistency filter*, which could extract information about the relative lengths of the time intervals (*duration facts*) in the output of qualitative simulation and use the inconsistencies among them to eliminate spurious behaviors. In this paper, we describe a new method for extracting duration facts that cannot be obtained by our previously described method. The power of the duration consistency filter is therefore increased. To prove the correctness of our approach, we have developed a new sign algebra, *the Extended Sign Algebra* SR1* and a new comparison formalism; *Comparison Calculus*.

Introduction

Qualitative simulation algorithms (Weld and Kleer 1990) symbolically solve sets of ordinary differential equations, predicting a set of trajectory descriptions, such that any actual solution of the equations in the input set is guaranteed to match one of these predictions in the output.

Along with qualitative descriptions of all possible behaviors that can be exhibited by systems in the input model, qualitative simulators may produce *spurious predictions*, behavior descriptions which no such system would exhibit. These spurious predictions limit the usefulness of qualitative reasoners in applications like design and diagnosis (Kuipers 1994). Several techniques (Kuipers 1994) (Say and Kuru 1993) (Say 1998) (Say 2001) for reducing the number of spurious behaviors in the algorithms' output have been developed. As a result of these studies, new filtering mechanisms have been proposed. The incorporation of some of these filters to the algorithm requires addition of extra items of information to the input set (like system-specific knowledge) or restrictions on the space of possible relationships between the system's variables (like the second derivative sign-equality assumption (Kuipers 1994).)

In (Könik and Say 1998), we described the *duration consistency filter*, which was able to extract information about the relative lengths of the time intervals (*duration facts*) in the output of qualitative simulation. This filter were using simple techniques employing concepts like symmetry, periodicity, and comparison of the circumstances during multiple traversals of the same interval to build a list of facts representing the deduced information about relative durations. These facts were fed to a simple linear inequality consistency checker, which eliminates proposed spurious behaviors leading to inconsistent duration data. In this paper, we propose a new method, *point-wise comparison*, for extracting duration facts that cannot be obtained by our previous method. The power of the duration consistency filter is therefore increased. To prove the consistency of this new method, we describe the relevant parts of the two mathematical formalisms that we have developed, namely, *the Extended Sign Algebra SR1**, and *Comparison Calculus*. Due to space limitations, some proofs will be skipped while some will only be outlined. A more thorough treatment can be found in (Könik 2000).



$$v = \frac{dx}{dt} \quad a = \frac{dv}{dt} \quad a' = \frac{da}{dt} \qquad a' = -c$$

x: height of the ball (ref. frame: elevator)
v: velocity of the ball (ref. frame: elevator)
a: acceleration of the ball(ref. frame: elevator)
c: a positive constant

Figure 1. Upwards Thrown Ball in Elevator with Increasing Acceleration

The Idea

Consider the system in Figure 1 depicting an upwards thrown ball in an elevator with increasing upwards acceleration. Figure 2 shows the behaviors of the position *x*, velocity *v* and the acceleration *a* of the ball with respect to the reference frame of the elevator[1]. In this problem, we want to be able to extract the duration facts $t_2 - t_1 < t_1 - t_0$ automatically. If this model is part of a

[1]To keep the model simple, we have assumed a constant value for a', but our technique is equally applicable as long as a is decreasing during the compared intervals.

bigger system that produces conflicting duration facts, we can eliminate spurious behaviors.

We show in (Könik 2000), that the constraints we presented in (Könik and Say 1998) are not sufficiently strong to extract this duration fact, although intuitively it is not very difficult to verify. If we consider the behavior of two imaginary balls starting at time t_1 at the maximum height, with zero velocity and moving according to the graph in Figure 2, one backwards in time and the other forward, we observe that the ball moving forward in time will have higher speed at each corresponding time point, since its acceleration is greater in magnitude at each corresponding time point. Consequently, we can conclude that the imaginary ball in the forward direction will hit the ground in a shorter time compared to the one in the backward direction, therefore $t_2 - t_1 < t_1 - t_0$. In the rest of the paper, we will develop the mathematical framework that enables us to obtain such results automatically.

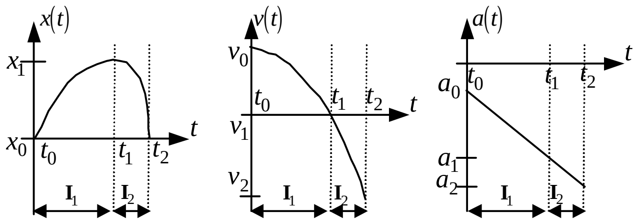


Figure 2. Comparison At Change of Direction

The Extended Sign Algebra SR1*

Qualitative reasoning algorithms make heavy use of signs for representing quantities. A brief overview of the new sign algebra SR1* that we use in our techniques is presented here.

A *sign* is often defined as one of the following four subsets of the set of real numbers $\Re$: $[+] =_{def} (0, +\infty)$, $[-] =_{def} (-\infty, 0)$, $[?] =_{def} (-\infty, +\infty)$, $[0] =_{def} \{0\}$. Williams (1991) has formalized the Sign Algebra S1 on the set $S' = \{[-], [0], [+], [?]\}$, with the sign addition and multiplication operations. For example, we have $[+]+[-]=[?]$ and $[+]\cdot[-]=[-]$ in S1. Here are some simple properties of S1 that we are going to use in this paper:

Proposition 1. Some Simple Properties of S1
For the signs s_1, s_2, $s_3 \in$ S1

i. $s_1 \subseteq s_2 \quad \leftrightarrow \quad s_1 = s_2 \quad$ if $s_2 \neq [?]$

ii. $s_1 = s_2 \cdot s_3 \quad \leftrightarrow \quad s_1 \cdot s_2 = s_3 \quad$ if $s_2 \neq [0],[?]$ □

Williams' SR1 (1991) is an extension of S1 where real numbers are elements of the domain in addition to signs of S'. For example, an expression such as $[+]+3=[+]$ is valid in SR1. Building on SR1, we have developed a new sign/real hybrid algebra called SR1* which proves useful in the forthcoming discussion. The domain of SR1* is the set of all nonempty subsets of $\Re$. The following set operators are well-defined on SR1*.

Definition 1. Set Operators
For A and B, two nonempty subsets of $\Re$,

i. $A+B =_{def} \{a+b: \ a \in A, \ b \in B\}$

ii. $-B =_{def} \{-b: \ b \in B\}$

iii. $A-B =_{def} A+(-B)$

iv. $A \times B =_{def} \{a \cdot b: \ a \in A, \ b \in B\}$

v. $|A| =_{def} \{|a|: \ a \in A\}$ □

Like SR1, SR1* contains all signs of S' and all real numbers[2] as elements. Unlike SR1, SR1* contains arbitrary subsets of $\Re$ as elements, and the operator semantics is slightly different. $[+]+(-3)$ is mapped to [?] in SR1 while it is mapped to $(-3, \infty)$ in SR1*. More information about SR1* and its relation with SR1 can be found in (Könik 2000).

*Proposition 2. A Set Theoretical Property of SR1**
Given $A_1, A_2, ..., A_n, B_1, B_2, ..., B_n \in$ SR1* such that $A_1 \subseteq B_1, A_2 \subseteq B_2, ..., A_n \subseteq B_n$ and Φ, a formula written using the binary and unary operators in Definition 1, the following relationship holds:

$$\Phi(A_1, A_2, ..., A_n) \subseteq \Phi(B_1, B_2, ..., B_n) \quad □$$

The sign abstraction operator [.] in SR1* is similar to its counterpart in SR1. Given $A \in$ SR1*, $[A]$ is the smallest sign in S' such that $A \subseteq [A]$. For example, the expressions $[(0,+\infty)]=[+]$, $[\{-1, +1\}]=[?]$, and $[5]=[+]$ are valid in SR1*. See (Könik 2000) for a formal definition of the abstraction operator in SR1*. The following are some properties of the abstraction operator of SR1*. Although Williams (1991) has presented similar statements for SR1, they deserve a separate proof in SR1*.

*Proposition 3. Abstraction Properties in SR1**
For any A, $B \in$ SR1*,

i. $[A+B] \subseteq [A]+[B]$

ii. $[A \cdot B] = [A] \cdot [B]$

iii. $[-A] = -[A]$

iv. $[A-B] \subseteq [A]-[B]$

Proof:

i. $A+B \subseteq [A]+[B]$ since $A \subseteq [A]$, $B \subseteq [B]$ (Prop. 2)
$[A+B] \subseteq [[A]+[B]] = [A]+[B]$
by first abstracting both sides then using $[A]+[B] \in S'$

ii. $[A \cdot B] \subseteq [A] \cdot [B]$ is similar to i but full equality has a

[2]As in SR1, the real numbers are considered to be in SR1*, because all singleton sets of real numbers are there, and the real numbers are isomorphic to them. For example 3+2=5 is considered to be in SR1* since {3}+{2} = {5} is in it.

long proof in (Könik 2000)

iii. using ii, by inserting B = {–1} □

Theorem 1. Abstraction of an SR1 Expression*

Given $A_1,...,A_n \in$ SR1* such that $A_1 \subseteq \Phi(A_2,...,A_n)$ where Φ is a formula written using the operators $\{\cdot,-,+\}$, we get: $[A_1] \subseteq \Phi([A_2],...,[A_n])$

Proof:

$[A_1] \subseteq [\Phi(A_2,...,A_n)]$

$\subseteq \Phi([A_2],...,[A_n])$ using Prop. 3 repeatedly □

Definition 2 Abstraction of a Function

Given a function *f*, and *F*, the image of *f* on a domain **I**, we call the sign valued expression $[F]$ *the image abstraction of f on the domain* **I** and for a $t\in$**I** we call $[f(t)]$, *the point abstraction of f at t.* □

Theorem 2. Abstraction of Functional Relations

Given *n* functions f_1, f_2, f_3, …, f_n and their images F_1, F_2, F_3, …, F_n on a domain $\mathbf{I}$, if we have

$$f_1(t) = \Phi\big(f_2(t), f_3(t),..., f_n(t)\big) \quad \forall t \in \mathbf{I}$$

such that Φ is a formula written using the operators $\{\cdot,-,+\}$, we get:

$$[F_1] \subseteq \Phi\big([F_2],[F_3],...,[F_n]\big)$$

Proof:

Using $F_1 \subseteq \Phi\big(F_2,F_3,...,F_n\big)$ and Theorem 1. □

Proposition 4. Image Abstraction of Sign Multiplication of Functions

Given three functions f, g, h and their images F, G, H on **I** such that $[f(t)] = [g(t)]\cdot[h(t)] \quad \forall t \in \mathbf{I}$, we get:

$[F] = [G]\cdot[H]$ if $[G] \neq [?] \vee [H] \neq [?]$ □

Comparison Calculus

Comparison Calculus is designed to accommodate automatic reasoning based on comparisons of behaviors of dynamic systems. It is built on top of SR1* Algebra described in previous section. Given a QSIM simulation, we will have algebraic comparison facts that can be extracted directly from a QSIM behavior. These facts will be fed in the algebraic constraints that we will prove in this section to deduce new algebraic facts that represent information about relative durations of time intervals in the simulation. These duration facts, which can be also represented as linear inequalities of time points, can be checked for consistency using a simple algorithm such as the one used in (Clarke and Zhao 1992) and if an inconsistency is detected, the behavior can be eliminated.

In the remainder of this section, we present a subset of Comparison Calculus that enables one to prove duration facts such as the one in the accelerating elevator problem. See (Könik 2000) for a full presentation of the formalism, and its other applications such as QSIM filter representation or Comparative Analysis.

Definition 3. Real Comparison Variables

For two real numbers x_1 and x_2, we define the *real comparison variables* Δx, Σx, and $\Delta|x|$ such that:

i. $\Delta x =_{def} x_2 - x_1$

ii. $\Sigma x =_{def} x_2 + x_1$

iii. $\Delta|x| =_{def} |x_2| - |x_1|$ (|*x*|: absolute value of *x*) □

Definition 4. Real Quantities that Summarize Some Property of a Function on an Interval

For two continuous functions f_1 and f_2 on intervals $\mathbf{I}_1 = (t_{b1},\ t_{e1})$ and $\mathbf{I}_2 = (t_{b2},\ t_{e2})$, we define the following quantities (for $j = 1, 2$):

i. $f_{bj} =_{def} \lim_{t\to t_{bj}} f_j(t)$: initial value of f_j on $\mathbf{I}_j$

ii. $f_{ej} =_{def} \lim_{t\to t_{ej}} f_j(t)$: final value of f_j on $\mathbf{I}_j$

iii. $f_{\Delta j} =_{def} f_{ej} - f_{bj}$: change of f_j on $\mathbf{I}_j$

iv. $t_{\Delta j} =_{def} t_{ej} - t_{bj}$: duration of the time interval $\mathbf{I}_j$

v. $\overline{f}_j =_{def} \frac{1}{t_{\Delta j}} \cdot \int_{t_{bj}}^{t_{ej}} f_j(t)\cdot dt$:average value of f_j on $\mathbf{I}_j$

f_{bj} and f_{ej} will also be called the end points of f_j on $\mathbf{I}_j$ □

The real quantities above will be inserted into Definition 3 to obtain expressions that are useful to make comparisons. For example for two time intervals $\mathbf{I}_1$ and $\mathbf{I}_2$ and their durations $t_{\Delta 1}$ and $t_{\Delta 2}$, we get well-defined sign valued expressions such as $[\Delta|t_\Delta|] = [|t_{\Delta 2}| - |t_{\Delta 1}|]$. Expressions like these allow us to represent comparison facts using algebraic formulas. For example if we consider $\mathbf{I}_1 = (t_0,\ t_1)$ and $\mathbf{I}_2 = (t_1,\ t_2)$ in Figure 2, the duration fact, $t_2 - t_1 < t_1 - t_0$ we want to extract can be represented as $[\Delta|t_\Delta|] = [-]$. Similarly, if we consider the functions x_1 and x_2 to be the restriction of the function *x* on the intervals $\mathbf{I}_1$ and $\mathbf{I}_2$, using the change variables $x_{\Delta 1}$ and $x_{\Delta 2}$ we obtain the algebraic expression $[\Delta|x_\Delta|] = [|x_{\Delta 2}| - |x_{\Delta 1}|]$. This expression gives us the change-of-*x* comparison in two intervals. For example in Figure 2, we have $[\Delta|x_\Delta|] = [0]$, meaning that the change of *x* in the interval $\mathbf{I}_1$ is equal in magnitude to the change of *x* in $\mathbf{I}_2$.

Theorem 3. Real Comparison Constraints

For two real numbers x_1 and x_2

i.a. $[\Delta|x|] = [\Delta x \cdot \Sigma x] = [\Delta x]\cdot[\Sigma x]$

i.b. $[\Sigma x] = [\Delta|x|]\cdot[\Delta x]$ if $[\Delta x] \neq [0]$

i.c. $[\Delta x]=[\Delta|x|]\cdot[\Sigma x]$ if $[\Sigma x]\neq[0]$

ii.a. $[\Sigma x]=\Sigma[x]$ if $\Sigma[x]\neq[?]$

ii.b. $[\Delta x]=\Delta[x]$ if $\Delta[x]\neq[?]$

Proof:

i.a. $[\Delta|x|]=[|x_2|-|x_1|]=[|x_2|^2-|x_1|^2]=[x_2{}^2-x_1{}^2]=$
$[(x_2-x_1)\cdot(x_2+x_1)]=[\Delta x\cdot\Sigma x]=[\Delta x]\cdot[\Sigma x]$ (Prop. 3.ii)

i.b. using i.a and Prop. 1.ii since $[\Delta x]\neq[?]$ (Δx is real)

i.c. similar to i.b.

ii.a. $[\Sigma x]=[x_2+x_1]\subseteq[x_2]+[x_1]=\Sigma[x]$ (Prop. 3.i)
Implying $[\Sigma x]=\Sigma[x]$ since $\Sigma[x]\neq[?]$ (Prop. 1.i)

ii.b. similar to ii.a □

The constraints in Thm. 3, are useful to compare real valued variables such as time durations, functions at a point, or change of a functions in an interval. Next, we describe how we can compare functions throughout intervals.

Definition 5. Comparison Functions on the Same Domain
For two functions f_1 and f_2, we construct *the comparison functions* Δf, Σf, and $\Delta|f|$ on a common interval $\mathbf{I}$ as follows :

i. $\Sigma f(t) =_{def} f_2(t)+f_1(t)$ $(\forall t\in\mathbf{I})$

ii. $\Delta f(t) =_{def} f_2(t)-f_1(t)$ $(\forall t\in\mathbf{I})$

iii. $\Delta|f|(t)=_{def} |f_2(t)|-|f_1(t)|$ $(\forall t\in\mathbf{I})$ □

Definition 6. Same Domain Pointwise Comparison Signs
Given two functions f_1 and f_2 and their comparison functions Σf, Δf, and $\Delta|f|$ on a common interval $\mathbf{I}$, if we let F_1, F_2, ΣF, ΔF, and $\Delta|F|$ be their images on $\mathbf{I}$, the well-defined sign expressions $[\Delta F]$, $[\Sigma F]$, and $[\Delta|F|]$ are called *the pointwise comparison signs*. Specifically, $[\Delta|F|]$ is called the *pointwise magnitude comparison sign*. Moreover, we call $[F_1]$ and $[F_2]$ the *simple sign constants* and using them we define *the compound sign constants* as: $\Sigma[F]=_{def}[F_2]+[F_1]$ and $\Delta[F]=_{def}[F_2]-[F_1]$. Similarly, for two real numbers x_1 and x_2, we will call the well-defined expressions $[\Delta x]$, $[\Sigma x]$, and $[\Delta|x|]$ *the real comparison signs* and $[x_1]$, $[x_2]$, $\Delta[x]$, and $\Sigma[x]$, *the real sign constants*. □

Proposition 5. Trivial Pointwise Comparison Constraints
Given two functions f_1, f_2 compared on a common interval **I,** the following constraints hold:

i. (a) $[F_1]=[F_2]=[0]$ $\leftrightarrow$
(b) $\Delta[F]=[0]$ $\leftrightarrow$
(c) $\Sigma[F]=[0]$ $\leftrightarrow$
(d) $[\Delta|F|]=[\Delta F]=[\Sigma F]=[0]$

ii. $[\Sigma F]\subseteq\Sigma[F]$

iii. $[\Delta F]\subseteq\Delta[F]$

iv. (a) $[F_1]\neq[?]\wedge[F_2]\neq[?]$ $\leftrightarrow$
(b) $\Delta[F]\neq[?]\vee\Sigma[F]\neq[?]$ $\rightarrow$
(c) $[\Delta F]\neq[?]\vee[\Sigma F]\neq[?]$

Proof:

ii. $\Sigma f(t)=f_2(t)+f_1(t)\rightarrow[\Sigma F]\subseteq[F_1]+[F_2]$ (Thm. 2)

iii. similar to ii □

The next theorem establishes the relation between the comparison signs $[\Delta F]$, $[\Sigma F]$, and $[\Delta|F|]$, given the sign constants.

Theorem 4. Pointwise Comparison Constraints
Given two functions f_1, f_2 compared on a common interval **I,**

i. $[\Delta|F|]=[\Delta F]\cdot[\Sigma F]$ if $\Delta[F]\neq[?]\vee\Sigma[F]\neq[?]$

ii. $[\Sigma F]=\Sigma[F]$ if $\Sigma[F]\neq[?]$

iii. $[\Delta F]=\Delta[F]$ if $\Delta[F]\neq[?]$

iv. $[\Sigma F]=[\Delta|F|]\cdot\Delta[F]$ if $\Delta[F]\neq[?]$

v. $[\Delta F]=[\Delta|F|]\cdot\Sigma[F]$ if $\Sigma[F]\neq[?]$

Proof:

i. For two functions f_1 and f_2 we get $(\forall t\in\mathbf{I})$:
$[\Delta|f|(t)]=[\Delta f(t)]\cdot[\Sigma f(t)]$ (Thm.3.i.a, Def. 5)
Proved by Prop. 4 using (b)→(c) in Prop.5.iv

ii, iii. by Proposition 5 ii, iii using Proposition 1.i

iv. $[\Delta F]=\Delta[F]\neq[?]$ using iii
$[\Delta F]=\Delta[F]=[0]$ is trivial by Prop.5.i. otherwise:
Since $[\Delta F]\neq[0],[?]\rightarrow[\Delta f(t)]\neq[0]$ $\forall t\in\mathbf{I}$
For f_1 and f_2 we get $(\forall t\in\mathbf{I})$: (Theorem 3.i.b)
$[\Sigma f(t)]=[\Delta|f|(t)]\cdot[\Delta f(t)]$ and,
proved by Prop. 4 since $[\Delta F]=\Delta[F]\neq[?]$ □

In application of Comparison Calculus we will describe in this paper, when we compare two functions f_1 and f_2 on $\mathbf{I}$, the sign constants $[F_1]$, $[F_2]$, and therefore $\Delta[F]$, $\Sigma[F]$ will be trivially extractable from a QSIM behavior. Moreover, the comparison constraints are usually useful when compared functions f_1 and f_2 don't change signs on $\mathbf{I}$ ($[F_1],[F_2]\neq[?]$). If that is the case, either $\Sigma[F]\neq[?]$ or $\Delta[F]\neq[?]$ will be true (Prop 5.iv. (a)→(b)) ensuring that some of the constraints in Theorem 4 will always be applicable. Closer inspection of these constraints reveals that, given the sign constants, the two comparison signs $[\Delta F]$ and $[\Sigma F]$ can be calculated from $[\Delta|F|]$. Therefore, obtaining $[\Delta|F|]$ will be more important then obtaining $[\Delta F]$ and $[\Sigma F]$.

Theorem 5.Qualitative Fundamental Theorem Of Calculus
For a function h with a continuous derivative h', if H and H' are their images on the interval $\mathbf{I}=(t_b,\ t_e)$, we

get: $[H] \subseteq [h_b]+[H']$ $\quad (h_b = h(t_b))$ □

The next theorem establishes the link between the comparison signs of functions with the comparison signs of their derivative.

Theorem 6. Comparison Propagation Over Derivative
For two functions f_1 and f_2 with continuous derivatives f_1' and f_2', we get:

i. $[\Delta F] \subseteq [\Delta f_b]+[\Delta F']$ $\quad (\Delta f_b = f(t_{b2}) - f(t_{b1}))$

ii. $[\Sigma F] \subseteq [\Sigma f_b]+[\Sigma F']$ $\quad (\Sigma f_b = f(t_{b2}) + f(t_{b1}))$

Proof:
Proved by replacing h in Thm. 5 with Δf and Σf □

So far, we have assumed that the functions we compare are defined on the same interval. Our strategy for comparing two functions on different intervals will be to construct a single standard comparison interval, and to transform the compared functions so that they are defined on it.

Definition 7. Minimum Interval
Given two intervals $\mathbf{I}_1 = (t_{b1},\ t_{e1})$ and $\mathbf{I}_2 = (t_{b2},\ t_{e2})$ we define the *minimum interval* of $\mathbf{I}_1$ and $\mathbf{I}_2$ as $\mathbf{I} = (0,\ t_\Delta)$, where $t_\Delta = min(t_{\Delta 1},\ t_{\Delta 2})$ and $t_{\Delta 1}$, $t_{\Delta 2}$ are the lengths the two intervals. □

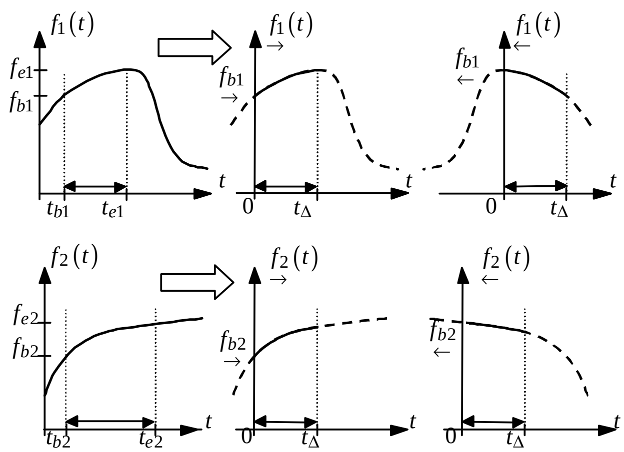


Figure 3. Constructing Directional Functions

Definition 8. Directional Functions
Given a pair of functions f_1, f_2 on the domains $\mathbf{I}_1 = (t_{b1},\ t_{e1})$, $\mathbf{I}_2 = (t_{b2},\ t_{e2})$, for each f_i we define two functions, the *forward directional function* $\underset{\rightarrow}{f_i}$, and the *backward directional function* $\underset{\leftarrow}{f_i}$ on the minimum interval $\mathbf{I}$ of $\mathbf{I}_1$ and $\mathbf{I}_2$ as follows:

i. $\underset{\rightarrow}{f_i}(t) = f_i(t_{bi}+t)$

ii. $\underset{\leftarrow}{f_i}(t) = f_i(t_{ei}-t)$ □

For a function f_i, the function $\underset{\rightarrow}{f_i}$ starts at the initial point of f_i and traces its values in the forward direction for the duration of the minimum interval. Similarly, $\underset{\leftarrow}{f_i}$ starts at the end point of f_i and traces it backwards. Figure 3 shows two functions f_1 and f_2 with their corresponding directional functions.

By comparing each directional function of f_1 with each directional function of f_2, we get four different sets of comparison functions. To distinguish these, we add the directions of the directional functions as subscripts to the comparison functions.

Definition 9. Comparison Functions on Different Domains
Given a pair of functions f_1, and f_2 on $\mathbf{I}_1$, and $\mathbf{I}_2$, we define the *directional comparison functions* $\underset{\alpha\beta}{\Delta f}(t)$, $\underset{\alpha\beta}{\Sigma f}(t)$ and $\Delta|\underset{\alpha\beta}{f}|(t)$ on the minimum interval $\mathbf{I}$ to be the comparison functions of the directional functions $\underset{\alpha}{f_1}$ and $\underset{\beta}{f_2}$ on $\mathbf{I}$ where α and β are two directions from the set {"→", "←"} such that we get:

i. $\underset{\alpha\beta}{\Delta f}(t) = \underset{\beta}{f_2}(t) - \underset{\alpha}{f_1}(t)$

ii. $\underset{\alpha\beta}{\Sigma f}(t) = \underset{\beta}{f_2}(t) + \underset{\alpha}{f_1}(t)$

iii. $\Delta|\underset{\alpha\beta}{f}|(t) = |\underset{\beta}{f_2}(t)| - |\underset{\alpha}{f_1}(t)|$ □

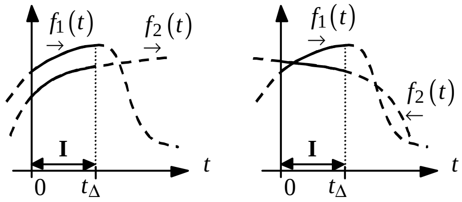
Table 1. Directional Comparison Functions and Signs

f_1, f_2	*direction*	*comparison signs*		
→,→	forward	$[\Delta\lvert\underset{\rightarrow}{F}\rvert]$	$[\Sigma\underset{\rightarrow}{F}]$	$[\Delta\underset{\rightarrow}{F}]$
←,←	backward	$[\Delta\lvert\underset{\leftarrow}{F}\rvert]$	$[\Sigma\underset{\leftarrow}{F}]$	$[\Delta\underset{\leftarrow}{F}]$
←,→	outward	$[\Delta\lvert\underset{\leftrightarrow}{F}\rvert]$	$[\Sigma\underset{\leftrightarrow}{F}]$	$[\Delta\underset{\leftrightarrow}{F}]$
→, ←	inward	$[\Delta\lvert\underset{\rightarrow\leftarrow}{F}\rvert]$	$[\Sigma\underset{\rightarrow\leftarrow}{F}]$	$[\Delta\underset{\rightarrow\leftarrow}{F}]$

For example, if we compare $\underset{\leftarrow}{f_1}$ and $\underset{\rightarrow}{f_2}$, the backward function of f_1, with the forward function of f_2, we will get the outward comparison functions and signs, which have the subscript "↔", a simplification of "←, →". The names and simplified subscripts of all comparison signs are given in Table 1.

In Figure 4, we compare the directional functions from Figure 3 in the forward and inward directions. For (a), we get: $[\Delta|\underset{\rightarrow}{F}|]$=[–], $[\Delta\underset{\rightarrow}{F}]$=[–], $[\Sigma\underset{\rightarrow}{F}]$=[+] and for (b) we get: $[\Delta|\underset{\rightarrow\leftarrow}{F}|]$=[?], $[\Delta\underset{\rightarrow\leftarrow}{F}]$=[?], $[\Sigma\underset{\rightarrow\leftarrow}{F}]$=[+].

Next, we will present the main pointwise comparison constraint that is key to the kind of reasoning described in second section. We first explain the kind of reasoning we want to obtain, on a simple example.



(a) (b)

Figure 4. Directional Pointwise Comparison Example

Let us assume we observe two cars with positions x_1, x_2 and velocities v_1, v_2 on the intervals $\mathbf{I}_1$, $\mathbf{I}_2$. Assume that we know:

$[\Delta\,|\underset{\rightarrow}{V}|] = [+]$: the speed of the second car is pointwise greater in the minimum interval

$[V_i] \neq [?],\ [0]$: The cars don't stop or change direction

$[\Delta\,|\,x_\Delta\,|] = [-]$: The first car has traveled a longer distance.

The second car should travel more distance during the minimum interval **I** in the first part of the comparison **(**since $[\Delta\,|\underset{\rightarrow}{V}|] = [+]$**)**. On the other hand, the first car should travel more in total during $\mathbf{I}_1$ ($[\Delta\,|\,x_\Delta\,|] = [-]$). Since the cars cannot stop or change direction ($[V_i] \neq [?],\ [0]$), after the end of **I,** the first car should continue to travel for overcoming the distance traveled by the second car. As a result, we can conclude that the first car has traveled longer**,** that is: $[\Delta\,|\,t_\Delta\,|] = [-]$.

Theorem 7. Main Pointwise Comparison Constraint

Let x_1, x_2 be two functions and v_1, v_2 their continuous derivatives on $\mathbf{I}_1$, $\mathbf{I}_2$. For two directions α and β, we have:

$$[\Delta\,|\,x_\Delta\,|] \subseteq [\Delta\,|\underset{\alpha\,\beta}{V}|] + [\Delta\,|\,t_\Delta\,|] \quad (\text{if } [V_i] \neq [?],[0] \quad i=1,\,2)$$

□

We can get the same result that we obtained intuitively above, using our new constraint. Since we have $[\Delta\,|\underset{\rightarrow}{V}|] = [+]$, $[\Delta\,|\,x_\Delta\,|] = [-]$, and since the precondition of the constraint is satisfied, we get $[-] \subseteq [+] + [\Delta\,|\,t_\Delta\,|]$ which implies $[\Delta\,|\,t_\Delta\,|] = [-]$.

Although the main pointwise comparison constraint looks somewhat similar to the one that we have presented in (Könik and Say 1998), that constraint cannot be applied in this situation. It requires information about the average velocity comparison $[\Delta\,|\,\overline{v}\,|]$ and, we show in (Könik 2000) that in this situation more than one possible value for $[\Delta\,|\,\overline{v}\,|]$ is consistent with the available information.

Similarly, if we knew $[\Delta\,|\underset{\leftrightarrow}{V}|] = [+]$ in the accelerating elevator problem, since we could detect $[\Delta\,|\,x_\Delta\,|] = [0]$ by observing the QSIM behavior, we could conclude $[\Delta\,|\,t_\Delta\,|] = [-]$. $[\Delta\,|\underset{\leftrightarrow}{A}|] = [+]$ is easy to detect because acceleration at any point in the second interval is greater than the acceleration at any point in the first interval, and this fact can be extracted from a QSIM trace by ordinal comparison of the end points of *a*. Theorem 6 is the link between the comparison sign of a function and its derivative, and Theorem 4 is useful for conversions between the three comparison sign types. Using them, we will show how one can obtain the value of $[\Delta\,|\underset{\leftrightarrow}{V}|]$ from the value of $[\Delta\,|\underset{\leftrightarrow}{A}|] = [+]$. The only remaining problem is that these two theorems are not stated in terms of directional pointwise comparison signs in Table 1. The next theorem fills in this gap.

Theorem 8. Mapping Between Directional Functions and Derivative Rules

If we let f_1 and f_2 be the directional functions of v_1 and v_2 on **I** in the directions α_1 and α_2 such that $f_1(t) = \underset{\alpha_1}{v_1}(t)$, $f_2(t) = \underset{\alpha_2}{v_2}(t)$ and a_1, a_2 are the derivatives of v_1 and v_2, we will get:

i. $$f_i'(t) = \begin{cases} \underset{\rightarrow}{a_i}(t) & \text{if } \alpha_i = \text{'}\rightarrow\text{'} \\ -\underset{\leftarrow}{a_i}(t) & \text{if } \alpha_i = \text{'}\leftarrow\text{'} \end{cases}$$

Inputs:

ii. $$f_{bi} = \begin{cases} v_{bi} & \text{if } \alpha_i = \text{'}\rightarrow\text{'} \\ v_{ei} & \text{if } \alpha_i = \text{'}\leftarrow\text{'} \end{cases}$$

iii. $[\Delta\,|\,F'\,|] = [\Delta\,|\underset{\alpha_1\,\alpha_2}{A}|]$

iv. $[F_i] = [V_i] \qquad \text{if } [V_i] \neq [?]$

v. $$[F_i'] = \begin{cases} [A_i] & \text{if } \alpha_i = \text{'}\rightarrow\text{'} \text{ and } [A_i] \neq [?] \\ -[A_i] & \text{if } \alpha_i = \text{'}\leftarrow\text{'} \text{ and } [A_i] \neq [?] \end{cases}$$

Output:

vi. $[\Delta\,|\underset{\alpha_1\,\alpha_2}{V}|] = [\Delta\,|\,F\,|]$

Proof:

i. $$f_i'(t) = \begin{cases} v_i'(t_{bi} + t) = a_i(t_{bi} + t) = \underset{\rightarrow}{a_i}(t) & \text{if } \alpha_i = \text{'}\rightarrow\text{'} \\ v_i'(t_{ei} - t) = -a_i(t_{ei} - t) = -\underset{\leftarrow}{a_i}(t) & \text{if } \alpha_i = \text{'}\leftarrow\text{'} \end{cases}$$

ii. $$f_{bi} = \begin{cases} \underset{\rightarrow}{v_i}(0) = v_i(t_{bi}) = v_{bi} & \text{if } \alpha_i = \text{'}\rightarrow\text{'} \\ \underset{\leftarrow}{v_i}(0) = v_i(t_{ei}) = v_{ei} & \text{if } \alpha_i = \text{'}\leftarrow\text{'} \end{cases}$$

iii. We have $|\,f_i'(t)\,| = |\underset{\alpha_i}{a_i}(t)|$ using i therefore we get the result using: $\Delta\,|\,f'\,|(t) = |\,f_2'(t)\,| - |\,f_1'(t)\,| = |\underset{\alpha_2}{a_2}(t)| - |\underset{\alpha_1}{a_1}(t)| = \Delta\,|\underset{\alpha_1\,\alpha_2}{a}|\,(t)$

iv. Using Prop. 1.i with $[F_i\,] \subseteq [V_i]$ (since $F_i = \underset{\alpha_i}{V_i} \subseteq V_i$)

v. Using i and $[A_i] \neq [?]$ we get:

$$[F_i'] = \begin{cases} [\underset{\rightarrow}{A_i}] = [A_i] & \text{if } \alpha_i = '\rightarrow' \\ -[\underset{\leftarrow}{A_i}] = -[A_i] & \text{if } \alpha_i = '\leftarrow' \end{cases}$$

□

We are finally ready to apply our techniques to extract the desired duration facts from the accelerating elevator problem (Figure 1). We let: $\mathbf{I}_1 = (t_0, t_1)$ and $\mathbf{I}_2 = (t_1, t_2)$ and make an outward comparison. The following quantities can be extracted from the behavior in a straightforward way (Figure 2).

$$[v_{e1}] = [v_{b2}] = [v(t_1)] = [0], \quad [V_1] = [+], \quad [V_2] = [-],$$
$$[A_1] = [-], \quad [A_2] = [-], \quad [\Delta \,|\, \underset{\leftrightarrow}{A} \,|] = [+]$$

To apply the mapping in Thm. 8**,** we let $(\alpha_1, \alpha_2) = (\leftarrow, \rightarrow)$:

$f_1(t) = \underset{\leftarrow}{v_1}(t)$ and $f_2(t) = \underset{\rightarrow}{v_2}(t)$ and we get:

$$[f_{b1}] = [v_{e1}] = [0], \quad [f_{b2}] = [v_{b2}] = [0] \qquad \text{(ii)}$$
$$[\Delta \,|\, F' \,|] = [\Delta \,|\, \underset{\leftrightarrow}{A} \,|] = [+] \qquad \text{(iii)}$$
$$[F_1] = [V_1] = [+], \; [F_2] = [V_2] = [-] \qquad \text{(iv)}$$
$$[F_1'] = -[A_1] = [+], \; [F_2'] = [A_2] = [-] \qquad \text{(v)}$$

Using these values we can apply the rules derived earlier and they fire as depicted in Table 2.

Table 2. Derivative Pointwise Comparison Rules Firing Example

Fired Rule	*Reason*	*Result*
Def. 6	$[f_{b1}] = [f_{b2}] = [0]$	$\Sigma[f_b] = [0]$
Thm.3.ii.a	$\Sigma[f_b] = [0]$	$[\Sigma f_b] = [0]$
Def. 6	$[F_1'] = [+]$ and $[F_2'] = [-]$	$\Delta[F'] = [-]$
Thm.4.iv	$[\Delta \,|\, F' \,|] = [+]$, $\Delta[F'] = [-]$	$[\Sigma F'] = [-]$
Thm.6.ii	$[\Sigma f_b] = [0]$, $[\Sigma F'] = [-]$	$[\Sigma F] = [-]$
Def. 6	$[F_1] = [+]$, $[F_2] = [-]$	$\Delta[F] = [-]$
Thm.4.iii	$\Delta[F] = [-]$	$[\Delta F] = [-]$
Thm.4.i	$[\Sigma F] = [-]$, $[\Delta F] = [-]$	$[\Delta \,|\, F \,|] = [+]$

The output obtained is $[\Delta \,|\, \underset{\leftrightarrow}{V} \,|] = [\Delta \,|\, F \,|] = [+]$ (Theorem 8.vi). Since we also know that the distances traveled in the two intervals are the same ($[\Delta \,|\, x_\Delta \,|]$=[0]) and since the other requirements of Theorem 7 are satisfied, we can simplify the constraint $[\Delta \,|\, x_\Delta \,|] \subseteq [\Delta \,|\, t_\Delta \,|] + [\Delta \,|\, \underset{\leftrightarrow}{V} \,|]$ to $[0] \subseteq [+] + [\Delta \,|\, t_\Delta \,|]$, which implies $[\Delta \,|\, t_\Delta \,|]$=[–], leading to the extraction of the duration fact $t_2 - t_1 \; < \; t_1 - t_0$.

To calculate the comparison sign $[\Delta \,|\, \underset{\leftrightarrow}{V} \,|]$, we have used the comparison sign $[\Delta \,|\, \underset{\leftrightarrow}{A} \,|]$ and the sign constants $[v_{e1}]$, $[v_{b2}]$, $[V_1]$, $[V_2]$, $[A_1]$, and $[A_2]$. The sign constants can be easily extracted from the behavior. A pointwise comparison sign is either calculated from a total comparison as in the case of $[\Delta \,|\, \underset{\leftrightarrow}{A} \,|]$, or, as in the case of $[\Delta \,|\, \underset{\leftrightarrow}{V} \,|]$, its value can propagate over a derivative constraint by applying the comparison constraints (Theorem 3, Theorem 4 and Theorem 6) together with the mapping in Theorem 8. It is also possible for a pointwise comparison sign to be computed from its higher order derivatives using the above scheme in a recursive way. Moreover, pointwise comparison can propagate over other constraints such as addition, multiplication, M^+ and M^- relations. (Könik 2000)

Conclusion

We have presented a method for extracting relative duration facts from pure qualitative simulation outputs which, when combined with other duration extraction methods, may lead to spurious behavior elimination. As a useful “side effect”, we have developed a new sign algebra and a comparison formalism, which can be applied to other qualitative reasoning problems such as comparative analysis as well.

In this paper, we have focused on the formal description and consistency proofs of the duration fact extraction system. Experiments to test the effectiveness of this system for eliminating spurious behaviors will be performed in further stages of our research.

In (Könik 2000) we started to formalize a more general mathematical framework, *comparison filters*. It combines the pointwise comparison techniques described here and the symmetry extraction methods described in (Könik and Say 1998) in a unified framework based on Comparison Calculus and SR1* Algebra. In (Könik 2000), we have proven consistency of additional constraints and obtained some theoretical results about their usefulness.

Relative duration fact extraction was first implemented by Çivi (1992), who presents a postprocessor which annotates QSIM outputs with deduced temporal interval comparisons for some fixed models. Çivi’s work does not deal with spurious behaviors.

Weld’s(1988) *differential qualitative* (DQ) *analysis* technique involves conceptually comparing two behaviors of the same variable for purposes of perturbation analysis. When comparing multiple traversals of the same interval, we use similar mathematical foundations, albeit for a different purpose. Some theorems and concepts of DQ analysis are special instances of Comparison Calculus, and some are complementary to the theorems that we have proved here. Specifically, DQ analysis does not deal with comparison in different directions, while Comparison Calculus does not deal with comparison with respect to a variable other than time. In (Könik 2000), we showed how Comparison Calculus can solve a perturbation analysis problem that DQ cannot solve.

Some of the simulations improved by the duration consistency filter involve *occurrence branching,* in which multiple branches are added to the behavior tree to represent different possible time-orderings of two “unrelated” variables reaching their respective landmarks. “History”-based reasoners like Williams’ TCP (1986) were designed with the purpose of eliminating this phenomenon. There has been some work (Clancy and Kuipers 1993) (Tokuda 1996) to modify the QSIM framework in this direction. Our approach would be useful in cases where the distinctions created by the “global state”-based branching mechanisms are relevant from the user’s point of view, and incorrect predictions in this format need to be minimized.

Hybrid qualitative-quantitative reasoners (Berleant and Kuipers 1997) (Kay 1996) enable the association of numerical values with the time-points in the qualitative simulation output, rendering the comparison of interval lengths trivial. Our work shows that such comparisons are possible and useful in pure qualitative simulation as well.

Since the introduction of "pure" QSIM in (Kuipers 1986), several other global filters (Kuipers 1994) (Say and Kuru 1993) (Say 1998) (Say 2001) have been added to the repertory, dealing with different classes of spurious predictions. The duration consistency filter eliminates the set of predictions from which inconsistent conclusions about the relative lengths of the time intervals can be drawn, improving the predictive performance of the overall simulator, and increasing the level of complexity of the systems that can be reasoned about. The work reported here further improves the power of duration consistency filter.

Availability of the program

The PROLOG source code of our implementation of QSIM with the duration consistency filter is available to interested researchers. Contact: konik@umich.edu

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