# A formal theory of qualitative size and distance relations between regions

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**Abstract**

We present a formal theory of qualitative distances between regions based on qualitative size relations. Using standard mereological relations, a sphere-predicate, and qualitative size relations such as roughly-the-same-size-as and negligible-in-size-with-respect-to, we define qualitative distance relations such as close-to, near-to, away-from, and far-away-from.

Relations such as roughly-the-same-size-as and negligible-in-size-with-respect-to are context-dependent and vague. The primary focus in the formal theory presented in this paper is on the *context-independent* logical properties of these sorts of qualitative size and distance relations. We are especially interested in how these relations interact with familiar mereological relations. In developing our formal theory, we draw upon work on order of magnitude reasoning in Artificial Intelligence.

## Introduction

Qualitative distance relations such as close-to, near-to, and far-away are important in geography (Tobler 1970), in Artificial Intelligence (Hernandez, Clementini, & Di Felice 1995; Clementini, Di Felice, & Hernández 1997; Davis 1989; 1999), spatial cognition (Talmy 1983; Herskowitz 1986), and other disciplines. Most attempts to formalize qualitative distance relations are based on the order of magnitude reasoning pioneered in (Raiman 1988; 1991; Mavrovouniotis & Stephanopoulos 1988; Dague 1993a; 1993b). Order of magnitude reasoning deals with qualitative relations between quantities, such as roughly-the-same-size-as and negligible-with-respect-to.

In this paper we present a mereological theory for domains of spatial regions and extend this theory by adding qualitative size relations and a 'sphere' predicate. In the resulting theory we are able to define qualitative distance relations such as close-to, near-to and far-away-from, etc. It is important for characterizing qualitative *distance* relations between spatially extended regions to take the *size* of the regions into account. Whether, for example, the relation near-to holds between regions $x$ and $y$ which are a fixed quantitative distance apart depends in part on the sizes of $x$ or $y$. For example, let $x$ be negligible in size with respect to $y$ and suppose that the least distance between points in $x$ and $y$ is very small with respect to the size of $y$ but large with respect to the size of $x$. Then $y$ may be near to $x$ (on $y$'s scale) but $x$ might not be near $y$ (on $x$'s scale). As pointed out in (Worboys 2001), in many cases utterances involving qualitative distance relations between extended objects can be understood only if the size of the objects is taken into account.

The theory presented in this paper combines a version of region-based qualitative geometry (RBG) (Tarski 1956; Borgo, Guarino, & Masolo 1996; Bennett *et al.* 2000) with work from order of magnitude reasoning, especially (Dague 1993b). It gives a detailed account of the logical properties of qualitative size and distance relations. We show that qualitative size relations (essentially ordering relations) can be defined within a mereological framework extended by the primitive relations same-size-as and roughly-the-same-size-as, while qualitative distance relations need to be defined within the stronger framework of region-based geometry.[1]

## Mereology

We present our formal theory of qualitative size and distance relations in a first-order predicate logic with identity. Variables range over regions of space. Spatial regions are here assumed to be parts of an independent background space in which all objects are located. On the intended interpretation, regions are the non-empty regular closed subsets of a three-dimensional Euclidean space.

We introduce the primitive binary predicate $P$, where $Pxy$ is interpreted as: $x$ is part of $y$. We define: $x$



[1] (Bennett 2002) sketches logical properties of region size measures within the framework of RBG by introducing the primitive 'sphere of insignificant size'.

Qualitative size relations are also discussed in (Gerevini & Renz 1998) in the context of a constraint based framework based on the RCC theory. The paper does not give an explicit axiomatization of relations such as roughly-the-same-size-as and negligible-with-respect-to. Neither does it consider qualitative distance relations.

*overlaps* $y$ if and only if there is a $z$ such that $z$ is part of both $x$ and $y$ ($D_O$); $x$ is a *proper part* of $y$ if and only if $x$ is a part of $y$ and $y$ is not a part of $x$ ($D_{PP}$); $z$ is the sum of $x$ and $y$ if and only if for all $w$, $w$ overlaps $z$ if and only if $w$ overlaps $x$ or $w$ overlaps $y$ ($D_+$); $z$ is the *difference* of $y$ in $x$ if and only if any region $w$ overlaps $z$ if and only if $w$ overlaps some part of $x$ that does not overlap $y$ ($D_-$).

$$\begin{array}{ll} D_O & O\ xy \equiv (\exists z)(P\ zx \wedge P\ zy) \\ D_{PP} & PP\ xy \equiv P\ xy \wedge \neg P\ yx \\ D_+ & +xyz \equiv (w)(Owz \leftrightarrow (O\ wx \vee O\ wy)) \\ D_- & -\ xyz \equiv (w)(O\ wz \leftrightarrow \\ & \qquad (\exists w_1)(P\ w_1x \wedge \neg O\ w_1y \wedge O\ w_1w)) \end{array}$$

We add the usual axioms of reflexivity (A1), antisymmetry (A2), and transitivity (A3). We also assume the following existence axioms: if $x$ is not a part of $y$ then there is a $z$ such that $z$ is a difference of $y$ in $x$ (A4), for any regions $x$ and $y$ there is a region $z$ that is the sum of $x$ and $y$ (A5).

$$\begin{array}{ll} A1 & P\ xx \\ A2 & P\ xy \wedge P\ yx \rightarrow x = y \\ A3 & P\ xy \wedge P\ yz \rightarrow P\ xz \\ A4 & \neg P\ xy \rightarrow (\exists z)(-\ xyz) \\ A5 & (\exists z)(+xyz) \end{array}$$

We can prove: $x$ and $y$ are identical if and only if they overlap exactly the same regions (T1). We can also prove that sums and differences are unique whenever they exist (T2-T3). Together, A4 and T2 ensure that summation is a functional operator.

$$\begin{array}{ll} T1 & x = y \leftrightarrow (z)(O\ zx \leftrightarrow O\ zy) \\ T2 & +\ xyz_1 \wedge +\ xyz_2 \rightarrow z_1 = z_2 \\ T3 & -\ xyz_1 \wedge -\ xyz_2 \rightarrow z_1 = z_2 \end{array}$$

EMR, extensional mereology for regions, is the theory axiomatized A1-A5 (Simons 1987; Varzi 1996).

## Ordering based on the exact size

In the next two sections, we present a modified version of our theory of granular parthood and qualitative cardinalities (Bittner & Donnelly 2006).[2]

We use $\|x\|$ in the meta-language to refer to the exact volume size of region $x$. In the formal theory we introduce the *same size* relation $\sim$ where, on the intended interpretation, $x \sim y$ holds if and only if $\|x\| = \|y\|$. We then define that the size of $x$ is *less than or equal* to the size of $y$ if and only if there is a region $z$ that is a part of $y$ and has the same size as $x$ ($D_\leq$).

$$D_\leq \quad x \leq y \equiv (\exists z)(z \sim x \wedge P\ zy)$$

On the intended interpretation, $x \leq y$ holds if and only if $\|x\|$ is less than or equal to $\|y\|$.

We require: $\sim$ is reflexive (A6); $\sim$ is symmetric (A7); $\sim$ is transitive (A8); if $x$ is part of $y$ and $x$ and $y$ have the same size then $y$ is part of $x$ (A9); for any $x$ and $y$, the size of $x$ is less than or equal to the size of $y$ or the size of $y$ is less than or equal to the size of $x$ (A10); if the size of $x$ is less than or equal to the size of $y$ and the size of $y$ is less than or equal to the size of $x$, then $x$ and $y$ have the same size (A11).

$$\begin{array}{ll} A6 & x \sim x \\ A7 & x \sim y \rightarrow y \sim x \\ A8 & x \sim y \wedge y \sim z \rightarrow x \sim z \\ A9 & P\ xy \wedge x \sim y \rightarrow P\ yx \\ A10 & x \leq y \vee y \leq x \\ A11 & x \leq y \wedge y \leq x \rightarrow x \sim y \end{array}$$

We can prove: if $x$ is identical to $y$, then $x$ and $y$ are of the same size (T4); if $x$ is part of $y$ and $y$ is part of $x$, then $x$ and $y$ have the same size (T5); if $x$ is part of $y$ and $x$ and $y$ have the same size then $x$ and $y$ are identical (T6); if $x$ is a part of $y$, then the size of $x$ is less than or equal to the size of $y$ (T7); $\leq$ is reflexive (T8); $\leq$ is transitive (T9); if the size of $x$ is less than or equal to the size of $y$ and $y$ and $z$ have the same size, then the size of $x$ is less than or equal to the size of $z$ (T10); if $z$ and $x$ have the same size and the size of $z$ is less than or equal to the size of $y$ then the size of $z$ is less than or equal to the size of $x$ (T11).

$$\begin{array}{ll} T4 & x = y \rightarrow x \sim y \\ T5 & P\ xy \wedge P\ yx \rightarrow x \sim y \\ T6 & P\ xy \wedge x \sim y \rightarrow x = y \\ T7 & P\ xy \rightarrow x \leq y \\ T8 & x \leq x \\ T9 & x \leq y \wedge y \leq z \rightarrow x \leq z \\ T10 & x \leq y \wedge y \sim z \rightarrow x \leq z \\ T11 & z \sim x \wedge x \leq y \rightarrow z \leq y \end{array}$$

Thus, $\sim$ is an equivalence relation, $\leq$ is reflexive and transitive, and $\sim$, $\leq$, $P$, and $=$ are logically interrelated in the expected ways.

## Roughly the same size, negligible in size

We introduce the relations *roughly the same size* ($\approx$) and *negligible in size* ($\ll$) as in (Bittner & Donnelly 2006). Let $\omega$ be a parameter such that $0 < \omega < 0.5$. On one possible class of interpretations, $x$ has *roughly same size as* $y$ if and only if $1/(1+\omega) \leq \|x\|/\|y\| \leq 1+\omega$. $x$ is a *negligible in size* with respect to $y$ if and only if $\|x\|/\|y\|$ is less than $\omega/(1+\omega)$.

Consider Figure 1. Values for the size of $x$ range along the positive horizontal axis and values for the size of $y$

[2] Axioms A6-A11 correspond to AC7-AC10,AC12,AC13 in (Bittner & Donnelly 2006) (BD06). The axioms are similar in structure. However in this paper we work in a mereological framework formalizing size relations between regions while in BD06 we worked in the framework of finite non-empty collections formalizing relations on cardinalities of collections. Similarly, A12-A16 correspond to AC14,AC15,AC17,AC18 in BD06. The axiom A14 of this paper is a theorem in BD06. Unlike in GP06 we do not include an axiom constraining the relationships between size relations the and summation operation. In BD06 granular parthood for material objects is formalized. The size relations discussed in the present paper hold between arbitrary regions and not only between parts.

range along the positive vertical axis. If $x$ and $y$ have the same size then $(\|x\|, \|y\|)$ represents a point on the dotted line. If $1/(1+\omega) \leq \|x\|/\|y\| \leq 1+\omega$ (i.e., $x$ has roughly the same size as $y$), then $(\|x\|, \|y\|)$ represents a point lying within the area delimited by the dashed lines. If $\|x\|/\|y\|$ is smaller than $\omega/(1+\omega)$ (i.e., $x$ is negligible with respect to $y$), then $(\|x\|, \|y\|)$ represents a point lying between the positive vertical axis and the solid diagonal line.

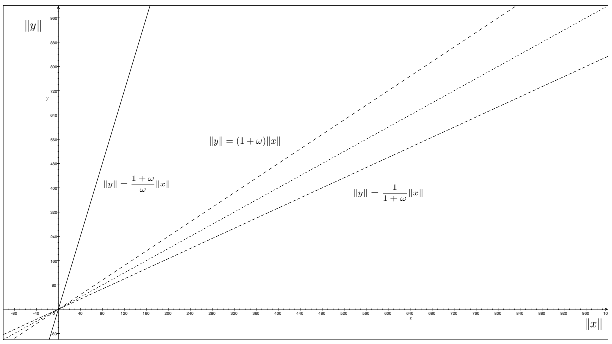


Figure 1: Graph for $\omega = 0.2$

Now consider a fixed region $y$ and imagine that different values of $\omega$ are appropriate for different contexts. The smaller the value of $\omega$, the smaller the value of $|\|x\| - \|y\||$ must be for $x$ to count as close in size to $y$ and the smaller $\|x\|$ must be for $x$ to count as negligible in size with respect to $y$. To picture this situation graphically: the smaller the value of $\omega$, the narrower the corridor between the dashed diagonal lines in Figure 1 and also the narrower the corridor between the solid diagonal line and the positive vertical axis.

We require: $\approx$ is reflexive (A12); $\approx$ is symmetric (A13); if $x$ and $y$ have roughly the same size and $y$ and $z$ have the same size, then $x$ and $z$ have roughly the same size (A14); if $x$ and $y$ have roughly the same size and $x$ is a part of $z$ and $z$ is a part of $y$, then $z$ and $x$, as well as $z$ and $y$, have roughly the same size (A15).

$$
\begin{array}{ll}
A12 & x \approx x \\
A13 & x \approx y \rightarrow y \approx x \\
A14 & x \approx y \wedge y \sim z \rightarrow x \approx z \\
A15 & x \approx y \wedge P\ xz \wedge P\ zy \rightarrow (z \approx x \wedge z \approx y)
\end{array}
$$

Notice that unlike (Raiman 1991) and (Dague 1993b) we do not require $\approx$ to be transitive. In many of the intended models of our theory, it is possible to find regions $z_1, \ldots, z_n$ such that $x \approx z_1$, $z_1 \approx z_2$, ... and $z_n \approx y$ and but NOT $x \approx y$. Hence, adding a transitivity axiom for $\approx$ would give rise to a version of the Sorites paradox (Hyde 1996; van Deemter 1995).

We can prove: if $x$ and $y$ have the same size and $y$ and $z$ have roughly the same size, then $x$ and $z$ have roughly the same size (T12); if $x$ and $y$ have the same size, then $x$ and $y$ have roughly the same size (T13).

$$
\begin{array}{ll}
T12 & x \sim y \wedge y \approx z \rightarrow x \approx z \\
T13 & x \sim y \rightarrow x \approx y
\end{array}
$$

Region $x$ is *negligible in size with respect to* region $y$ if and only if there are regions $z_1$ and $z_2$ such that (i) $x$ and $z_1$ have the same size, (ii) $z_1$ is a part of $y$, (iii) $z_2$ is the difference of $z_1$ in $y$ and (iii) $z_2$ and $y$ have roughly the same size ($D_{\ll}$).

$$
\begin{array}{ll}
D_{\ll} & x \ll y \equiv (\exists z_1)(\exists z_2)(z_1 \sim x \wedge P\ z_1 y \wedge \\
& \qquad -y z_1 z_2 \wedge z_2 \approx y)
\end{array}
$$

As pointed out above, when $\approx$ is interpreted so that $z \approx y$ holds if and only if $1/(1+\omega) \leq \|z\|/\|y\| \leq 1+\omega$, then $x \ll y$ holds if and only if $\|x\|/\|y\|$ is smaller than $\omega/(1+\omega)$.

We require that if $x$ is negligible with respect to $y$ and the size of $y$ is less than or equal to the size of $z$, then $x$ is negligible with respect to $z$ (A16).

$$
A16 \quad x \ll y \wedge y \leq z \rightarrow x \ll z
$$

We can prove: if $x$ is negligible with respect to $y$, then $x$ is smaller than $y$ (T14); if the size of $x$ is less than or equal to the size of $y$ and $y$ is negligible with respect to $z$, then $x$ is negligible with respect to $z$ (T15); if $x$ is a part of $y$ and $y$ is negligible with respect to $z$, then $x$ is negligible with respect to $z$ (T16); if $x$ is negligible with respect to $y$ and $y$ is part of $z$, then $x$ is negligible with respect to $z$ (T17); $\ll$ is transitive (T18).

$$
\begin{array}{ll}
T14 & x \ll y \rightarrow (x \leq y \wedge x \not\approx y) \\
T15 & x \leq y \wedge y \ll z \rightarrow x \ll z \\
T16 & P\ xy \wedge y \ll z \rightarrow x \ll z \\
T17 & x \ll y \wedge P\ yz \rightarrow x \ll z \\
T18 & x \ll y \wedge y \ll z \rightarrow x \ll z
\end{array}
$$

Thus, the relation negligible-in-size-with-respect-to has the expected logical properties. We call the theory, which extends EMR by axioms A6-A16, QSizeR.

## Spheres and connectedness

We introduce the primitive predicate $S$ where $S\ x$ is interpreted as $x$ is a sphere. We define: $x$ is *maximal with respect to $y$ in $z$* if and only if (i) $x$, $y$, and $z$ are spheres, (ii) $x$ and $y$ are non-overlapping parts of $z$, and (iii) every sphere $u$ that has $x$ as a part either is identical to $x$, overlaps $y$, or is not a part of $z$ ($D_{Mx}$). $x$ is a *concentric* proper part of $y$ if and only if (i) $x$ and $y$ are spheres, (ii) $x$ is a proper part $y$ and (iii) all spheres that are maximal with respect to $x$ in $y$ have the same size ($D_{CoPP}$).

$$
\begin{array}{ll}
D_{Mx} & Mx\ xyz \equiv S\ x \wedge S\ y \wedge S\ z \wedge \\
& \quad P\ xz \wedge P\ yz \wedge \neg O\ xy \wedge \\
& \quad (u)(S\ u \wedge P\ xu \rightarrow (x = u \vee O\ uy \vee \neg Puz) \\
D_{CoPP} & CoPP\ xy \equiv S\ x \wedge S\ y \wedge PP\ xy \wedge \\
& \quad (u)(v)(Mx\ uxy \wedge Mx\ vxy \rightarrow u \sim v))
\end{array}
$$

We require that the following spheres exist: Every region has a sphere as a part (A17). Every sphere has a concentric proper part (A18). If sphere $x$ is a proper part of sphere $y$ then there is a sphere $z$ that is maximal with respect to $x$ in $y$ (A19).

$$
\begin{array}{ll}
A17 & (\exists z)(S\ z \wedge P\ zx) \\
A18 & S\ x \rightarrow (\exists y)(S\ y \wedge CoPP\ yx) \\
A19 & S\ x \wedge S\ y \wedge PP\ xy \rightarrow (\exists z)(Mx\ zxy)
\end{array}
$$

Similar to (Bennett *et al.* 2000) we then define that two regions $x$ and $y$ are *connected* if and only if there is a sphere $z$ that overlap $x$ and $y$ and all spheres that are concentric proper parts of $z$ also overlap $x$ and $y$ ($D_C$).

$$D_C \quad C\ xy \equiv (\exists z)(S\ z \wedge O\ zx \wedge O\ zy \wedge (u)(CoPP\ uz \rightarrow (O\ ux \wedge O\ uy))$$

On the intended interpretation, the connection relation $C$ holds between regions $x$ and $y$ if and only if the distance between them is zero (where the distance between regions is here understood as the greatest lower bound of the distance between any point of the first region and any point of the second region).

We can prove that $C$ is reflexive (T18a), symmetric (T18b), and that if $x$ is part of $y$, then everything connected to $x$ is connected to $y$ (T18c).

$$T18a \quad C\ xx \qquad T18b \quad C\ xy \rightarrow C\ yx$$
$$T18c \quad P\ xy \rightarrow (z)(Czx \rightarrow Czy)$$

We call the theory formed by axioms A1-A11 and A17-A19 *region-connection geometry* RCG.

## Qualitative distance relations

We now use the sphere primitive, the connectedness relation, and the qualitative *size* relations of QSizeR to define qualitative *distance* relations such as close-to, near-to, and away-from.

Region $x$ is *close* to region $y$ if and only if either $x$ and $y$ are connected or there is a sphere $z$ such that $z$ is connected to both $x$ and $y$ and $z$ is negligible in size with respect to $x$ ($D_{Cl}$). $x$ is *strictly close* to $y$ if and only if $x$ is close to $y$ but not connected to $y$ ($D_{SN}$). $x$ is *near* to $y$ if and only if either $x$ and $y$ are connected or there is a sphere $z$ such that $z$ is connected to $x$ and $y$ and the size of $z$ is less than or equal to the size of $x$ ($D_N$). $x$ is *strictly near* to $y$ if and only if $x$ is near to $y$ but not close to $y$ ($D_{SN}$). $x$ is *away* from $y$ if and only if $x$ is not near to $y$ ($D_A$). $x$ is *far away* from $y$ if and only if (i) $x$ and $y$ are not connected and (ii) there is a sphere $z$ such that $z$ is connected to $x$ and $y$, and (iii) $x$ is negligible in size with respect to all spheres $w$ that are connected to $x$ and $y$ ($D_{FA}$). $x$ is *moderately away* from $y$ if and only if $x$ is away from $y$ but not far away from $y$ ($D_{MA}$).[3]

$$D_{Cl} \quad Cl\ xy \equiv C\ xy \vee (\exists z)(S\ z \wedge C\ zx \wedge C\ zy \wedge z \ll x)$$
$$D_{SCl} \quad SCl\ xy \equiv Cl\ xy \wedge \neg C\ xy$$
$$D_N \quad N\ xy \equiv C\ xy \vee (\exists z)(S\ z \wedge C\ zx \wedge C\ zy \wedge z \leq x)$$
$$D_{SN} \quad SN\ xy \equiv N\ xy \wedge \neg Cl\ xy$$
$$D_A \quad A\ xy \equiv \neg N\ xy$$
$$D_{FA} \quad FA\ xy \equiv \neg C\ xy \wedge (\exists z)(S\ z \wedge C\ zx \wedge C\ zy) \wedge (w)(S\ w \wedge C\ wx \wedge C\ wy \rightarrow x \ll w)$$
$$D_{MA} \quad MA\ xy \equiv A\ xy \wedge \neg FA\ xy$$

[3]Notice that, unlike the other distance relations, $N$ and $A$ are crisp, i.e., their interpretations do not depend on $\omega$ (See also Table 1). A possible definition that takes the vagueness of 'near' better into account may be $N'\ xy \equiv C\ xy \vee (\exists z)(S\ z \wedge C\ zx \wedge C\ zy \wedge z \approx x)$.

Let $d(x, y)$ be the greatest lower bound of the distance between any point of $x$ and any point of $y$ and let $d_{\|x\|}$ be the diameter of a sphere of size $\|x\|$. When $\approx$ is interpreted so that $z \approx y$ holds if and only if $1/(1+\omega) \leq \|z\|/\|y\| \leq 1 + \omega$, then the distance relations defined above hold for the distance ranges specified in Table 1.

| Relation | holds for distance ranges |
|---|---|
| $Cl\ xy$ | $0 \leq d(x, y) \leq (\omega * d_{\|x\|})/(1 + \omega)$ |
| $SCl\ xy$ | $0 < d(x, y) \leq (\omega * d_{\|x\|})/(1 + \omega)$ |
| $N\ xy$ | $0 \leq d(x, y) \leq d_{\|x\|}$ |
| $SN\ xy$ | $(\omega * d_{\|x\|})/(1 + \omega) < d(x, y) \leq d_{\|x\|}$ |
| $A\ xy$ | $d_{\|x\|} < d(x, y)$ |
| $FA\ xy$ | $(d_{\|x\|} * (1 + \omega))/\omega < d(x, y)$ |
| $MA\ xy$ | $d_{\|x\|} < d(x, y) \leq (d_{\|x\|} * (1 + \omega))/\omega$ |

Table 1: Distance ranges for which the qualitative distance relations hold on the intended interpretation in context $\omega$. (In this table $<$ and $\leq$ refer to the total (strict) ordering on the real numbers.)

Consider Figure 2. In the center of the concentric circles there is the circle-shaped region $x$ of size $\|x\|$ and radius $r(x)$. In the center of $x$ is the origin of our coordinate system. Using our qualitative distance relations we can identify the following nested ring structure around $x$ for every context $\omega$: The relation strictly-close holds between $x$ and any region $y$ which has points in the *SCl*-ring (the ring between $r(x)$ and $r(x)+(\omega * d_{\|x\|})/(1+\omega)$ excluding the boundary $r(x)$).

The relation strictly-near holds between $x$ and any region $y$ which has points in the *SN*-ring (in the ring between $r(x) + (\omega * d_{\|x\|})/(1 + \omega)$ and $r(x) + d_{\|x\|}$, excluding the boundary $r(x) + (\omega * d_{\|x\|})/(1 + \omega)$).

The relation moderately-far-away holds between $x$ and any region $y$ which has points in the *MA*-ring (the ring between $r(x) + d_{\|x\|}$ and $r(x) + (d_{\|x\|} * (1 + \omega))/\omega$, excluding the boundary $r(x) + d_{\|x\|}$).

The relation far-away holds between $x$ and any region $y$ which has all points in the *FA*-ring (outside the circle with radius $r(x) + (d_{\|x\|} * (1 + \omega))/\omega$).

The following theorems are immediate consequences of our definitions: $x$ is close to $y$ if and only if $x$ is connected to $y$ or $x$ is strictly close to $y$ (T19); if $x$ and $y$ are connected then $x$ and $y$ are not strictly close (T20); $x$ is near to $y$ if and only if $x$ is close to $y$ or $x$ is strictly near to $y$ (T21); if $x$ and $y$ are close then $x$ and $y$ are not strictly near (T22); $x$ is away from $y$ if and only if $x$ moderately away from $y$ or $x$ is far away from $y$ (T23); if $x$ and $y$ are moderately away then $x$ and $y$ are not far away (T24).

$$T19 \quad Cl\ xy \leftrightarrow (C\ xy \vee SCl\ xy)$$
$$T20 \quad C\ xy \rightarrow \neg SCl\ xy$$
$$T21 \quad N\ xy \leftrightarrow (Cl\ xy \vee SN\ xy)$$
$$T22 \quad Cl\ xy \rightarrow \neg SN\ xy$$
$$T23 \quad A\ xy \leftrightarrow (MA\ xy \vee FA\ xy)$$
$$T24 \quad MA\ xy \rightarrow \neg FA\ xy$$

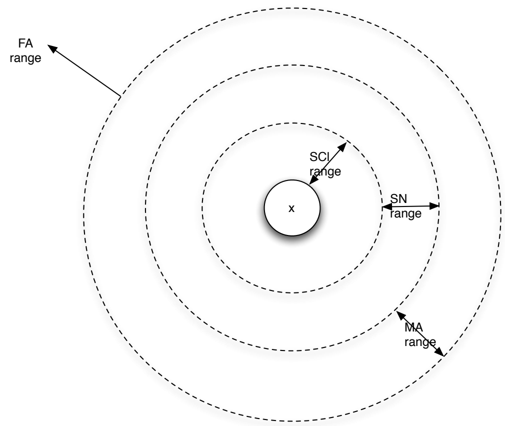


Figure 2: Qualitative distance diagram for a circular region $x$.

The implication hierarchy and the sets of jointly exhaustive and pair-wise disjoint relations which follow from these theorems are pictured graphically in Figure 3.

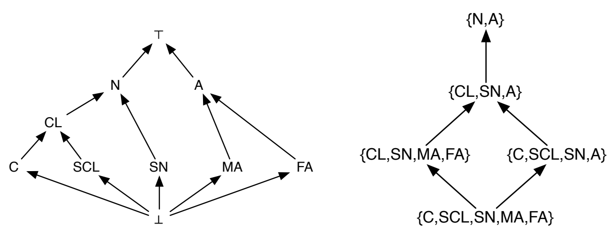


Figure 3: Implication hierarchy of the qualitative distance relations (left). Sets of JEPD qualitative distance relations (right).

We can also prove that $Cl$ and $N$ are reflexive and that $SCl$, $SN$, $A$, $MA$, and $FA$ are irreflexive.

Notice that NONE of the defined distance relations is symmetric. For example, a road-sized region may be (on the scale of the road) close to a pebble-sized region in an adjacent ditch, even if the pebble-sized region is not (on the scale of the pebble) close to the road-sized region. However we can prove: if the size of $x$ is less than or equal to $y$ and $x$ is close to $y$ then $y$ is also close to $x$ (T25); if the size of $x$ is less than or equal to $y$ and $x$ is strictly close to $y$ then $y$ is strictly close to $x$ (T26); if the size of $x$ is less than or equal to $y$ and $x$ is near to $y$ then $y$ is near to $x$ (T27); if the size of $x$ is less than or equal to $y$ and $x$ is strictly near to $y$ then $y$ is near to $x$ (T28); if the size of $x$ is equal to $y$ and $x$ is strictly near to $y$ then $y$ is strictly near to $x$ (T29); if the size of $y$ is less than or equal to $x$ and $x$ and $y$ are away from one another then $y$ and $x$ are away from one another (T30); if the size of $y$ is less than or equal to $x$ and $x$ and $y$ are far away then $y$ and $x$ are far away (T31); if the size of $y$ is less than or equal to $x$ and $x$ and $y$ are moderately away then $y$ and $x$ are away (T32); if the size of $y$ is equal to the size of $x$ and $x$ and $y$ are moderately away then $y$ and $x$ are moderately away (T33).

$$\begin{array}{ll}
T25 & x \leq y \wedge Cl\,xy \rightarrow Cl\,yx \\
T26 & x \leq y \wedge SCl\,xy \rightarrow SCl\,yx \\
T27 & x \leq y \wedge N\,xy \rightarrow N\,yx \\
T28 & x \leq y \wedge SN\,xy \rightarrow N\,yx \\
T29 & x \sim y \wedge SN\,xy \rightarrow SN\,yx \\
T30 & y \leq x \wedge A\,xy \rightarrow A\,yx \\
T31 & y \leq x \wedge FA\,xy \rightarrow FA\,yx \\
T32 & y \leq x \wedge MA\,xy \rightarrow A\,yx \\
T33 & x \sim y \wedge MA\,xy \rightarrow MA\,yx
\end{array}$$

Theorems T25-T33 reflect the logical interrelationships between the qualitative distance relations and the relative size of the regions involved.

We can also prove the following theorems about logical interrelationships between parthood and the various qualitative distance relations: if $x$ and $y$ are close and $z$ has $y$ as a part then $x$ and $z$ are close (T34); if $x$ and $y$ are near and $z$ has $y$ as part then $x$ and $z$ are near (T35); if $x$ is a part of $y$ and $y$ and $z$ are away then $x$ and $z$ are away (T36).

$$\begin{array}{ll}
T34 & Cl\,xy \wedge P\,yz \rightarrow Cl\,xz \\
T35 & N\,xy \wedge P\,yz \rightarrow N\,xz \\
T36 & P\,xy \wedge A\,yz \rightarrow A\,xz
\end{array}$$

We call the theory which extends QSizeR and RCG by the definitions for qualitative distance relations QDistR.

## Conclusions

We have presented an axiomatic theory of qualitative size and distance relations between regions. The theory is based on the formal characterization of the primitive predicates and relations: part-of ($P$), sphere ($S$), exactly-the-same-size ($\sim$), and roughly-the-same-size ($\approx$). In our theory, we are able to formally distinguish: i) regions that are negligible in size with respect to one another, ii) regions that are close, near, far away, etc. We thereby extend existing work on mereo-geometries and order of magnitude reasoning.

The axiomatic theory presented in this paper is part of the top-level ontology 'Basic Formal Ontology' (BFO). BFO is developed using *Isabelle*, a computational system for implementing logical formalisms (Nipkow, Paulson, & Wenzel 2002). The computational representation of BFO consists of several hierarchically organized sub-theories. An automatically generated WEB presentation of the theory containing all axioms, definitions, theorems, and the computer-verified proofs can be accessed at `http://www.ifomis.org/bfo/fol`.

Relations such as roughly-the-same-size-as, negligible-in-size-with-respect-to, close-to, far-away, etc, are *context-dependent* and *vague*. Context is represented abstractly in numerical parameters which determine the canonical interpretations of the qualitative size and distance relations of the formal theory.

Although the canonical models use precise numerical parameters for fixing the interpretations, it is not expected that precise numerical parameters are fixed in actual practical contexts. Since the qualitative size and distance relations are vague, in many cases (at best) we can associate contexts demanding high precision with a different range of numerical parameters than contexts requiring only loose precision.

Since the logical properties of the relations of our theory are valid over a range of numerical parameters, the formal theory can be used for reasoning even where qualitative size and distance relations lack precise numerical definitions. Thus the primary focus in the formal theory presented in this paper is on the *context-independent logical* properties of these sorts of qualitative size and distance relations and the logical interrelations among one another and the mereotopological relations.

## References


Bennett, B.; Cohn, A. G.; Torrini, P.; and Hazarika, S. M. 2000. A foundation for region-based qualitative geometry. In Horn, W., ed., *Proceedings of ECAI-2000*, 204–208.

Bennett, B. 2002. Physical objects, identity and vagueness. In Fensel, D.; McGuinness, D.; and Williams, M.-A., eds., *Principles of Knowledge Representation and Reasoning: Proceedings of the Eighth International Conference (KR2002)*. San Francisco, CA: Morgan Kaufmann.

Bittner, T., and Donnelly, M. 2006. A theory of granular parthood based on qualitative cardinality and size measures. In Bennett, B., and Fellbaum, C., eds., *Proc. 4th Int. Conference on Formal Ontology in Information Systems*, 65–76.

Borgo, S.; Guarino, N.; and Masolo, C. 1996. A pointless theory of space based on strong connection and congruence. In Aiello, L. C.; Doyle, J.; and Shapiro, S., eds., *Principles of Knowledge Representation and Reasoning (KR96)*, 220–229. Morgan Kaufmann.

Clementini, E.; Di Felice, P.; and Hernández, D. 1997. Qualitative representation of positional information. *Artificial Intelligence* 95(2):317–356.

Dague, P. 1993a. Numeric reasoning with relative orders of magnitude. In *Proceedings of the National Conference on Artificial Intelligence*, 541–547.

Dague, P. 1993b. Symbolic reasoning with relative orders of magnitude. In *Proc. 13th Intl. Joint Conference on Artificial Intelligence*, 1509–1515. Morgan Kaufmann.

Davis, E. 1989. Order of magnitude reasoning in qualitative differential equations. In de Kleer, J., and Weld, D., eds., *Readings in Qualitative Physical Reasoning*. Morgan Kaufmann. 424–434.

Davis, E. 1999. Order of magnitude comparisons of distance. *Journal of AI Research* 10:1–38.

Gerevini, A., and Renz, J. 1998. Combining topological and qualitative size constraints for spatial reasoning. In *Proceedings of the 4th international conference on principles and practice of constraint programming (CP'98)*. Berlin: Springer-Verlag.

Hernandez, D.; Clementini, E.; and Di Felice, P. 1995. Qualitative distances. In Frank, A. U., and Kuhn, W., eds., *Spatial Information Theory, A Theoretical Basis for GIS*, LNCS. Semmering, Austria: Springer-Verlag.

Herskowitz, A. 1986. *Language and Spatial Cognition - An Interdisciplinary Study of the Propositions in English*. Studies in natural language processing. Cambridge University Press.

Hyde, D. 1996. Sorites paradox. In *Stanford Encyclopedia of Philosophy*.

Mavrovouniotis, M., and Stephanopoulos, G. 1988. Formal order-of-magnitude reasoning in process engineering. *Computers and Chemical Engineering* 12:867–881.

Nipkow, T.; Paulson, L. C.; and Wenzel, M. 2002. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*, volume 2283 of *LNCS*. Springer.

Raiman, O. 1988. Order of magnitude reasoning. In Weld, D., and de Kleer, J., eds., *Readings in Qualitative Reasoning about Physical Systems*. San Mateo, Calefornia: Morgan Kaufmann Publishers, INC.

Raiman, O. 1991. Order of magnitude reasoning. *Artificial Intelligence* 51:11–38.

Simons, P. 1987. *Parts, A Study in Ontology*. Oxford: Clarendon Press.

Talmy, L. 1983. How language structures space. In Pick, H., and Acredolo, L., eds., *Spatial Orientation: Theory, Research, and Application*. New York, NY: Plenum Press.

Tarski, A. 1956. Foundations of the geometry of solids. In *Logic, Semantic, Metamathematics*. Oxford Clarendon Press.

Tobler, W. 1970. A computer movie simulating urban growth in the detroit region. *Economic Geography* 46(2):234–240.

van Deemter, K. 1995. The sorites fallacy and the context-dependence of vague predicates. In Kanazawa, M.; Pinon, C.; and de Swart, H., eds., *Quantifiers, Deduction, and Context*. CSLI Publications, Stanford, CA. 59–86.

Varzi, A. 1996. Parts, wholes, and part-whole relations: The prospects of mereotopology. *Data and Knowledge Engineering* 20(3):259–86.

Worboys, M. F. 2001. Nearness relations in environmental space. *International Journal of Geographical Information Science* 15(7):633–651.