Can We Do Trigonometry Qualitatively?

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Abstract

This paper proposes fuzzy qualitative representation of trigonometry (FQT) in order to bridge the gap between qualitative and quantitative representation of physical systems using Trigonometry. Fuzzy qualitative coordinates are defined by replacing a unit circle with a fuzzy qualitative circle; the Cartesian translation and orientation are replaced by their fuzzy membership functions. Trigonometric functions, rules and the extensions to triangles in Euclidean space are converted into their counterparts in fuzzy qualitative coordinates using fuzzy logic and qualitative reasoning techniques. We developed a MATLAB toolbox XTrig in terms of 4-tuple fuzzy numbers to demonstrate the characteristics of the FQT. This approach addresses a representation transformation interface to connect qualitative and quantitative descriptions of trigonometry-related systems (e.g., robotic systems).

1 Introduction

Trigonometry is a branch of mathematics that deals with the relationships between the sides and angles of triangles and with the properties and application of trigonometric functions of angles. It began as the computational component of geometry in the second century BC and plays a crucial role in mathematics, engineering, etc. In order to bridge the gap between qualitative and quantitative description of physical systems, we propose a fuzzy qualitative representation of trigonometry (FQT), which provides theoretical foundations for their representation transformation of trigonometry.

It is often desirable and sometimes necessary to reason about the behaviour of a system on the basis of incomplete or sparse information. The methods of model-based technology provide a means of doing this [Kuipers, 1994]. The initial approaches to model-based technology were seminal but focused on qualitative reasoning only, providing a means whereby the global picture of how a system might behave could be generated using only the sign of the magnitude and direction of change of the system variables. This made qualitative reasoning complementary to quantitative simulation. However, quantitative and qualitative simulation forms the two ends of a spectrum; and semi-quantitative methods were developed to fill the gap. For the most part these were interval reasoners bolted on to existing qualitative reasoning systems (e.g. [Berleant, 1997]); however, one exception to this was fuzzy qualitative reasoning which integrated the strengths of approximate reasoning with those of qualitative reasoning to form a more coherent semi-quantitative approach than their predecessors [Shen and Leitch, 1993; Coghill, 1996]. Model-based technology methods have been successfully applied to a number of tasks in the process domain. However, while some effort has been expended on developing qualitative kinematic models, the results have been limited [Blackwell, 1988; Faltings, 1992] etc. The basic requirement for progressing in this domain is the development of qualitative version of the trigonometric rules. Buckley and Eslami [Buckley and Eslami, 2002] proposed the definition of fuzzy trigonometry from fuzzy perspective without consideration of the geometric meaning of trigonometry. Some progress has been made in this direction by Parsons [Parsons, 2001] and Liu [Liu, 1998], but as with other applications of qualitative reasoning, the flexibility gained in variable precision by integrating fuzzy and qualitative approaches is no less important in the kinematic domain. In this paper we present an extension of the rules of trigonometry to the fuzzy qualitative case, which will serve as the basis for fuzzy qualitative reasoning about the behaviour and possible diagnosis of kinematic robot devices.

1.1 Quantity Representation in Fuzzy Qualitative Reasoning

Qualitative reasoning has explored tradeoffs in representations for continuous parameters ranging in resolution from sign algebras to the hyperreals. Intervals are a well-known variable-resolution representation for numerical values, and have been heavily used in qualitative reasoning [Forbus, 1996]. A quantity space is to represent continuous values via sets of ordinal relations, it can be thought of as partial information about a set of intervals [Lee *et al.*, 2002]. The natural mapping between quantity spaces and intervals has been exploited by a variety of systems that use intervals whose endpoints are known numerical values to refine predictions produced by purely qualitative reasoning [Kuipers, 1994]. Fuzzy intervals have also been used in fuzzy reasoning about mechatronics systems [Shen and Leitch, 1993]. Fuzzy qualitative trigonometry has choosen the concept of a quantity space due to the fact that it is semi-qualitative qualitative reasoning and the success of fuzzy qualitative simulation. In this case the quantity space is a set of overlapping fuzzy numbers, an example of which in FuSim Fuzzy Simulation [Shen and Leitch, 1993] is shown in Figure 1.

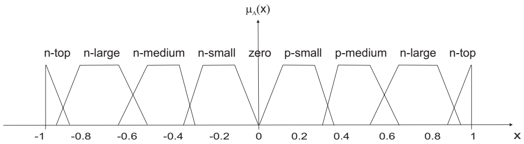

Figure 1: A fuzzy quantity space

The quantity space for every variable in the system is a finite and convex discretisation of the real number line. The quantity space for each variable individually is a subset of the real number line, which still covers the real number line. The quantity space in Figure 1 would be suitable for a variable whose domain has been normalised.

1.2 Fuzzy Reasoning & Qualitative Reasoning

Fuzzy reasoning and qualitative reasoning are two major streams in the theory of approximate reasoning. Fuzzy reasoning theory and its applications have been greatly developed since a famous controversy on fuzzy logic in 1993 [Shastri, 1994]. The use of fuzzy reasoning methods are becoming more and more popular in intelligent systems [Fuller, 1999; Chen *et al.*, 2003], especially hybrid methods and their applications integrating with evolutionary computing [Pedrycz and Reformat, 2003], decision trees [Tsang *et al.*, 2000], neural networks [Vuorimaa *et al.*, 1995], data mining [Smith and Eloff, 2000], and so on [Kwan and Cai, 1994; Ben Ghalia and Wang, 2000; Bae *et al.*, 2003]. Qualitative reasoning can be reviewed by [Weld and de Kleer, 1990; Williams and de Kleer, 1991; Faltings and Struss, 1992; Bredeweg and Struss, 2003]. The integration of fuzzy reasoning and qualitative reasoning (i.e., fuzzy qualitative reasoning) provides an opportunity to explore research (e.g., spatial reasoning) with both advantages of fuzzy reasoning & qualitative reasoning. Some of fuzzy qualitative reasoning contributions can be found in [Shen and Leitch, 1993; Lee *et al.*, 2002; Ali *et al.*, 2003; Li and Li, 2004]. Shen & Leitch [Shen and Leitch, 1993] use a fuzzy quantity space shown in Figure 1. This allows for a more detailed description of the values of the variables. Such an approach relies on the extension principle and approximation principle in order to express the results of calculations in terms of the fuzzy sets of the fuzzy quantity space. Relative order of magnitude using fuzzy relations [Ali *et al.*, 2003] is a promising approach for solving some ambiguity problems in qualitative reasoning. Not only can it mechanize the commonsense reasoning of engineers simplifying complex equations and computing approximate solutions, but also it can applied to provide a fuzzy semantics to plausible reasoning with qualitative probabilities.

2 Fuzzy Qualitative Cartesian Coordinates

2.1 Angle and distance measurement

Angle and distance measurement plays an important role in fuzzy qualitative trigonometry due to the fact that trigonometry is centred on angle measurement and quantities determined by the measure of an angle. Fuzzy qualitative quantity space Q_X is introduced to represent qualitative states of a Cartesian orientation Q_X^a and translation Q_X^d. It means that angle and distance measurement in fuzzy qualitative coordinates depends on the numbers and fuzzy characteristic of the elements of a fuzzy qualitative quantity space. For example, a fuzzy qualitative version of a Cartesian position is given for an orientation range $[0\ \Theta]$ and a translation range $[0\ L]$,

$$Q_X = \{Q_X^a, Q_X^d\} \tag{1}$$

where

$$Q_X^a = [QS_a(\theta_1), \cdots, QS_a(\theta_i), \cdots QS_a(\theta_m)]$$
$$Q_X^d = [QS_d(l_1), \cdots, QS_d(l_j), \cdots QS_d(l_n)]$$

$QS_a(\theta_i)$ denotes the state of an angle θ_i, $QS_d(l_j)$ denotes the state of a distance l_j; m & n are the number of the elements of the two quantity spaces. It is noted that $\theta_1,\ l_1$ and $\theta_m,\ l_n$ are equal to 0 and $\Theta,\ L$ respectively to ensure that the description is closed. The measurement of qualitative position P can be denoted by $P(QS_a(\theta_i), QS_d(l_j))$.

2.2 Qualitative fuzzy trigonometric circle

The geometric meaning of fuzzy qualitative trigonometry is described in a proposed fuzzy qualitative circle, Cartesian orientations and translations are constructed in the circle. For simplicity, 4-tuple fuzzy numbers are employed in this paper. Let O be the origin, a fuzzy qualitative trigonometric circle centred on O and with a radius equal to a 4-tuple fuzzy number $[1,\ 1,\ 0,\ 0]$. Compared with the trigonometric circle, its Cartesian translation is replaced by the membership distributions, $u_x,\ u_y$, of a set of 4-tuple fuzzy numbers on the distance $[-1,\ 1]$; Cartesian orientation is replaced by u_θ on the circle range $[0,\ 2\pi)$. Counterclockwise is the positive orientation, the number of qualitative orientation states in a full circle starting from the u_x axis is denoted by p, that of qualitative translation states by q_{axis} (i.e. $q_x,\ q_y$). The qualitative description of an orthogonal angle is given by $QS_a(\frac{p}{4}+1)$.

2.3 Example I

A MATLAB toolbox named the *XTrig* has been developed in terms of 4-tuple fuzzy numbers to implement the proposed FQT. The original aim of the toolbox is to provide a program to connect motion control and symbolic planning for a robotic system. The left-hand side of Fig. 2 shows that the descriptions of the Cartesian translation and orientation of a fuzzy qualitative circle are replaced by quantity spaces instead. The elements of quantity spaces are fuzzy membership functions of real numbers of Cartesian coordinates. For example, let the number of the quantity space Cartesian orientation p be 16, and Cartesian translation, q_{u_x} and q_{u_y}, be 21, the *XTrig* generates quantity spaces, Q_X^a and Q_X^d. The quantity spaces show that there are 16 and 21 4-tuple fuzzy numbers available to describe a Cartesian orientation and translation. That

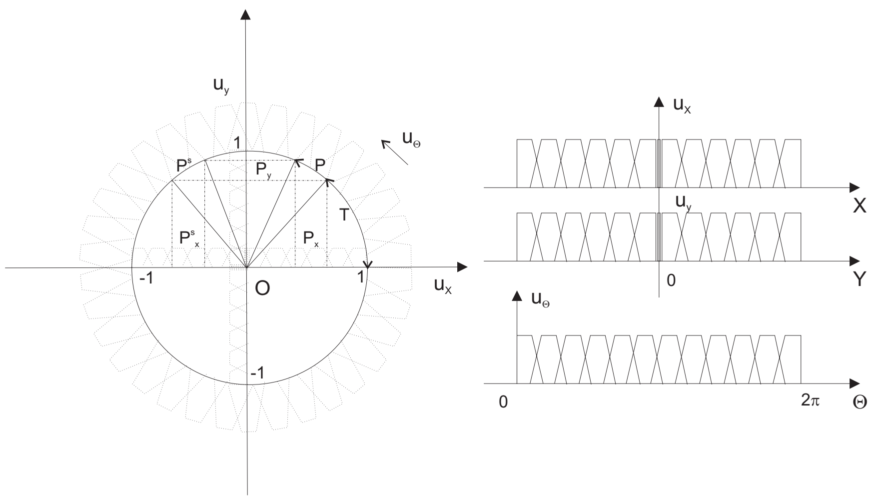

Figure 2: Qualitative Fuzzy Cartesian Coordinates

is, there are 16 fuzzy qualitative orientation angles and 21 translation positions. If one wants to calculate the qualitative position of the third orientation state in the FQT circle, one has to project a line segment, whose ends are crossing points of the boundary lines of the orientation state and a fuzzy qualitative circle, into the u_x and u_y axes. The same applies to the rest. Fuzzy numbers in the Cartesian translation quantity space, Q_X^d, fall in the ranges of the projections of the line segment in the fuzzy qualitative axes are the results for the fuzzy qualitative positions of the orientation state at the fuzzy qualitative coordinates. The results of the positions of the third orientation state $P(QS_d(3)_{u_x}, QS_d(3)_{u_x})$ are,

$$QS_a(3)_{u_x} = \begin{bmatrix} 0.7119 & 0.7966 & 0.0169 & 0.0169 \\ 0.8136 & 0.8983 & 0.0169 & 0.0169 \\ 0.9153 & 0.8983 & 0.0169 & 0 \end{bmatrix}$$

$$QS_a(3)_{u_y} = \begin{bmatrix} 0.3051 & 0.3898 & 0.0169 & 0.0169 \\ 0.4068 & 0.4915 & 0.0169 & 0.0169 \\ 0.5085 & 0.5932 & 0.0169 & 0.0169 \\ 0.6102 & 0.6949 & 0.0169 & 0.0169 \\ 0.7119 & 0.7966 & 0.0169 & 0.0169 \end{bmatrix}$$

It shows the calculation of the positions of the fuzzy qualitative angle is based on its corresponding fuzzy numbers rather than real numbers.

3 Fuzzy Qualitative Trigonometric Functions

FQT functions provide a fuzzy qualitative description of their quantitative counterparts. Each trigonometric function is derived and illustrated using quantity spaces of fuzzy qualitative coordinates. For the simplicity, symbols of quantitative trigonometric functions are used to describe their counterparts in fuzzy qualitative trigonometry but with qualitative variables instead (e.g., $QS_i(j)$).

3.1 Trigonometric Functions

In what follows, the fuzzy qualitative sine function is explained as an example. The sine of the qualitative arc T is defined by the u_y coordinate of the qualitative position P in Fig. 2, in which T is a set of quantitative arcs describing the distance from the qualitative position P to the crossing point between the positive axis of u_x and the circle along the fuzzy qualitative circle. The formulation is,

$$\sin\left(QS_a\left(\angle POP_x\right)\right) = \frac{QS_d\left(|P_yO|\right)}{\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \end{array}\right]} = QS_d\left(|P_yO|\right) \tag{2}$$

It clearly indicates that the sine of a qualitative angle or qualitative arc is the u_y coordinate of the qualitative state. The other trigonometric functions can be similarly derived,

$$\begin{aligned} &\cos\left(QS_a\left(\angle POP_x\right)\right) = \frac{QS_d\left(|P_xO|\right)}{\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \end{array}\right]} = QS_d\left(|P_xO|\right) \\ &\sec\left(QS_a\left(\angle POP_x\right)\right) = \frac{\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \end{array}\right]}{\cos\left(QS_a\left(\angle POP_x\right)\right)} \\ &\csc\left(QS_a\left(\angle POP_x\right)\right) = \frac{\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \end{array}\right]}{\sin\left(QS_a\left(\angle POP_x\right)\right)} \\ &\arcsin\left(QS_d\left(|PP_x|\right)\right) = QS_a\left(\angle POP_x\right) \\ &\arccos\left(QS_d\left(|PP_y|\right)\right) = QS_a\left(\angle POP_x\right). \end{aligned} \tag{3}$$

It should be noted that there are limits for fuzzy qualitative arcsine and arccosine functions. They are $\arcsin\left(QS_d\left(|PP_x|\right)\right) \in \left[-\frac{p}{4}-1, \frac{p}{4}+1\right]$, and $\arccos\left(QS_d\left(|PP_x|\right)\right) \in \left[0, \frac{p}{2}+1\right]$. In addition, before deriving more FQT functions, a Pythagorean lemma is introduced as,

Pythagorean Lemma: The sum of the squares of the qualitative sides, $QS_d(a)$, $QS_d(b)$, including a right angle is α equal to the square of the qualitative hypotenuse, $QS_d(c)$, opposite to the right angle.

After a little algebra calculation, the fundamental trigonometric identity can be reached from the lemma,

$$\cos^2\left(QS_a(P)\right) + \sin^2\left(QS_a(P)\right) =_\alpha [1\ 1\ 0\ 0] \tag{4}$$

Where $QS_a(P)$ is the qualitative value of any non right-angled angles of the right-angled triangle. In terms of the lemma, fuzzy qualitative tangent and cotangent functions can be derived,

$$\tan\left(QS_a\left(\angle POP_x\right)\right) = \frac{\sin\left(QS_a\left(\angle POP_x\right)\right)}{\cos\left(QS_a\left(\angle POP_x\right)\right)}$$
$$\cot\left(QS_a\left(\angle POP_x\right)\right) = \frac{\cos\left(QS_a\left(\angle POP_x\right)\right)}{\sin\left(QS_a\left(\angle POP_x\right)\right)} \quad (5)$$

3.2 The relevance index

The relationship between two adjacent fuzzy qualitative states is complex due to the ambiguous boundary of their fuzzy numbers. A relevance index is introduced to clear such confusion and presents the relationship between them, its formula is as follows,

$$QS_r\left(i\right) = \left[\begin{array}{cccc} \lambda_1 & X & X & \lambda_2 \end{array}\right] \quad (6)$$

Where λ_1 is the forward relevance index that denotes the relationship between the first fuzzy number of $QS_r\left(i\right)$ and the last fuzzy number of $QS_r\left(i-1\right)$, and λ_2 is the backward relevance index that denotes the relationship between the last fuzzy number of $QS_r\left(i\right)$ and the first fuzzy number of $QS_r\left(i+1\right)$. The elements X could be any number in order to keep a four-tuple fuzzy number form. X is replaced by NaN in the *XTrig* toolbox, where NaN means not a number. It is very complex to analyze the relationship of any two fuzzy numbers; it is an open problem in the Fuzzy community. For simplicity, the relation of fuzzy numbers is analyzed for 4-tuple fuzzy numbers only here. Consider two adjacent 4-tuple fuzzy numbers m and n, where $m = [a,\ b, \tau,\ \beta]$, $n = [c,\ d, \gamma,\ \delta]$, ($a \leq c$ and $b \leq d$). A crossing point $p(u, v)$ is defined as shown in Fig. 3, the value of u is the underlying real point where the membership distribution of m intersects with the distribution of n and v is the degree of membership of μ within m or n. A boundary point $p_b(u_b, v_b)$ is defined as the crossing point when $v_b = 0.5$. The elements of the relevance index can be defined

$$\lambda = \begin{cases} fuzzy-equality & v \geq 1 \\ strong-equality & v_b \leq v < 1 \\ weak-equality & 0 \leq v \leq v_b \end{cases}$$

where

$$v = -\frac{1}{\gamma+\beta}\left(c-b-\gamma-\beta\right)$$

The relationship of adjacent fuzzy number is fuzzy-equality when their crossing point lies above line $v = 1$; it is strong-equality when the point is between lines $v = 1$ and $v = 0.5$; it is weak-equality when the point is satisfied with the constraint, $0 \leq v \leq v_b$. Four numbers 0, 1, 2 and 3 are employed in the *XTrig* toolbox to denote fuzzy-equality, strong-equality, weak-equality and inequality.

3.3 Example II

Examples of fuzzy qualitative trigonometric functions following the quantity spaces generated in Section 2.3 are given. First, let us consider the fuzzy qualitative sine function with the 3rd fuzzy qualitative orientation angle (i.e., $QS_a(3)$). The result shows that $sin(QS_a(3))$ denotes fuzzy numbers within the range of fuzzy qualitative side P_yO, which is opposite

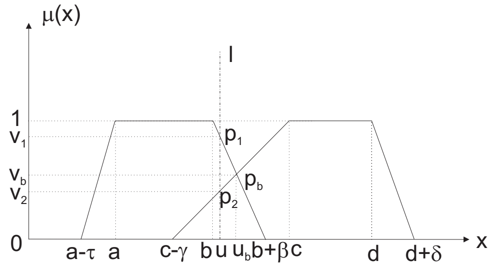

Figure 3: Relation analysis of two 4-tuple fuzzy numbers

to $QS_a(3)$. There are three fuzzy numbers within that fuzzy qualitative range, and its forward relevant relation is weak-equality (i.e., $\lambda_1 = 2.000$) and its backward relevant relation is fuzzy-equality (i.e., $\lambda_2 = 0$).

$$sin(QS_a(3)) = \left[\begin{array}{cccc} 0.7119 & 0.7966 & 0.0169 & 0.0169 \\ 0.8136 & 0.8983 & 0.0169 & 0.0169 \\ 0.9153 & 0.8983 & 0.0169 & 0 \\ 2.000 & NaN & NaN & 0 \end{array}\right]$$

And more examples,

$$cos(QS_a(112)) = \left[\begin{array}{cccc} 0.9153 & 1.0000 & 0.0169 & 0 \\ 0 & NaN & NaN & 0 \end{array}\right]$$

$$tan(QS_a(3)) = \left[\begin{array}{cccc} 1.0244 & 1.3056 & 0.0482 & 0.0659 \\ 1.6552 & 2.2083 & 0.0885 & 0.1395 \\ 1.3714 & 1.7667 & 0.0659 & 0.0954 \\ 2.3478 & 3.2778 & 0.1395 & 0.1928 \end{array}\right]$$

There are two points worth noting in above examples. One is that fuzzy qualitative trigonometric functions have period characteristic, e.g., $cos(QS_a(112)) = cos(QS_a(1+16\times 7))$. The other is that fuzzy qualitative tangent and cotangent functions are calculated based on fuzzy qualitative sine and cosine functions and the Pythagorean lemma. The latter ensures the removal those fuzzy numbers that are the result of fuzzy qualitative sine and cosine functions but do not have geometric meaning.

4 Fuzzy Qualitative Trigonometric Rules

4.1 Related Values

FQT divides its Cartesian orientation into 4-times fuzzy regions in order to have the characteristic of related values that its counterpart have. Let $QS_a(i)$, $QS_a(j)$ be the ith and jth qualitative states of Cartesian orientation angles, their relationship can be derived,

1. FQT supplementary $\leftrightarrow QS_a(i) + QS_a(j) = \frac{p}{2} + 1$
2. FQT complementary $\leftrightarrow QS_a(i) + QS_a(j) = \frac{p}{4} + 1$
3. FQT opposite $\leftrightarrow QS_a(i) + QS_a(j) = p + 1$
4. FQT anti supplementary $\leftrightarrow QS_a(i) - QS_a(j) = \frac{p}{2}$

For instance, consider two fuzzy qualitative states, $QS_a(P)$ and $QS_a(P^s)$, where they are fuzzy qualitative supplementary in Fig. 2. It clearly presents the relation,

$$\sin\left(QS_a\left(P\right)\right)=\frac{QS_d\left(\left|P^sP^s_x\right|\right)}{\left[\begin{array}{cccc}1&1&0&0\end{array}\right]}=\sin\left(QS_a\left(P^s\right)\right)$$
$$\cos\left(QS_a\left(P\right)\right)=-\frac{QS_d\left(\left|P^sP^s_y\right|\right)}{\left[\begin{array}{cccc}1&1&0&0\end{array}\right]}=-\cos\left(QS_a\left(P^s\right)\right) \tag{7}$$

where $QS_a(P)+QS_a(P^s)=\frac{p}{2}+1$. Then it reaches the following by substituting equation 7 into 5,

$$\tan\left(QS_a\left(P\right)\right)=-\tan\left(QS_a\left(P^s\right)\right)$$
$$\cot\left(QS_a\left(P\right)\right)=-\cot\left(QS_a\left(P^s\right)\right)$$

Further the other related values of fuzzy qualitative functions have been derived and have the same characteristic that its quantitative trigonometry has. The equality relationship of the related values is fuzzy qualitative equality since both sides of each equation describe the fuzzy qualitative function of a same qualitative state. The definition of fuzzy qualitative equality leads to that the fuzzy terms in both sides are exchangeable. However, strong-equality and weak-equality only allow part of fuzzy numbers to replace one another.

4.2 Example III

Examples of FQT related values using the quantity spaces generated in Section 2.3 are given in this section. Consider the fuzzy qualitative sine function and the result of the sine function of the 3rd fuzzy qualitative orientation angle in Section 3.3 again, its supplementary value, complementary, opposite and anti-supplementary are calculated to prove the correctness of rules of FQT related values. The following equations correspond to FQT supplementary, complementary, opposite and anti-supplementary, respectively.

$$\sin\left(QS_a\left(\frac{p}{2}+1-3\right)\right)=\sin\left(QS_a\left(6\right)\right)$$
$$\cos\left(QS_a\left(\frac{p}{4}+1-3\right)\right)=\cos\left(QS_a\left(2\right)\right)$$
$$\sin\left(QS_a\left(p+1-3\right)\right)=\sin\left(QS_a\left(14\right)\right)$$
$$\sin\left(QS_a\left(\frac{p}{2}+3\right)\right)=\sin\left(QS_a\left(11\right)\right)$$

5 Fuzzy Qualitative Triangle Theorems

The role that the counterparts of fuzzy qualitative triangle theorems play in quantitative geometry indicates its contribution to fuzzy qualitative calculation and analysis. The notation of a fuzzy qualitative triangle as shown in Fig. 4 can be given, its angles are denoted as $QS_a(A)$, $QS_a(B)$ and $QS_a(C)$, its sides are denoted as $QS_d(a)$, $QS_d(b)$ and $QS_d(c)$.

5.1 Sine and Cosine Rules

The area S of fuzzy qualitative triangle ABC above can be calculated from the perspective of the three sides,

$$S_1=\frac{QS_d\left(a\right)QS_d\left(c\right)\sin\left(QS_a\left(B\right)\right)}{2}$$
$$S_2=\frac{QS_d\left(b\right)QS_d\left(c\right)\sin\left(QS_a\left(A\right)\right)}{2} \tag{8}$$
$$S_3=\frac{QS_d\left(a\right)QS_d\left(b\right)\sin\left(QS_a\left(C\right)\right)}{2},$$

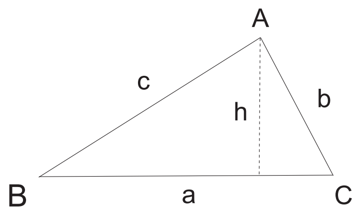

Figure 4: A fuzzy qualitative triangle

Clearly $S_1=S_2=S_3$ can be proved since all three describe the same area of a fuzzy qualitative angle. The fuzzy qualitative version of sine rule can be derived by dividing $QS_d\left(a\right)QS_d\left(b\right)QS_d\left(c\right)$ into equation 8,

$$\frac{QS_d\left(a\right)}{QS_a\left(A\right)}=\frac{QS_d\left(b\right)}{QS_a\left(B\right)}=\frac{QS_d\left(c\right)}{QS_a\left(C\right)} \tag{9}$$

The sine rule allows working out any unknown length and angle, provided that some of the lengths and angles are known in the triangle. The cosine rule can be derived using the dot product of vectors, the vector $QS_d(b)$ can be extended as,

$$\left\|QS_d\left(b\right)\right\|^2=QS_d\left(b\right)\cdot QS_d\left(b\right)$$
$$=\left(QS_d\left(a\right)-QS_d\left(c\right)\right)\cdot\left(QS_d\left(a\right)-QS_d\left(c\right)\right)$$
$$=QS_d^2\left(a\right)+QS_d^2\left(c\right)-2\cdot QS_d\left(a\right)\cdot QS_d\left(c\right)$$
$$=\left\|QS_d\left(a\right)\right\|^2+\left\|QS_d\left(c\right)\right\|^2-2\left\|QS_d\left(a\right)\right\|\left\|QS_d\left(c\right)\right\|\cos\left(\angle B\right) \tag{10}$$

The other two fuzzy qualitative sides can be derived in the same way. The cosine rule also provides the same facility as sine rule does to work out any unknown length and angle, provided that a minimum of two (e.g., lengths and angles) to be known in a triangle.

5.2 Triangle Theorems

Triangle theorems can be converted into those in terms of fuzzy qualitative trigonometry including AAA, AAS, ASA, ASS, SAS and SSS, where A stands for a fuzzy qualitative angle of a fuzzy qualitative triangle, S stands for a side.

- AAA theorem

$$QS_a\left(C\right)=\frac{p}{2}-QS_a\left(A\right)-QS_a\left(B\right). \tag{11}$$

- AAS theorem

$$QS_d\left(b\right)=QS_d\left(a\right)\frac{\sin\left(QS_a\left(B\right)\right)}{\sin\left(QS_a\left(A\right)\right)}, \tag{12}$$

$$QS_d\left(c\right)=QS_d\left(b\right)\cos\left(QS_a\left(A\right)\right)+QS_d\left(a\right)\cos\left(QS_a\left(B\right)\right). \tag{13}$$

- ASA theorem

$$QS_d\left(a\right)=\frac{\sin\left(QS_a\left(A\right)\right)}{\sin\left(\frac{p}{2}-QS_a\left(A\right)-QS_a\left(B\right)\right)}QS_d\left(c\right)$$
$$QS_d\left(b\right)=\frac{\sin\left(QS_a\left(B\right)\right)}{\sin\left(\frac{p}{2}-QS_a\left(A\right)-QS_a\left(B\right)\right)}QS_d\left(c\right). \tag{14}$$

- ASS theorem

$$n = \begin{cases} 2 & If \sin\left(QS_a\left(A\right)\right) < \frac{QS_d(a)}{QS_d(c)} \\ 1 & If \sin\left(QS_a\left(A\right)\right) = \frac{QS_d(a)}{QS_d(c)} \\ 0 & If \sin\left(QS_a\left(A\right)\right) > \frac{QS_d(a)}{QS_d(c)} \end{cases} \qquad (15)$$

- SAS theorem

$$\begin{aligned} & QS_a\left(A\right) = \sin^{-1}\left(QS_d\left(a\right)\sin\left(QS_a\left(B\right)\right)\right)/ \\ & \sqrt{QS_d^2\left(a\right) + QS_d^2\left(c\right) - 2QS_d^2\left(a\right)QS_d^2\left(c\right)\cos\left(QS_a^2\left(B\right)\right)} \\ & QS_a\left(C\right) = \sin^{-1}\left(QS_d\left(c\right)\sin\left(QS_a\left(B\right)\right)\right)/ \\ & \sqrt{QS_d^2\left(a\right) + QS_d^2\left(c\right) - 2QS_d^2\left(a\right)QS_d^2\left(c\right)\cos\left(QS_a^2\left(B\right)\right)}. \end{aligned} \qquad (16)$$

- SSS theorem

$$\begin{aligned} QS_a\left(A\right) &= \cos^{-1}\left(\frac{QS_d^2\left(b\right) + QS_d^2\left(c\right) - QS_d^2\left(a\right)}{2QS_d\left(b\right)QS_d\left(c\right)}\right) \\ QS_a\left(B\right) &= \cos^{-1}\left(\frac{QS_d^2\left(a\right) + QS_d^2\left(c\right) - QS_d^2\left(b\right)}{2QS_l\left(a\right)QS_d\left(c\right)}\right) \\ QS_a\left(C\right) &= \cos^{-1}\left(\frac{QS_d^2\left(a\right) + QS_d^2\left(b\right) - QS_d^2\left(c\right)}{2QS_d\left(a\right)QS_d\left(b\right)}\right) \end{aligned} \qquad (17)$$

5.3 Example IV

An example is presented in this section applying the proposed FQT to a 2-link planar robot shown in Fig. 5, the notation in Fig. 4 is applied here. The example aims to calculate the parameters of the end-effector (e.g., position, velocity and acceleration) given a scenario as follows,

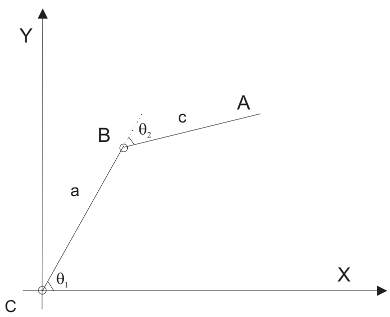

Figure 5: A 2-link robot

$$\begin{aligned} QS_d\left(a\right) &= \left[\begin{array}{cccc} 0.4068 & 0.4915 & 0.0169 & 0.0169 \end{array}\right] \\ QS_d\left(c\right) &= \left[\begin{array}{cccc} 0.5085 & 0.5932 & 0.0169 & 0.0169 \end{array}\right] \\ QS_a\left(\theta_1\right) &= \left[\begin{array}{cccc} 0.1263 & 0.1789 & 0.0105 & 0.0105 \end{array}\right] \\ QS_a\left(B\right) &= \left[\begin{array}{cccc} 0.3158 & 0.3684 & 0.0105 & 0.0105 \end{array}\right] \end{aligned}$$

Where $QS_d(a)$ and $QS_d(c)$ are the length of the two links, $QS_a(\theta_1)$ and $QS_a(\angle ABC)$ are the angles of link 1 and crossing angle of the two links, respectively. In accordance with SAS theorem, the follow results can be reached,

$$QS_d^{'}\left(b\right) = \left[\begin{array}{cccc} 0.8071 & 0.8484 & 0.0033 & 0.0125 \\ 0.8430 & 0.8728 & 0.0010 & 0.0104 \\ 0.8775 & 0.8966 & -0.0012 & 0.0085 \\ 0.9107 & 0.9197 & -0.0033 & 0.0066 \\ 0.9427 & 0.9423 & -0.0052 & 0.0048 \end{array}\right]$$

Then, by mapping the above value back to Cartesian translation quantity space, we can produce the following,

$$QS_d\left(b\right) = \left[\begin{array}{cccc} 0.8136 & 0.8983 & 0.0169 & 0.0169 \\ 0.9153 & 1.000 & 0.0169 & 0 \end{array}\right]$$

Next, we can have the following values by applying FQT Sine rule,

$$\begin{aligned} Sin\left(QS_a^{'}\left(\angle ACX\right)\right) &= \frac{QS_d\left(c\right)Sin\left(QS_a\left(B\right)\right)}{QS_d\left(b\right)} \\ &= \left[\begin{array}{cccc} 0.4037 & 0.5385 & 0.0263 & 0.0263 \\ 0.04147 & 0.05605 & 0.0280 & 0.0289 \\ 0.4267 & 0.5855 & 0.0299 & 0.0321 \\ 0.4614 & 0.06073 & 0.0287 & 0.0282 \\ 0.4740 & 0.6321 & 0.0306 & 0.0310 \\ 0.4876 & 0.6602 & 0.0328 & 0.0345 \\ 0.5191 & 0.6760 & 0.0312 & 0.0183 \\ 0.5332 & 0.7036 & 0.0733 & 0.0209 \\ 0.5486 & 0.7350 & 0.0356 & 0.0241 \end{array}\right] \end{aligned}$$

Further, applying FQT arcsin(), $QS_a\left(\angle ACX\right)$ is calculated as,

$$QS_a\left(\angle ACX\right) = \left[\begin{array}{cccc} 0.0632 & 0.1158 & 0.0105 & 0.0105 \end{array}\right]$$

Finally, the end-effector of the two-link robot can be described by its angle and distance from base C, i.e., $\{QS_a\left(\angle ACX\right), QS_d\left(b\right)\}$, where $QS_a\left(\angle ACX\right) = QS_a\left(\theta_1\right) - QS_a\left(C\right)$, see Table 1.

Table 1: Qualitative position description of the end-effector

	QS_1	QS_2	QS_3	QS_4
$QS_a\left(\angle ACX\right)$	$QS_a(1)$	$QS_a(1)$	$QS_a(2)$	$QS_a(2)$
$QS_d\left(b\right)$	$QS_d(9)$	$QS_d(10)$	$QS_d(9)$	$QS_d(10)$

6 Discussions and Conclusions

A fuzzy qualitative version of traditional trigonometry has been proposed in this paper. Trigonometry in Cartesian coordinates is mapped into a fuzzy qualitative coordinate system by using fuzzy logic and qualitative reasoning techniques. FQT functions and their characteristics are derived and proved with examples through the paper. The trigonometry extension (i.e., FQT) could provide a general interface to easily communicate between the numeric world and qualitative world. Future work will focus on applying FQT to the robotics domain (e.g., qualitative kinematics and robotic communication) and process systems (e.g., reasoning about behaviours of dynamic systems).

References

[Ali *et al.*, 2003] A.H. Ali, D. Dubois, and H. Prade. Qualitative reasoning based on relative orders of magnitude. *IEEE transactions on Fuzzy Systems*, 11(1):9–23, 2003.

[Bae *et al.*, 2003] S.M. Bae, S.C. Park, and S.H. Ha. Fuzzy web ad selector based on web usage mining. *IEEE intelligent Systems*, 18(6):62–69, 2003.

[Ben Ghalia and Wang, 2000] B. Ben Ghalia and P.P. Wang. Itelligent system to support judgmental business forecasting: the case of estimating hotel room demand. *IEEE transactions on Fuzzy Systems*, 8(4):380–397, 2000.

[Berleant, 1997] D. Berleant. Qualitative and quantitative simulation: bridging the gap. *Artificial Intelligence Journal*, 95(2):215–255, 1997.

[Blackwell, 1988] A.F. Blackwell. Spatial reasoning for robots: A qualitative approach. *MSc Thesis, Victoria University of Wellington*, 1988.

[Bredeweg and Struss, 2003] B. Bredeweg and P. (eds.) Struss. Special issues on qualitative reasoning. *AI magzine*, 2003.

[Buckley and Eslami, 2002] J.J. Buckley and E. Eslami. An introduction to fuzzy logic and fuzzy sets. *Springer-Verlag*, 2002.

[Chen *et al.*, 2003] B.S. Chen, Yang Y.S., B.K. Lee, and Lee T.H. Fuzzy adaptive predictive flow control of atm network traffic. *IEEE transactions on Fuzzy Systems*, 11(4):568–581, 2003.

[Coghill, 1996] G. M. Coghill. *Mycroft: A Framework for Constraint-Based Fuzzy Qualitative Reasoning*. PhD thesis, Heriot-Watt University, 1996.

[Faltings and Struss, 1992] B. Faltings and P. (eds) Struss. Recent advacnes in qualitative physics. *MIT Press*, 1992.

[Faltings, 1992] B. Faltings. A symbolic approach to qualitative kinematics. *Artificial Intelligence*, 56(2-3):139–170, 1992.

[Forbus, 1996] K. Forbus. Qualitative reasoning crc handbook of computer science and engineering. *CRC Press*, 1996.

[Fuller, 1999] R. Fuller. On fuzzy reasoning schemes. *The state of the art of information systems application in 2007, TUCS General Publications, Carlsson,C.(eds)*, 16:85–112, 1999.

[Kuipers, 1994] B. Kuipers. Qualitative reasoning. *MIT Press*, 1994.

[Kwan and Cai, 1994] H.K. Kwan and Y. Cai. A fuzzy neural network and its application to pattern recognition. *IEEE transactions on Fuzzy Systems*, 2(30:185–193, 1994.

[Lee *et al.*, 2002] J.W.T. Lee, D.S. Yeung, and E.C.C. Tsang. Ordinal fuzzy sets. *IEEE transactions on Fuzzy Systems*, 10(6):767–778, 2002.

[Li and Li, 2004] Y. Li and S. Li. A fuzzy sets theoretic approach to approximate spatial reasoning. *IEEE transactions on Fuzzy Systems*, 12(6):745–754, 2004.

[Liu, 1998] J. Liu. A method of spatial reasoning based on qualitative trigonometry. *Artificial Intelligence*, 98(1-2):137–168, 1998.

[Parsons, 2001] S. Parsons. Qualitative methods for reasoning under uncertainty. *MIT Press*, 2001.

[Pedrycz and Reformat, 2003] E. Pedrycz and M. Reformat. Evolutionary fuzzy modelling. *IEEE transactions on Fuzzy Systems*, 11(5):652–655, 2003.

[Shastri, 1994] L. (eds) Shastri. A fuzzy logic symposium. *IEEE Expert*, 9(4), 1994.

[Shen and Leitch, 1993] Q. Shen and R. Leitch. Fuzzy qualitative simulation. *IEEE Transactions on Systems, Man, and Cybernetics*, 23(4):1038–1061, 1993.

[Smith and Eloff, 2000] E. Smith and J. Eloff. Cognitive fuzzy modelling for enhanced risk assessment in a health care institution. *IEEE Intelligent Systems*, 15(2):69–75, 2000.

[Tsang *et al.*, 2000] E.C.C. Tsang, X.Z. Wang, and D.S. Yeung. Improving learning accuracy of fuzzy decision trees by hybrid neural networks. *IEEE transactions on Fuzzy Systems*, 8(5):601–614, 2000.

[Vuorimaa *et al.*, 1995] P. Vuorimaa, T. Jukarainen, and E. Karpanoja. A neuro-fuzzy system for chemical agent detection. *IEEE transactions on Fuzzy Systems*, 3(4):415–424, 1995.

[Weld and de Kleer, 1990] D.S. Weld and J. (eds.) de Kleer. Reading in qualitative reasoning about physical systems. , *Morgan Kaufman, San Mateo, CA*, 1990.

[Williams and de Kleer, 1991] B.C. Williams and J. (eds.) de Kleer. Special issue on qualitative reasoning about physical systems. *Artificial Intelligence*, 51(1-3), 1991.