Using Orders of Magnitude in Multi-attribute Decision-making

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Abstract

This paper presents a method for evaluating qualitative orders of magnitude information in multi-attribute decision-making. It allows the selection of an alternative from among a set of alternatives. These are characterized by having all descriptors defined in orders of magnitude. A representation for the different alternatives by means of k-dimensional labels is proposed, each of these standing for the conjunction of k labels corresponding to the qualitative information considered. A method is given for choosing the best alternative based on comparing distances against a reference k-dimensional label. For this reason, a distance is introduced that enables a total order to be defined for the set of alternatives. Finally, the method based on this order is proposed and its consistency established.

Introduction

In multiple attribute decision-making processes, the evaluation of alternatives depends on the previous valuation of input factors or variables [4], [9]. In some cases, the available information cannot be expressed with real numbers. The choice of the methodology used is dictated by the scale type on which the information is represented. The multi-attribute decision method presented in this work is especially suitable when aiming at an evaluation from the qualitative ordinal descriptions of the variables involved. These ordinal scaled descriptions are considered in the case in which numerical values are not precisely known, or when orders of magnitude and variable tendencies are more relevant than their exact values.

This work is the adaptation of previous work, based on interval algebra, to the case of variables defined over absolute orders of magnitude spaces [6].

In addition, this work considers the hypothesis that the value assigned to the alternatives is an increasing function with respect to the input variables, i.e. the greater the input variable values the higher the corresponding alternative values. In the case of having decreasing dependency with respect to some variables, each of these variables will be replaced by its opposite, changing the positive/negative sign of its values.

The proposed method is based on the synthesization of the initial information via a k-dimensional label, which can be seen as a rectangle, and its evaluation by means of a distance to a reference k-dimensional label. This in turn is based on a qualitative generalization of a type of goal programming method known as the reference point method for vectorial optimization and decision-making support [3], [6], [7]. In general, reference point methods for optimization in R^n choose the points at the shortest distance from a previously fixed reference point as the optimal alternative (the "goal" to be reached) [2].

In this work, optimization in the set of existing alternatives is performed by selecting not an independent fixed reference, but a "realistic" reference for the problem to be solved: with respect to the natural order, the proposed reference is the supreme reference for the set of available alternatives, guaranteeing consistency with the order between rectangles.

The proposed methodology may be of interest to very different areas. Concretely, applications in candidate assessments (students in learning processes, recruiting processes) as well as project management (architectonic, civil engineering and business projects) can be considered [5]. It may also be of interest in decision-making processes in areas such as finance and marketing [1].

Section 2 presents some features related to the qualitative models of absolute orders of magnitude, and Section 3 provides a qualitative representation of alternatives in a partially-ordered set. Section 4 defines some possible distances in R^n, whether weighted or not, and one of these in particular is proposed. In Section 5, a total order in the set of alternatives is defined in such a way that the set of labels corresponding to the available alternatives becomes a chain (ranking), and the alternative chosen is that represented by the maximum for this chain.

The consistency property for the method of choice is established. Analysis of the necessary conditions under which consistency can hold leads to the determination of the reference as the supreme of the set of labels corresponding to the available alternatives. Lastly, conclusions and open problems are presented.

Absolute Orders of Magnitude Models

The one-dimensional absolute order of magnitude model [8] works with a finite number of qualitative labels obtained via a discretization of the real line. The number of labels chosen to describe a real problem is not fixed, but depends on the

characteristics of each represented variable.

Let us consider a set of landmarks $\{a_1, \ldots, a_{n+1}\}$ to define the set of basic labels $S_* = \{B_1, \ldots, B_n\}$ (see Figure 1), where $B_i = [a_i, a_{i+1}],\ i = 1, \ldots, n$.

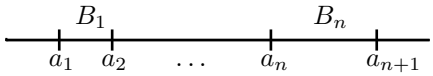

Figure 1: The discretization

The complete Orders of Magnitude Space (OM) space S is defined as $S = S_* \cup \{[B_i, B_j]/B_i, B_j \in S_*,$ with $i < j\}$, being $[B_i, B_j]$ the label corresponding to the minimum closed interval of the real line that contains B_i and B_j. So, if $B_i = [a_i, a_{i+1}], B_j = [a_j, a_{j+1}]$, then $[B_i, B_j] = [a_i, a_{j+1}]$ [8].

In this paper, variables can be defined in spaces with different granularity (different number of basic labels) and they can therefore have different sets of landmarks.

Alternatives representation: the partially ordered set $\mathcal{E}$

In the proposed multi-attribute decision making problem, each alternative is characterized by the values of k attributes or input variables, and these values are given by means of qualitative labels belonging to different orders of magnitude spaces. So, each alternative is represented by a k-dimensional label.

Let S_i be the orders of magnitude space associated to the variable i with set of basic labels S_{i*}.

The set of alternatives $\mathcal{E}$ is defined as:

$$\mathcal{E} = S_1 \times \ldots \times S_k = \\ = \{(E_1, \ldots, E_k) \mid E_i \in S_i\ \forall i = 1, \ldots k\}. \quad (1)$$

Each k-dimensional label $(E_1, \ldots, E_k)$ is interpreted as a set of k qualitative labels (each associated to an input variable) that define an alternative in such a way that, on every variable, higher values always mean better results [2].

The order relation $\leq$ that respects this fact is considered: $E = (E_1, \ldots, E_k) \leq E' = (E'_1, \ldots, E'_k)$ means that alternative E' is better than alternative E. This order relation is built from the total order relation $\leq$ in $\mathbb{R}$, which in turn, induces an order relation between basic labels in each S_{h*}:

$$B_i \leq B_j \iff x \leq y\ \forall x \in B_i, \forall y \in B_j \iff a_{i+1} \leq a_j. \quad (2)$$

This induces a partial order in each component S_h (see Figure 2):
Let $E = [B_i, B_j]$ and $E' = [B'_i, B'_j]$ be two labels in S_h, with $B_i, B_j, B'_i, B'_j \in S_{h*}$; then:

$$E \leq E' \iff B_i \leq B'_i \text{ and } B_j \leq B'_j. \quad (3)$$

This order is extended to the cartesian product $\mathcal{E}$:

$$(E_1, \ldots, E_k) \leq (E'_1, \ldots, E'_k) \\ \iff E_i \leq E'_i,\ \forall i = 1, \ldots, k. \quad (4)$$

This relation is an order relation in $\mathcal{E}$, but a partial order, since there are pairs of non-comparable k-dimensional labels.

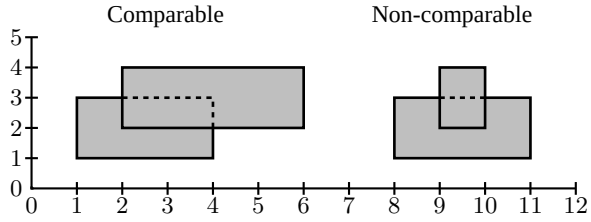

Figure 2: The partial order $\leq$ in $\mathcal{E}$

Distances in the set $\mathcal{E}$

A method is presented for computing distances between k-dimensional labels. Although the analytic expressions of these distances, and so their qualitative interpretations, are different, all are mathematically equivalent in the sense that the open sets of the induced topologies are the same.

Non-weighted distances

Let $(M_i, d_i)_{i=1,\ldots l}$ a family of matric spaces, where, for each i, M_i is a non-empty set and d_i a distance defined in M_i. Let's consider the three classical distances in $\mathbb{R}^l$, that is to say, the Euclidean distance d_e, the distance of the maximum d_m, and the distance of Manhattan d_s.

Proposition 1 The cartesian product $\prod_{i=1}^{l} M_i$ is a metric space with the distances built from the distances d_i in M_i and either the Euclidean distance or the distance of the maximum, or the distance of Manhattan in $\mathbb{R}^l$:

$$d_e((x_1, \ldots, x_l), (y_1, \ldots, y_l)) = \sqrt{\sum_{i=1}^{l} d_i^2(x_i, y_i)} \quad (5)$$

$$d_m((x_1, \ldots, x_l), (y_1, \ldots, y_l)) = \max_{i=1,\ldots,l} \{d_i(x_i, y_i)\} \quad (6)$$

$$d_s((x_1, \ldots, x_l), (y_1, \ldots, y_l)) = \sum_{i=1}^{l} d_i(x_i, y_i) \quad (7)$$

Proposition 2 Given a set X, and a metric space (M, d), any injective mapping $\Phi : X \hookrightarrow M$ induces a metric space structure in X, by means of $d_X(x, y) = d(\Phi(x), \Phi(y))$.

Therefore, it is possible to define a distance in the set of k-dimensional labels $\mathcal{E}$ by using the mapping Φ from $\mathcal{E}$ into the cartesian product of $\mathbb{R}^k$ for k copies of $\mathbb{R}$, $\Phi : \mathcal{E} \hookrightarrow \mathbb{R}^k \times \mathbb{R} \times \cdots \times \mathbb{R}$ given by

$$\Phi(E) = \Phi((E_1, \ldots, E_k)) = \\ \Phi([B_{i_1}, B_{j_1}], \ldots, [B_{i_k}, B_{j_k}]) = \\ \Phi([a_{i_1}, a_{j_1+1}], \ldots, [a_{i_k}, a_{j_k+1}]) = \\ (c(E), l_1(E), \ldots, l_k(E)), \quad (8)$$

where

$$c(E) = \left(\frac{a_{i_1} + a_{j_1+1}}{2}, \ldots, \frac{a_{i_k} + a_{j_k+1}}{2}\right) \quad (9)$$

is the center of the k-rectangle E and

$$l_1(E) = a_{j_1+1} - a_{i_1}, \ldots, l_k(E) = a_{j_k+1} - a_{i_k} \quad (10)$$

are the lengths of the sides of E.

This mapping is evidently injective, given that any rectangle with sides parallel to the coordinate axis is determined by its center and the lengths of its sides. Therefore, from proposition 2, any distance in $\mathbb{R}^k \times \mathbb{R} \times \cdots \times \mathbb{R}$ induces a distance in $\mathcal{E}$.

From proposition 1, distances in the product $\mathbb{R}^k \times \mathbb{R} \times \cdots \times \mathbb{R}$ can be built from d_e, d_m or d_s and different distances in each factor of $\mathbb{R}^k$, $\mathbb{R}$,...,$\mathbb{R}$.

When the Euclidean distance is taken in the first factor of the product $\mathbb{R}^k \times \mathbb{R} \times \cdots \times \mathbb{R}$, the distance of Manhattan in the other k factors $\mathbb{R}$ and the distance d_s of proposition 1 for their combination, the corresponding distance in $\mathcal{E}$ is:

$$d_{\mathcal{E}\,s}(E, E') = d_s(\Phi(E), \Phi(E')) = d_{euc}(c(E), c(E')) + \sum_{i=1}^{k} |l_i(E) - l_i(E')|. \quad (11)$$

This distance provides an intuitive notion of proximity between k-dimensional rectangles, since it takes into account the position of their centers, related to the magnitude of the alternatives, and the lengths of their sides, related to the imprecision of the data of the input variables. Moreover, this distance applied to the case of maximum precision, in which the intervals are reduced to points, is precisely the Euclidean distance in $\mathbb{R}^k$.

Weighted distances

If in the k-dimensional labels $E = (E_1, \ldots, E_k)$ each of the labels E_k has a different importance, due to the fact that different variables have different relevance for the evaluation, a set of positive weights $\alpha_1, \ldots, \alpha_k$ can be considered. Distances in $\mathcal{E}$ taking into account these weights can be built by changing the injection Φ by a weighted injection. Taking $\Psi(E) = (c(E), \alpha_1 l_1(E), \ldots, \alpha_k l_k(E))$, the following weighted distance is obtained:

$$d_{\mathcal{E}\,sw}(E, E') = d_s(\Psi(E), \Psi(E')) = d_{euc}(c(E), c(E')) + \sum_{i=1}^{k} \alpha_i \, |l_i(E) - l_i(E')|, \quad (12)$$

If it is also wished to weight the distance between the centers of the rectangles with respect to the lengths of the sides, it is only necessary to add a number $\beta > 0$ to the set of weights and choose a suitable weighted injection to obtain:

$$d_{\mathcal{E}\,sw}(E, E') = \beta d_{euc}(c(E), c(E')) + \sum_{i=1}^{k} \alpha_i \, |l_i(E) - l_i(E')|. \quad (13)$$

Choice of the best alternative

Starting from a distance in $\mathcal{E}$ and a reference k-dimensional label $\overline{E}$, a total order $\trianglelefteq$ can be defined in $\mathcal{E}$, such that the set of labels $E^1, \ldots, E^n$ corresponding to the available alternatives become a chain: $E^{i_1} \trianglelefteq \cdots \trianglelefteq E^{i_n}$. Then alternative E^{i_n} corresponding to the maximum of the chain will be chosen.

A total order in $\mathcal{E}$

Let $\overline{E} \in \mathcal{E}$ be any k-dimensional label and let us call it the reference label.

Let d be any of the distances defined in $\mathcal{E}$ in Section 4, then the following binary relation in $\mathcal{E}$:

$$E \preceq E' \Longleftrightarrow d(E', \overline{E}) \leq d(E, \overline{E}) \quad (14)$$

is a pre-order, i.e. it is reflexive and transitive.

This pre-order relation induces an equivalence relation in $\mathcal{E}$ by means of:

$$E \equiv E' \Longleftrightarrow [E \preceq E' \ , \ E' \preceq E] \Longleftrightarrow d(E', \overline{E}) = d(E, \overline{E}). \quad (15)$$

Then, in the quotient set $\mathcal{E}/\equiv$ the following relation between equivalence classes:

$$\text{class}\,(E) \trianglelefteq \text{class}\,(E') \Longleftrightarrow E \preceq E' \Longleftrightarrow d(E', \overline{E}) \leq d(E, \overline{E}) \quad (16)$$

is a total order relation.

In this way, given a set of alternatives $E^1, \ldots, E^n$, these can be ordered as a chain with respect to their proximity to the reference label: $\text{class}\,(E^{i_1}) \trianglelefteq \cdots \trianglelefteq \text{class}\,(E^{i_n})$. Alternatives belonging to the same equivalence class, i.e. alternatives at the same distance from $\overline{E}$, will be regarded as alternatives with the same value, therefore, from now on an abuse of notation will be made by changing $\preceq$ into $\trianglelefteq$: $E^{i_1} \trianglelefteq \cdots \trianglelefteq E^{i_n}$.

Consistency of the method of choice

The method of choice of the best alternative via a distance to a reference label is really necessary when no alternative is better than all the rest with respect to every variable, i.e., when the set $\{E^1, \ldots, E^n\}$ has no maximum with respect to the order relation $\leq$.

But when $\{E^1, \ldots, E^n\}$ has a maximum E^m with respect to $\leq$, that is to say, when there already exists a priori an alternative E^m better than the others, the proposed method for choice will be consistent if it provides the same E^m as the best alternative. Formally, given any $E^1, \ldots, E^n \in \mathcal{E}$:

$$\exists m \in \{1, \ldots, n\} \ \ E^i \leq E^m \ \forall\, i = 1, \ldots, n \Longrightarrow E^i \trianglelefteq E^m \ \forall i = 1, \ldots, n \quad (17)$$

The following proposition determines the construction of the reference label $\overline{E}$ for any set of labels $E^1, \ldots, E^n$ which have a maximum with respect to the partial order $\leq$:

Proposition 3 If for any set of k-dimensional labels $E^1, \ldots, E^n$ with maximum E^m the label $\overline{E} = E^m$ is chosen as reference, then the property of consistency is accomplished. Otherwise, this property can not be assured.

The first statement is trivial, because $d(\overline{E}, \overline{E}) = 0 \leq d(\overline{E}, E^i) \, \forall i = 1, \ldots, n$. In the case $\overline{E} \neq E^m$, a counterexample of the property of consistency can always be found. In particular, for the distance $d_{\mathcal{E}\,s}$ introduced in Section 4.1, the following counterexample can be considered (see Figure 3):

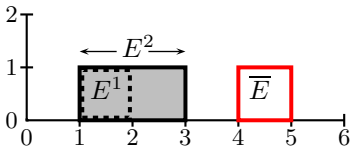

Figure 3. Counterexample of the consistency

Being E^1, E^2 the alternatives and $\overline{E}$ the chosen reference label, we have $d(E^1, \overline{E}) = 3$, $d(E^2, \overline{E}) = 3.5$, so the "nearest" to $\overline{E}$ is E^1, i.e. $E^2 \trianglelefteq E^1$, and, nevertheless, $E^1 \leq E^2$.

Selection of the reference label

The consistency of the proposed method determines the reference label in the case of a set of k-dimensional labels with a maximum, as has been proved. The natural generalization to any set of labels is the following:

Given any $E^1, \ldots, E^n$, the supreme of these hyperrectangles with respect to the partial order $\leq$ will be taken (see Figure 4):

$$\overline{E} = \sup\{E^1, \ldots, E^n\}. \quad (18)$$

That is to say, if $E^r = (E^r_1, \ldots, E^r_k)$, with $E^r_h = [B^r_{i_h}, B^r_{j_h}]$ for all $h = 1, \ldots, k$, and for all $r = 1, \ldots, n$, then $\overline{E} = (\overline{E}_1, \ldots, \overline{E}_k)$, where

$$\overline{E}_h = [\max\{B^1_{i_h}, \ldots, B^n_{i_h}\}, \max\{B^1_{j_h}, \ldots, B^n_{j_h}\}]. \quad (19)$$

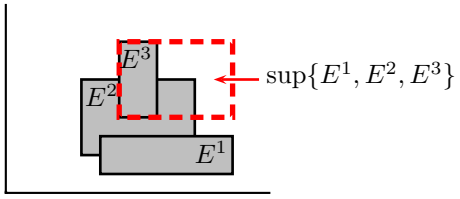

Figure 4: Reference label: the supreme

Note that in the case of rectangles with a maximum, this supreme is in fact the maximum, so this choice of the reference rectangle holds the property of consistency of the method.

Selection of the best alternative

Finally, the steps of the proposed method for selecting the best alternative are:

- To fix a distance d in $\mathcal{E}$: $d_{\mathcal{E}_s}$.
- To build a reference label $\overline{E}$: the supreme of the set of alternatives.
- To assign to each k-dimensional label E the value $d(E, \overline{E})$; so, the alternatives are ordered (with respect to the defined total order) as a chain,
- To choose as the best alternative the maximum of the chain, that is to say, the one (or ones) of minimum distance.

Conclusion

In this paper, a methodology is proposed for the evaluation of multi-attribute qualitative alternatives based on the use of distances to a reference point. The use of the appropriate distance provides an intuitive notion of proximity between alternatives and consistently maintains the maximum order of magnitude label chains.

The presented methodology allows, on the one hand, the imprecise concepts of the specific application to be handled, and, on the other, the methods of "goal programming" to be generalized without the need for previous knowledge of the ideal goal.

Future work will be carried out into the efficiency of other qualitative distances in the methodology and the applicability to different domains. In addition, a software tool implementing the presented methodology is being developed as part of the AURA research project.

Acknowledgements

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