Consistent Relative and Absolute Order-of-Magnitude Models

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Abstract

The aim of this paper is to analyse under which conditions Absolute Order-of-Magnitude and Relative Order-of-Magnitude models may be consistent and to determine the constraints which guarantee consistency. A graphical interpretation of the constraints is provided, bridging the absolute qualitative labels of two quantities into their corresponding relative relation(s), and conversely. The ROM relations are then characterized in the absolute order-of-magnitude world.

1. Introduction

Order-of-magnitude models are an essential piece among the theoretical tools available for qualitative reasoning about physical systems (Travé-Massuyès *et al.* 97), (Kalagnanam *et al.* 91), (Struss 88). They aim at capturing order-of-magnitude commonsense inferences, such as used in the engineering world. Order-of-magnitude knowledge may be of two types: absolute or relative. The absolute order-of-magnitudes (AOM) are represented by a partition of $\mathbb{R}$, each element of the partition standing for a basic qualitative class. A general algebraic structure, called Qualitative Algebra or Q-algebra, was defined on this framework (Travé&Piera 89)(Piera&Travé 89), providing a mathematical structure which unifies sign algebra and interval algebra through a continuum of qualitative structures built from the rougher to the finest partition of the real line. The most referenced order-of-magnitude Q-algebra partitions the real line into 7 classes, corresponding to the labels: Negative Large(NL), Negative Medium(NM), Negative Small(NS), Zero(0), Positive Small(PS), Positive Medium(PM) and Positive Large(PL). Q-algebras and their algebraic properties have been extensively studied (Missier *et al.* 89),(Piera&Travé 90), (Missier 91), (Agell 98).

Order-of-magnitude knowledge may also be of relative type (ROM), in the sense that a quantity is now qualified with respect to another quantity by means of a set of binary order-of-magnitude relations. The seminal ROM model was the formal system FOG (Raiman 91), based on three basic relations, used to represent the intuitive concepts of "negligible with respect to" (Ne), "close to" (Vo) and "comparable to" (Co), and described by 32 intuition-based inference rules. The ROM models that were proposed later improved FOG not only in the necessary aspect of a rigorous formalisation, but also permitting the incorporation of quantitative information when available and the control of the inference process, in order to obtain valid results in the real world (Mavrovouniotis&Stephanopoulos 87) and (Dague 93a),(Dague 93b).

The formal model ROM(K) (Dague 93a) was devised first, as an extension of FOG by adding a relation Di standing for "distant from". This addition provided the system with a nice symmetrical property and the ability to express gradual changes from one order-of-magnitude to another thanks to the existence of overlapping regions. (Dague 93b) then showed with ROM($\mathbb{R}$) how to transpose this system to $\mathbb{R}$ with a guarantee of soundness, solving hence the two above mentioned problems: possible incorporation of quantitative information and control of the inference process. ROM($\mathbb{R}$) subsumes the O(M) model by (Mavrovouniotis&Stephanopoulos 87).

Although one type of reasoning, based on AOM or ROM, might be better suited to a given application domain, it is generally the case that both are necessary to capture all the relevant information. This is why it would be most useful to bridge AOM and ROM. Few attempts have been made though. (Sánchez *et al.* 96) proposed a mixed model but their ROM relations, defined with respect to the AOM labels, do not provide a complete interpretation in $\mathbb{R}$. (Gasca 98) presented an operational system including both AOM and ROM relations based on interval constraint propagation. However, it missed a rigorous formalisation assuring consistency between AOM and ROM relations.

The aim of this paper is to analyse under which conditions consistency between AOM and ROM models can hold and to determine the constraints that it would imply. Consequently, the ROM relations would then be characterizable in the AOM world. Conversely, a pair of quantities described by AOM labels could be related by the corresponding ROM relation(s).

The paper is organised as follows. Section 2 presents the AOM and ROM frameworks. Section 3 provides the formulation of the problem approached in this paper. The constraints implied by and guaranteeing consistency are then outlined in sections 4 and 5. Section 6 provides a graphical interpretation of the constraints and bridges the absolute qualitative labels of two quantities into their corresponding relative relation(s). The ROM relations are then characterized in the absolute world in section 7. Finally, section 8 discusses the work and outlines several conclusions and lines for future work.

2. AOM and ROM models

2.1. Absolute OM models

The AOM models rely on a partition of $\mathbb{R}$ which defines the quantity space S_1, each element of the partition standing for a basic qualitative class to which a label is associated. The partition is defined by a set of real landmarks including 0 and generates the *Universe of Description* of the AOM model. It is referred to as an *absolute partition*.

In the following, we restrict ourselves to symmetrical absolute partitions. The symmetrical AOM model with n positive (negative) qualitative labels is denoted by OM(n) and it is referred as the AOM model of granularity n. The set of positive landmarks is denoted by L.

For instance, the OM(5) model is based on the following set of landmarks:

$$\{-\delta, -\beta, -\alpha, -\gamma, 0, \gamma, \alpha, \beta, \delta\},$$

with corresponding labels: NVL $=]\infty, -\delta]$ (Negative Very Large); NL = $]-\delta, -\beta]$ (Negative Large); NM = $]-\beta, -\alpha]$ (Negative Medium); NS = $]-\alpha, -\gamma]$ (Negative Small); NVS = $]-\gamma, 0[$ (Negative Very Small); [0]={0}; PVS $=]0, \gamma[$ (Positive Very Small); PS=$[\gamma, \alpha[$ (Positive Small) PM=$[\alpha, \beta[$ (Positive Medium); PL=$[\beta, \delta[$ (Positive Large); PVL=$[\delta, +\infty[$ (Positive Very Large).

The resulting absolute partition is the following:

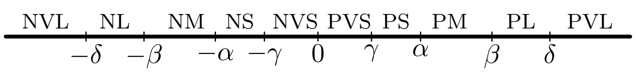

Figure 1: The OM(5) absolute partition

Q-algebras over these models and their algebraic properties have been extensively studied (Travé & Piera 89), (Missier *et al.* 89), (Dormoy 89), (Piera & Travé 90), (Missier 91), (Agell 98) and summarized in (Travé-Massuyès *et al.* 97).

2.2 Relative OM models

ROM models are based on the definition of a set of binary relations r_i which are invariant by homothety, i.e. Ar_iB only depends on the quotient A/B, the axiomatics of which are described by a set of rules. The first ROM model FOG (Raiman 91) was based on three relations, "negligible with respect to" (Ne), "close to" (Vo) and "comparable to" (Co, in the sense of "the same sign and order of magnitude as"), and included 32 inference rules.

The ROM models that were proposed later improved FOG not only in the necessary aspect of a rigorous formalisation, but also permitting the incorporation of quantitative information when available and the control of the inference process, in order to obtain valid results in the real world.

The formal model ROM(K) (Dague 93a) was proposed as an extension of FOG by adding a relation Di standing for "distant from". The four binary relations Ne, Vo, Co, Di, are defined by means of 15 axioms, which provide about 45 inference rules (Dague 93a). ROM(K) has a nice symmetrical property and the ability to express gradual changes from one order-of-magnitude to another thanks to the existence of overlapping regions when interpreted in N.S.A. (Non Standard Analysis).

(Dague 93b) then showed how to transpose ROM(K) to $\mathbb{R}$ with a guarantee of soundness, resulting in the system ROM($\mathbb{R}$). ROM($\mathbb{R}$) permits the incorporation of quantitative information and obtains sound results while maintaining the semantics of the inference paths in terms of the four symbolic relations of ROM(K).

The following ROM($\mathbb{R}$) relations are defined in $\mathbb{R}$, parametrized by a positive real k.

Definition 1 (Dague 93b).
Negligibility at order k: Given two real numbers x and y, then x is *negligible at order* k or k*-negligible with respect to* y, xN_ky, if $|x| \leq k|y|$.
Proximity at order k: Given two real numbers x and y, then x is *close at order* k *to* y, xP_ky, if $|x-y| \leq k \cdot \max\{|x|, |y|\}$.
Distance at order k: Given two real numbers x and y, then x is *distant at order* k *from* y, xD_ky, if $|x-y| \geq k \cdot \max\{|x|, |y|\}$.

Note that P_k can be interpreted in term of N_k as follows: xP_ky if $|x-y| \leq k \cdot \max\{|x|, |y|\}$ is equivalent to $|x-y| N_k \max\{|x|, |y|\}$, i.e. $(x = y = 0)$ or $(y \neq 0$ and $\frac{|x-y|}{\max\{|x|,|y|\}} \leq k)$, or equivalently, $(x = y = 0)$ or $(y \neq 0$ and $1-k \leq \frac{x}{y} \leq \frac{1}{1-k})$. This is in turn equivalent to $1-z \leq k$, for any $z \in \left\{\frac{x}{y}, \frac{y}{x}\right\}$.

Dague (1993b) matches the above relations to ROM(K) relations using two parameters k_1 and k_2 in the following way:

Vo $\leftrightarrow P_{k_1}$
Co $\leftrightarrow P_{1-k_2}$
Ne $\leftrightarrow N_{k_1}$
Di $\leftrightarrow D_{k_2}$

Note that $x \,\mathrm{Di}\, y$ is equivalent to $(x = y = 0)$ or $(x \neq y$ and $\frac{|x-y|}{\max\{|x|,|y|\}} \geq k_2)$, i.e., $(x/y \leq 1-k_2)$ or $(x/y \geq 1/(1-k_2))$ or $y = 0$. Consequently, $x \,\mathrm{Di}\, y$ is equivalent to $[(x-y) \,\mathrm{Co}\, x]$ or $[(y-x) \,\mathrm{Co}\, y]$.

A first group of ROM(K) axioms is satisfied for any k_1 and k_2. A second group requires the following constraint:

$$0 < k_1 \leq k_2 \leq 1/2.$$

The remaining axioms cannot be satisfied in $\mathbb{R}$. For these, (Dague 93b) proposes to calculate the order-of-magnitude precision loss of the conclusion in the worst case.

ROM($\mathbb{R}$) has two degrees of freedom, in the form of two parameters k_1 and k_2, which define a *relative partition* of the real line. This partition concerns the quotients of quantities, for instance, $x \,\mathrm{Ne}\, y$ if and only if x/y belongs to $[-k_1, k_1]$.

This partition is symmetrical with respect to 0: two quantities of the same sign on the right side of 0 and of opposite signs on the left side of 0. Restricting ourselves to the right side of 0, it is also symmetrical in a qualitative sense with respect to 1ttt. Consequently, the whole partition is fully characterized by the interval $[0, 1]$, which is in turn fully characterized by the set of relative landmarks $Q^* = \{k_1, k_2, 1 - k_2, 1 - k_1\}$.

3. AOM and ROM models consistency problem formulation

The aim of this paper is to analyse under which conditions AOM and ROM models can be consistent and to determine the constraints that it would imply on their degrees of freedom.

The AOM models that we consider are the ones presented in section 2.1. Their degrees of freedom include the number of landmarks defining the absolute partition and the landmark values themselves.

Concerning the ROM models, we do consider the ROM($\mathbb{R}$) model presented in section 2.2, for several reasons. The main reason is that it is fully interpretable in $\mathbb{R}$, which is a necessary condition to obtain a gateway with AOM models which are themselves fully interpretable in $\mathbb{R}$. The second reason is that it is the most general ROM model that satisfies the previous condition. The degrees of freedom of ROM($\mathbb{R}$) are the values of the parameters k_1 and k_2. Note that the number of landmarks of the relative partition depends on these two values.

Given two quantities, which are qualified in the absolute world, we want to be able to provide the corresponding ROM relations which link them. This problem will be referred as the AOM $\rightarrow$ ROM problem. Conversely, given two quantities known to be related by a given ROM relation, we want to be able to give their possible qualifications in terms of absolute qualitative labels. This latter problem will be referred as the ROM $\rightarrow$ AOM problem.

Labels in OM(n) are represented by intervals of $\mathbb{R}$ and ROM relations in ROM($\mathbb{R}$) are defined via quotients of real numbers. Then, from the fact that:

$$x \in [l_1, l_2], y \in [l_3, l_4] \implies \frac{x}{y} \in \left[\frac{l_1}{l_4}, \frac{l_2}{l_3}\right]$$

with $l_1,\ l_2,\ l_3,\ l_4 > 0$, it is necessary that the quotients of the landmarks of the absolute model coincide with the landmarks of the relative partition of ROM($\mathbb{R}$).

Definition 2: (Consistency Property) If the quotients between the landmarks of an absolute model OM(n) coincide with the landmarks of a relative partition of ROM($\mathbb{R}$), then the OM(n) and the ROM($\mathbb{R}$) models are said to be *consistent*[1].

We want to determine the constraints linking the landmarks of the absolute and relative partitions which guarantee the consistency property. Having obtained this result, it will be possible to characterize the ROM($\mathbb{R}$) relations in the AOM world, and conversely.

[1]The consistency property proposed in Definition 2 can be qualified as "strong" since it requires the identity on the atomic relationships on both sides.

4. Minimal granularity to match OM(n) and ROM($\mathbb{R}$)

The first necessary condition for guaranteeing the consistency property concerns the number of landmarks of the AOM model OM(n).

In order to maintain the maximum power to ROM($\mathbb{R}$), let us consider the case in which $k_1 < k_2 < 1/2$, resulting in four landmarks in the interval [0,1] of the corresponding relative partition: $0 < k_1 < k_2 < 1 - k_2 < 1 - k_1 < 1$. Let us call this case the *full ROM($\mathbb{R}$)*, noted *F-ROM($\mathbb{R}$)*.

Proposition 1. The minimum granularity to match OM(n) and F-ROM($\mathbb{R}$) is $n = 5$.

The proof consists in checking that for $n = 2$, $n = 3$ and $n = 4$ there are not enough landmarks to obtain four quotients in the interval $[0, 1]$. For OM(5), the set of positive landmarks is $L = \{\gamma, \alpha, \beta, \delta\}$. Hence, there are twelve quotients different from 0 and 1, and six of them are in the interval]0,1[. These are given by the set of non-ordered quotients:

$$Q = \left\{\frac{\gamma}{\alpha}, \frac{\gamma}{\beta}, \frac{\gamma}{\delta}, \frac{\alpha}{\beta}, \frac{\alpha}{\delta}, \frac{\beta}{\delta}\right\}$$

5. Matching between OM(5) and F-ROM($\mathbb{R}$)

This section is concerned with the determination of the constraints that must hold to turn OM(5) consistent with F-ROM($\mathbb{R}$). The first issue is to guarantee the required number of relative landmarks, i.e. card(Q)= card(Q^*); the second is to obtain the formal matching between the relative landmarks and their expression in terms of the absolute landmarks, i.e. $Q = Q^*$.

5.1 Cardinality

This section outlines the conditions to guaranty card(Q)=card(Q^*)=4.

Since card(Q) is potentially greater than 4, the following condition is applied iteratively until card(Q)=4:

$$\frac{x}{y} = \frac{z}{t}, \text{ for some } x, y, z, t \in L = \{\gamma, \alpha, \beta, \delta\} \qquad (1)$$

Let us hence consider $\frac{x}{y} = \frac{z}{t} = q$, or equivalently $x = qy, z = qt$, and, without loss of generality, $q > 1$.

Notation: By convention, the ordered 4-tuple (a_1, a_2, a_3, a_4) is used for $a_1 < a_2 < a_3 < a_4$.

The only 4-tuples $(\gamma, \alpha, \beta, \delta)$ that satisfy condition (1) are given by the following 6 cases:

- Case 1) $(\gamma, \alpha, \beta, \delta) = (\gamma, q\gamma, q^2\gamma, \delta)$
- Case 2) $(\gamma, \alpha, \beta, \delta) = (\gamma, q\gamma, \beta, q^2\gamma)$
- Case 3) $(\gamma, \alpha, \beta, \delta) = (\gamma, q\gamma, \beta, q\beta)$
- Case 4) $(\gamma, \alpha, \beta, \delta) = (\gamma, \alpha, q\gamma, q^2\gamma)$
- Case 5) $(\gamma, \alpha, \beta, \delta) = (\gamma, \alpha, q\gamma, q\alpha)$
- Case 6) $(\gamma, \alpha, \beta, \delta) = (\gamma, \alpha, q\alpha, q^2\alpha)$

After some calculations, all these cases give five quotients in Q, except for the cases 3) and 5) that give directly four.

Case 3)

The quotients in case 3) are $Q = \left\{\frac{1}{q}, \frac{\gamma}{\beta}, \frac{\gamma}{q\beta}, q\frac{\gamma}{\beta}\right\}$, or, in terms of α, β and q: $Q = \left\{\frac{1}{q}, \frac{\alpha}{q\beta}, \frac{\alpha}{q^2\beta}, \frac{\alpha}{\beta}\right\}$.

The landmarks of L are, again in terms of α, β and q: $L = \left\{\frac{\alpha}{q}, \alpha, \beta, q\beta\right\}$. This can be interpreted as the situation in which there exists $q > 1$ such that all the very small numbers multiplied by q are, at the most, small, and all the very large numbers divided by q are, at least, large.

There are two possible orderings of Q:

3.1) $\left(\frac{\alpha}{q^2\beta}, \frac{\alpha}{q\beta}, \frac{\alpha}{\beta}, \frac{1}{q}\right)$

3.2) $\left(\frac{\alpha}{q^2\beta}, \frac{\alpha}{q\beta}, \frac{1}{q}, \frac{\alpha}{\beta}\right)$

Case 5) The quotients in case 5) are $Q = \left\{\frac{\gamma}{\alpha}, \frac{1}{q}, \frac{\gamma}{q\alpha}, \frac{\alpha}{q\gamma}\right\}$, or, in terms of α, β and q: $Q = \left\{\frac{\beta}{q\alpha}, \frac{1}{q}, \frac{\beta}{q^2\alpha}, \frac{\alpha}{\beta}\right\}$. The landmarks of L are, again in terms of α, β and q: $L = \left\{\frac{\beta}{q}, \alpha, \beta, q\alpha\right\}$. This can be interpreted as the situation in which there exists $q > 1$ such that all the very small numbers multiplied by q are, at the most, medium, and, all the very large numbers divided by q are, at least, medium.

There are two possible orderings of Q:

5.1) $\left(\frac{\beta}{q^2\alpha}, \frac{1}{q}, \frac{\beta}{q\alpha}, \frac{\alpha}{\beta}\right)$

5.2) $\left(\frac{\beta}{q^2\alpha}, \frac{1}{q}, \frac{\alpha}{\beta}, \frac{\beta}{q\alpha}\right)$

Let us study the cases 1), 2), 4) and 6) in which there are still five quotients in Q. Condition (1) is applied again.

Let us hence consider $\frac{x}{y} = \frac{z}{t} = p$ for some $x, y, z, t \in L$, or equivalently $x = py, z = pt$, and, without loss of generality, $p > 1$.

Case 1) In the case 1), the absolute landmarks are $(\gamma, \alpha, \beta, \delta) = (\gamma, q\gamma, q^2\gamma, \delta)$ and five quotients less than 1 result:

$$Q = \left\{\frac{1}{q}, \frac{1}{q^2}, \frac{\gamma}{\delta}, q\frac{\gamma}{\delta}, q^2\frac{\gamma}{\delta}\right\}$$

By imposing condition (1) one more time, we obtain 6 cases:

- 1.1) $(\gamma, \alpha, \beta, \delta) = (\gamma, p\gamma, p^2\gamma, \delta)$
- 1.2) $(\gamma, \alpha, \beta, \delta) = (\gamma, p\gamma, q^2\gamma, p^2\gamma)$
- 1.3) $(\gamma, \alpha, \beta, \delta) = (\gamma, p\gamma, q^2\gamma, pq^2\gamma)$
- 1.4) $(\gamma, \alpha, \beta, \delta) = (\gamma, q\gamma, p\gamma, p^2\gamma)$
- 1.5) $(\gamma, \alpha, \beta, \delta) = (\gamma, q\gamma, p\gamma, pq\gamma)$
- 1.6) $(\gamma, \alpha, \beta, \delta) = (\gamma, q\gamma, pq\gamma, p^2q\gamma)$

The case 1.1) is a deadlock case, since $(\gamma, p\gamma, p^2\gamma, \delta)$ is structurally equal to the initial $(\gamma, q\gamma, q^2\gamma, \delta)$. Case 1.2) gives only two quotients and cases 1.3), 1.5) and 1.6) give only three quotients, and all of them can be discarded. Hence, the only remaining case is 1.4), that gives exactly four quotients.

By making equal the expression of the absolute landmarks in case 1) and case 1.4) $(\gamma, q\gamma, q^2\gamma, \delta) = (\gamma, q\gamma, p\gamma, p^2\gamma)$, we obtain $p = q^2$. Hence the case 1.4) corresponds to the absolute landmarks $(\gamma, q\gamma, q^2\gamma, q^4\gamma)$, or in terms of α, $L = \left(\frac{\alpha}{q}, \alpha, q\alpha, q^3\alpha\right)$ and the set of quotients is $Q = \left\{\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}\right\}$.

	landmarks of OM(5) $L = (\gamma, \alpha, \beta, \delta)$	quotients less than 1 $Q = (k_1, k_2, 1-k_2, 1-k_1)$
I.(3.1)	$(\frac{\alpha}{q}, \alpha, \beta, q\beta)$	$(\frac{\alpha}{q^2\beta}, \frac{\alpha}{q\beta}, \frac{\alpha}{\beta}, \frac{1}{q})$
II.(3.2)	$(\frac{\alpha}{q}, \alpha, \beta, q\beta)$	$(\frac{\alpha}{q^2\beta}, \frac{\alpha}{q\beta}, \frac{1}{q}, \frac{\alpha}{\beta})$
III.(5.1)	$(\frac{\beta}{q}, \alpha, \beta, q\alpha)$	$(\frac{\beta}{q^2\alpha}, \frac{1}{q}, \frac{\beta}{q\alpha}, \frac{\alpha}{\beta})$
IV.(5.2)	$(\frac{\beta}{q}, \alpha, \beta, q\alpha)$	$(\frac{\beta}{q^2\alpha}, \frac{1}{q}, \frac{\alpha}{\beta}, \frac{\beta}{q\alpha})$
V.(1)	$(\frac{\alpha}{q}, \alpha, q\alpha, q^3\alpha)$	$(\frac{1}{q^4}, \frac{1}{q^3}, \frac{1}{q^2}, \frac{1}{q})$
VI.(2)	$(\frac{\alpha}{q}, \alpha, \sqrt{q}\alpha, q\alpha)$	$(\frac{1}{q^2}, \frac{1}{q\sqrt{q}}, \frac{1}{q}, \frac{1}{\sqrt{q}})$
VII.(4)	$(\frac{\alpha}{\sqrt{q}}, \alpha, \sqrt{q}\alpha, q\sqrt{q}\alpha)$	$(\frac{1}{q^2}, \frac{1}{q\sqrt{q}}, \frac{1}{q}, \frac{1}{\sqrt{q}})$
VIII.(6)	$(\frac{\alpha}{q^2}, \alpha, q\alpha, q^2\alpha)$	$(\frac{1}{q^4}, \frac{1}{q^3}, \frac{1}{q^2}, \frac{1}{q})$

Table 1: Admissible cases for consistency

The following results are obtained by studying the cases 2), 4) and 6) in an analogous way:

Case 2) $L = (\gamma, \alpha, \beta, \delta) = (\gamma, q\gamma, q\sqrt{q}\gamma, q^2\gamma) = \left(\frac{\alpha}{q}, \alpha, \sqrt{q}\alpha, q\alpha\right)$; $Q = \left\{\frac{1}{q}, \frac{1}{q\sqrt{q}}, \frac{1}{q^2}, \frac{1}{\sqrt{q}}\right\}$.

Case 4) $L = (\gamma, \alpha, \beta, \delta) = (\gamma, \sqrt{q}\gamma, q\gamma, q^2\gamma) = \left(\frac{\alpha}{\sqrt{q}}, \alpha, \sqrt{q}\alpha, q\sqrt{q}\alpha\right)$; $Q = \left\{\frac{1}{\sqrt{q}}, \frac{1}{q}, \frac{1}{q^2}, \frac{1}{q\sqrt{q}}\right\}$.

Case 6) $L = (\gamma, \alpha, \beta, \delta) = (\gamma, q^2\gamma, q^3\gamma, q^4\gamma) = \left(\frac{\alpha}{q^2}, \alpha, q\alpha, q^2\alpha\right)$; $Q = \left\{\frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \frac{1}{q}\right\}$.

In summary, there are eight cases that allow one to establish a bijection between Q and Q^*. These are summarized in Table 1.

5.2 Formal matching

For each admissible case of table 1, it remains to impose the formal matching between the relative landmarks and their expression in terms of the absolute landmarks, i.e. $Q = Q^*$.

Let us consider case I, the other ones being dealt with in a similar way. We want the condition $Q = Q^*$, i.e. $(\frac{\alpha}{q^2\beta}, \frac{\alpha}{q\beta}, \frac{\alpha}{\beta}, \frac{1}{q}) = (k_1, k_2, 1-k_2, 1-k_1)$

It is hence enough to solve the following system:

$$\left.\begin{array}{l} \frac{\alpha}{q^2\beta} + \frac{1}{q} = 1 \\ \frac{\alpha}{q\beta} + \frac{\alpha}{\beta} = 1 \end{array}\right\}$$

The unique positive solution for q is $q = \sqrt{2}$, and then $\beta = \left(\frac{2+\sqrt{2}}{2}\right)\alpha$. The landmarks of OM(5) are:

$$\left(\gamma = \frac{\alpha}{q} = \frac{\alpha}{\sqrt{2}}, \quad \alpha, \quad \beta = \frac{2+\sqrt{2}}{2}\alpha, \quad \delta = (\sqrt{2}+1)\alpha\right)$$

The quotients less than 1 are:

$$(k_1 = \frac{\alpha}{q^2\beta} = \frac{2-\sqrt{2}}{2}, \, k_2 = \frac{\alpha}{q\beta} = \sqrt{2} - 1,$$

	landmarks OM(5) $(\gamma,\alpha,\beta,\delta)$	quotients less than 1 $(k_1,k_2,1-k_2,1-k_1)$	q,β
I	$(\frac{\alpha}{q},\ \alpha,\ \beta,\ q\beta)$	$(\frac{\alpha}{q^2\beta},\ \frac{\alpha}{q\beta},\ \frac{\alpha}{\beta},\ \frac{1}{q})$	$q=\sqrt{2}$ $\beta=\left(\frac{2+\sqrt{2}}{2}\right)\alpha$
II	$(\frac{\alpha}{q},\ \alpha,\ \beta,\ q\beta)$	$(\frac{\alpha}{q^2\beta},\ \frac{\alpha}{q\beta},\ \frac{1}{q},\ \frac{\alpha}{\beta})$	$q\simeq 1.75487$ $\beta=\frac{\alpha}{q-1}$
III	$(\frac{\beta}{q},\alpha,\beta,q\alpha)$	$(\frac{\beta}{q^2\alpha},\ \frac{1}{q},\ \frac{\beta}{q\alpha},\ \frac{\alpha}{\beta})$	$q\simeq 2.32472$ $\beta=(q-1)\alpha$

Table 2: Cases for consistency

$$1-k_2=\frac{\alpha}{\beta}=2-\sqrt{2},\ 1-k_1=\frac{1}{q}=\frac{1}{\sqrt{2}})$$

In case IV, the unique solution for q is the solution of the equation $q^3-q^2+1=0$, that is, $q\simeq 1.32$, which is not possible because $k_2=\frac{1}{q}\simeq\frac{1}{1.32}\simeq 0.76\not<\frac{1}{2}$.

In the cases V, VI, VII and VIII, the corresponding system of equations has no solution.

Hence, only cases I, II and III have solutions and are summarized in table 2 [1].

The conclusion of the above analysis is that consistency between OM(5) and F-ROM($\mathbb{R}$) is possible, although it is highly constrained. Indeed, only one degree of freedom remains out of four for the AOM model, the F-ROM($\mathbb{R}$) model resulting fully determined.

6. Bridging AOM and ROM models

This section is concerned with the AOM $\to$ ROM and the ROM $\to$ AOM problems, as defined in section 3. These correspondences are easily obtained graphically.

Figures 2a, 2b and 2c show simultaneously the landmarks of the absolute model and the landmarks of the relative partition in cases I, II and III respectively.

[1]In case II, q is obtained from the equation $q^3-2q^2+q-1=0$ and its real solution is $q=\frac{2}{3}+\frac{1}{3}\sqrt[3]{\frac{25-3\sqrt{69}}{2}}+\frac{1}{3}\sqrt[3]{\frac{25+3\sqrt{69}}{2}}\simeq 1.75487$. In case III, q is obtained from the equation $q^3-3q^2+2q-1=0$ and its real solution is $q=1+\frac{1}{3}\sqrt[3]{\frac{27-3\sqrt{69}}{2}}+\sqrt[3]{\frac{9+\sqrt{69}}{18}}\simeq 2.32472$

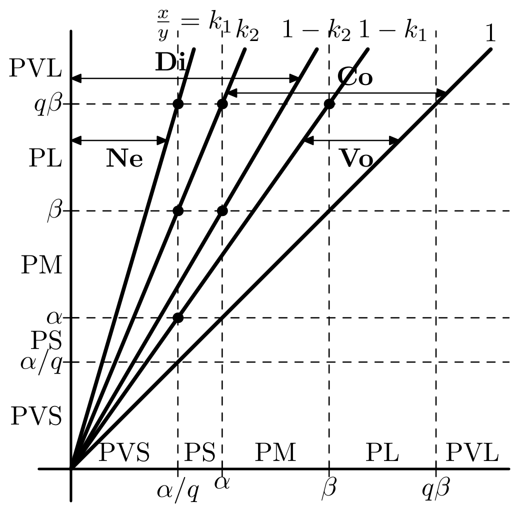

Figure 2a: Case I

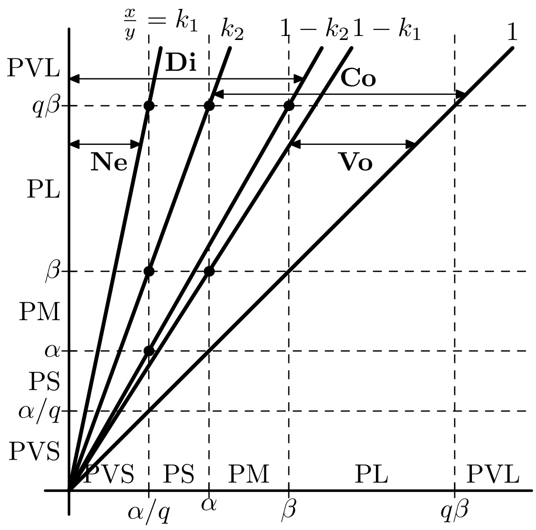

Figure 2b: Case II

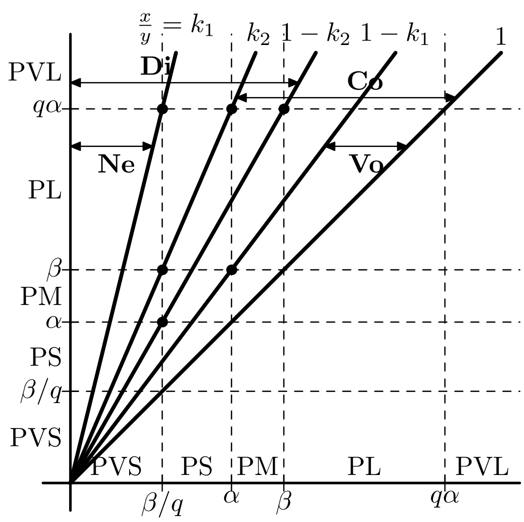

Figure 2c: Case III

The absolute landmarks are reported on the axes X and Y, and the relative ones are given by the straight lines of equations $\frac{x}{y} = k$, $k \in \{k_1, k_2, 1-k_2, 1-k_1\}$.

The ROM($\mathbb{R}$) relations Ne, Co, Vo and Di all lie between the Y axis and the straight line $x = y$. Each of them corresponds to a sector indicated by a double arrow between a pair of lines. Formally, each of the relations Ne, Co, Vo and Di, corresponds to the region of the plane $\{(x,y) \,|\, x \geq 0, y \geq 0, x \leq y, \text{ such that } x \; R \; y\}$, where R is any of the four relations.

In the following the four relations are characterized in terms of the corresponding region and a few examples of ROM $\rightarrow$ AOM and AOM $\rightarrow$ ROM correspondances are provided.

6.1 Relation Ne

$x \,\text{Ne}\, y \iff (x = y = 0)$ or ($y \neq 0$ and $\left|\frac{x}{y}\right| \leq k_1$), thus $x \,\text{Ne}\, y$ when the point (x, y) lies in the region between the straight lines $x = 0$ and $\frac{x}{y} = k_1$.
Examples of correspondances are:
1.1. Any very small number is negligible with respect to any very large number: $\forall x \in$ VS, $\forall y \in$ VL: $x \,\text{Ne}\, y$
1.2. The numbers that are not very small can be negligible only with respect to some very large numbers: $x \notin$ VS, $x \,\text{Ne}\, y \Longrightarrow y \in$ VL.

6.2 Relation Vo

$x \,\text{Vo}\, y \iff (x = y = 0)$ or ($y \neq 0$ and $\frac{|x-y|}{\max\{|x|,|y|\}} \leq k_1$), thus, for $x \leq y$, $x \,\text{Vo}\, y$ when the point (x, y) lies in the region between the straight lines $\frac{x}{y} = 1 - k_1$ and $\frac{x}{y} = 1$.
Examples of correspondances are:
2.1. In case I, any very small number is not Vo with respect to any medium, or large, or very large number: $x \in$ VS, $x \,\text{Vo}\, y \Longrightarrow y \in$ VS or $y \in$ S.

2.2. In case I, any small or medium number is not Vo with respect to any very large number.

2.3. In cases II and III, any very small or small number is not Vo with respect to any large, or very large number.

6.3 Relation Co

$x \,\text{Co}\, y \iff (x = y = 0)$ or ($y \neq 0$ and $\frac{|x-y|}{\max\{|x|,|y|\}} \leq 1 - k_2$), thus, for $x \leq y$, $x \,\text{Co}\, y$ when the point (x, y) lies in the region between the straight lines $\frac{x}{y} = k_2$ and $\frac{x}{y} = 1$.
Examples of correspondances are:
3.1. Any very small number is not Co to any large or very large number.

3.2. Any small number is not Co to any very large number.

6.4 Relation Di

$x \,\text{Di}\, y \iff (x = y = 0)$ or ($y \neq 0$ and $\frac{|x-y|}{\max\{|x|,|y|\}} \geq k_2$), thus, for $x \leq y$, $x \,\text{Di}\, y$ when the point (x, y) lies in the region between the straight lines $x = 0$ and $\frac{x}{y} = 1 - k_2$.
Examples of correspondances are:
4.1. In case I, any very small or small number is distant from any large or very large number.

4.2. In cases II and III, any very small number is distant from any medium or large or very large number.

4.3. In cases II and III, any small or medium number is distant from any very large number.

7. Characterization of the ROM relations in the absolute world

The consistency conditions established in section 5 and summarized in table 2 allow us to provide a characterization of the ROM($\mathbb{R}$) relations in terms of AOM concepts.

7.1 Negligibility

Proposition 2. x is negligible with respect to y, $x \,\text{Ne}\, y$, if and only if $x = y = 0$ or, in the case $y \neq 0$, there exists a very large number M ($M \in$ PVL) such that the quotient $\frac{x}{y}$ multiplied by M is still a very small number (PVS).

Proof: If $x \,\text{Ne}\, y$, and $y \neq 0$, then $\left|\frac{x}{y}\right| \leq k_1$ (remember that $k_1 = \frac{\alpha}{q^2\beta} = \frac{2-\sqrt{2}}{2}$ in case I, $k_1 = \frac{\alpha}{q^2\beta} = \frac{q-1}{q^2}$, $q \simeq 1.75487$ in case II, and $k_1 = \frac{\beta}{q^2\alpha} = \frac{q-1}{q^2}$ in case III).

In cases I and II, we obtain $q\beta\left|\frac{x}{y}\right| \leq \frac{\alpha}{q}$. When this inequality is strict[2], there exists M such that $q\beta\left|\frac{x}{y}\right| \leq M\left|\frac{x}{y}\right| < \left|\frac{\alpha}{q}\right|$.

Reciprocally, if there exists $M \geq q\beta$ such that $\left|\frac{Mx}{y}\right| < \frac{\alpha}{q}$, then $\left|\frac{q\beta x}{y}\right| \leq \left|\frac{Mx}{y}\right| < \frac{\alpha}{q}$, implying that $x \,\text{Ne}\, y$.

The proof in case III is analogous, by interchanging α and β ■

Since xN_ky is equivalent to $\left(\frac{k_1}{k} \cdot x\right) \,\text{Ne}\, y$, by the above characterization of Ne, we obtain the characterization of N_k.

Proposition 3. x is negligible at order k with respect to y, xN_ky, if and only if $x = y = 0$ or, in the case $y \neq 0$, there exists a very large number M such that the quotient $\frac{k_1}{k} \cdot \frac{x}{y}$ multiplied by M is still a very small number.

7.2 Proximity

The proximity at order k relation, xP_ky, is equivalent to:

$$(x = y = 0) \text{ or } \left(y \neq 0 \text{ and} \frac{|x-y|}{\max\{|x|,|y|\}} \leq k\right),$$

i.e. $|x-y| N_k \max\{|x|,|y|\}$. Hence, from Proposition 3 the following result is straightforward:

Proposition 4. x is close at order k to y, $x \, P_k \, y$, when $x = y = 0$ or, in the case $y \neq 0$, there exists a very large number M such that the quotient $\frac{k_1}{k} \cdot \frac{|x-y|}{\max\{|x|,|y|\}}$ multiplied by M is a very small number.

The following proposition gives a characterization for the relation Vo, which is a particular case of P_k with $k = k_1$.

[2] When $\left|\frac{\alpha y}{qx}\right| = q\beta$, the number M needs to be large but not very large. Nevertheless, since $q\beta$ is precisely the landmark between large and very large numbers, this does not change the nature of the result.

Proposition 5. When $x, y \neq 0$, let z be the quotient in the set $\left\{\frac{x}{y}, \frac{y}{x}\right\}$ such that $|z| \leq 1$. Then, $x \,\mathrm{Vo}\, y$ if and only if there exists a very large number M such that M multiplied by $(1-z)$ is a very small number.
Proof:

Since $x \,\mathrm{Vo}\, y$ means $x \, P_{k_1} \, y$, by proposition 4 this is equivalent to the existence of a very large number M such that the quotient $\frac{|x-y|}{\max\{|x|,|y|\}}$ multiplied by M is a very small number, and this is equivalent to the fact that the product $(1-z) \cdot M$ is a very small number, because $\frac{|x-y|}{\max\{|x|,|y|\}} = |1-z|$.

7.3 Comparability

The comparability relation Co is also a particular case of P_k with $k = 1 - k_2$.

Proposition 6. Let us assume $x, y \neq 0$ and let z be the quotient in the set $\left\{\frac{x}{y}, \frac{y}{x}\right\}$ such that $|z| \leq 1$. Then,

I) In case I, $x \,\mathrm{Co}\, y$ if and only if there exists a large number M such that M multiplied by $(1-z)$ is a small or a very small number.

II) In case II, $x \,\mathrm{Co}\, y$ if and only if there exists a medium number M such that M multiplied by $(1-z)$ is a very small number.

III) In case III, when $x \,\mathrm{Co}\, y$ there exists a medium number M such that M multiplied by $(1-z)$ is a small or very small number.

Proof:

- Case I. If $|x| \geq |y|$, i.e. $z = \frac{y}{x}$, the relation $x \,\mathrm{Co}\, y$ is equivalent to: $1 - \frac{y}{x} \leq 1 - k_2 = 1 - \frac{\alpha}{q\beta}$, and, taking into account that $1 - \frac{\alpha}{q\beta} = \frac{\alpha}{\beta}$ this is equivalent to: $\beta\left(1 - \frac{y}{x}\right) \leq \alpha$. If $x \,\mathrm{Co}\, y$, it is enough to consider $M = \beta$. Reciprocally, if M is a large number such that $M\left(1 - \frac{y}{x}\right) \leq \alpha$, then $\beta\left(1 - \frac{y}{x}\right) \leq \alpha$ because $M \geq \beta$. The proof in the case $|x| < |y|$ is analogous.
- Case II. f $|x| \geq |y|$, i.e. $z = \frac{y}{x}$, the relation $x \,\mathrm{Co}\, y$ is equivalent to: $1 - \frac{y}{x} \leq 1 - k_2 = 1 - \frac{\alpha}{q\beta}$, and, taking into account that $1 - \frac{\alpha}{q\beta} = \frac{1}{q}$ this is equivalent to: $\alpha\left(1 - \frac{y}{x}\right) \leq \frac{\alpha}{q}$. If $x \,\mathrm{Co}\, y$, it is enough to consider $M = \alpha$. Reciprocally, if M is a medium number such that $M\left(1 - \frac{y}{x}\right) \leq \frac{\alpha}{q}$, then $\alpha\left(1 - \frac{y}{x}\right) \leq \frac{\alpha}{q}$ because $M \geq \alpha$. The proof in the case $|x| < |y|$ is analogous.
- Case III. If $|x| \geq |y|$, i.e. $z = \frac{y}{x}$, $x \,\mathrm{Co}\, y$ is equivalent to: $1 - \frac{y}{x} \leq 1 - k_2 = \frac{\beta}{q\alpha}$, and this is equivalent to: $\frac{\alpha^2 q}{\beta}\left(1 - \frac{y}{x}\right) \leq \alpha$. It is enough to take $M = \frac{\alpha^2 q}{\beta} = \frac{\alpha q}{q-1} > \alpha$. The proof in the case $|x| < |y|$ is analogous ■

In case I, $1 - k_2 = 1 - \frac{\alpha}{q\beta} = \frac{\alpha}{\beta}$ is the quotient between the length of the interval corresponding to [PVS,PS] and the length of the interval [PVS,PM].

In case II, $1 - k_2 = 1 - \frac{\alpha}{q\beta} = \frac{1}{q}$ is the quotient between the length of the interval corresponding to PVS and the length of the interval [PVS,PS]. This is also the quotient between the length of the interval corresponding to [PVS,PM] and the length of the interval [PVS,PL].

In case III, $1 - k_2 = 1 - \frac{1}{q} = \frac{\beta}{\alpha q}$ is the quotient between the length of the interval corresponding to PVS and the length of the interval [PVS,PS]. This is also the quotient between the length of the interval corresponding to [PVS,PM] and the length of the interval [PVS,PL].

7.4 Distance

Proposition 7.
If $\frac{|x-y|}{\max\{|x|,|y|\}} \neq k_2$, then $x \,\mathrm{Di}\, y$ if and only if there exists some real number M neither large nor very large such that M multiplied by the quotient $\frac{|x-y|}{\max\{|x|,|y|\}}$ is not very small.

Proof: In cases I and II, $k_2 = \frac{\alpha}{q\beta}$, and in case III, $k_2 = \frac{1}{q}$. Let us give the proof in cases I and II, the one in case III being analogous.
Let us suppose that there exists M such that $|M| \leq \beta$ and $\frac{|M(x-y)|}{\max\{|x|,|y|\}} \geq \alpha/q$, then $\frac{|\beta(x-y)|}{\max\{|x|,|y|\}} \geq \alpha/q$, therefore $\frac{|x-y|}{\max\{|x|,|y|\}} \geq \frac{\alpha}{q\beta}$, i.e., $x \,\mathrm{Di}\, y$.

Reciprocally, if $x \,\mathrm{Di}\, y$, it suffices to take M such that $\frac{\alpha \max\{|x|,|y|\}}{q|x-y|} < |M| < \beta$. The case $\frac{\alpha \max\{|x|,|y|\}}{q|x-y|} = \beta$ is a limit case (cf. footnote 2) ■

Since $x D_k y$ is equivalent to $\frac{k_2}{k} \cdot \frac{|x-y|}{\max\{|x|,|y|\}} \geq k_2$, from proposition 7 the following result is straightforward:

Proposition 8.
If $\frac{|x-y|}{\max\{|x|,|y|\}} \neq k_2$, then $x D_k y$ if and only if there exists some real number M neither large nor very large such that M multiplied by the quotient $\frac{k_2}{k} \cdot \frac{|x-y|}{\max\{|x|,|y|\}}$ is not very small.

8. Conclusions

The aim of this paper is to analyse under which conditions an AOM and a ROM model are consistent and to determine the constraints that consistency implies. A graphical interpretation of the constraints is provided, bridging the absolute qualitative labels of two quantities into their corresponding relative relation(s), and conversely. The ROM relations are then characterized in the absolute world.

The study has been performed with a general AOM model OM(n) on one hand and the most general interpretable in $\mathbb{R}$ ROM model, F-ROM($\mathbb{R}$), on the other hand. The obtained results show that consistency is highly constrained. Indeed, only one degree of freedom remains out of four for the AOM model, the F-ROM($\mathbb{R}$) model resulting fully determined.

It is our opinion that the definition of consistency that has been adopted is the most intuitive and the one that guarantees the best correspondances between the absolute and the relative world, hence providing the best determined characterization of ROM($\mathbb{R}$) relations in terms of absolute concepts. Indeed, consistency requires that the quotients between the landmarks of the AOM model coincide with the landmarks of F-ROM($\mathbb{R}$). This requires first, the number of quotient landmarks to be 4 and second, the formal matching of their corresponding expressions. The first condition guarantees full expressivity in the relative world and implies an AOM model OM(n) of granularity n superior to 5. The second condition provides the most deterministic AOM $\rightarrow$ ROM and ROM $\rightarrow$ AOM correspondances, and hence the most deterministic characterization of ROM relations in the absolute world.

However, the problem could be formulated with a weaker definition of consistency. Preserving full expressivity in the relative world with an OM(5) absolute model, a *weak-consistency* property could be defined by relaxing the second condition, leading to weaker correspondances and characterizations. Another option could be to consider a higher granularity AOM model, but at the price of poorer semantics for the absolute labels.

Both two conditions could be relaxed as well. A special case would be to consider Mavroviouniotis and Stephanopoulos (1997) O(M) model instead of F-ROM($\mathbb{R}$). O(M) is a particular case of ROM($\mathbb{R}$) with $k_1 = k_2$, therefore in this model there are only two quotients less than 1: k_1 and $1 - k_1$. In this case, an OM(4) is sufficient to construct the matching. With the absolute landmarks γ, α and β, only two quotients less than 1 are obtained if and only if $\frac{\gamma}{\alpha} = \frac{\alpha}{\beta}$, i.e., α is the geometric mean of γ and β ($\gamma = \frac{\alpha^2}{\beta}$).

By imposing that the addition of the two quotients $\frac{\alpha^2}{\beta^2}$ and $\frac{\alpha}{\beta}$ is 1, the equation $\frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta} - 1 = 0$ reduces to one case: $\beta = q\alpha$, with $q = \frac{\sqrt{5}+1}{2}$.

A detailed study of this case could be performed in a way similar to sections 5, 6 and 7. However, one should notice that the absolute landmarks are also highly constrained by the numeric value of q.

Finally, we believe that our study could be advantageously completed by reinterpreting, in the case of consistency, the ROM(K) axioms in the absolute world. This would exhibit the semantics of these rules in the absolute world.

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