A Common Framework for Qualitative and Quantitative Modelling

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Abstract

The paper introduces model ensembles as a common framework for understanding qualitative differential equations (QDEs) and differential inclusions in a precise mathematical sense. It provides basic insights into the communalities and differences of both approaches to model under uncertainty. On this basis, a set of established methods for QDEs, some hybrid methods and standard quantitative methods can be classified, the notion of a "spurious behaviour" is clarified more thoroughly, and the importance of generality as a concept complementary to uncertainty is underpinned. Further paths for extending qualitative reasoning are outlined.

Introduction

Although much progress has been achieved in integrating qualitative differential equations (QDEs) with quantitative knowledge (Kay & Kuipers 1993; Kuipers 1994; Kay 1998; Berleant & Kuipers 1997), hybrid systems that combine different modelling approaches and types of knowledge from a coherent framework are still urgently needed (Travé-Massuyès, Ironi, & Dague 2004; Price *et al.* 2005). There is no common mathematical theory to my knowledge that describes, e.g. qualitative, semi-qualitative, set-valued, interval-based and order of magnitude reasoning. Relevant approaches as those of Bradley *et al.* (2001) integrate various reasoning techniques hierarchically in a more pragmatic way.

Up to now, it has been an open issue whether QDEs and differential inclusions (DIs) are essentially the same way of representing uncertainty (Kuipers 2000; Saint-Pierre 2004), although methods as Q3 closely resemble the numerical analysis of differential inclusions by considering numerical envelopes on functions and landmarks in an efficient way (Berleant & Kuipers 1997). DIs represent a similar approach to account for uncertainty (Aubin & Cellina 1984), since contingent dynamics can be computed even if no probabilistic knowledge is available. The question is whether QDEs can be mathematically described by differential inclusions, which would provide a valueable bridge between set-valued analysis and computer sience. The works of Dordan (1992; 1995), Aubin (1996), and Hüllermeyer (1997) make considerable steps in that direction. Based on an ordinary differential equation $\dot{x} = f(x)$ they introduce the concept of a monotonic cell, consisting of all states x such that $f(x)$ has a given sign vector. A trajectory can be described qualitatively by the sequence of the monotonic cells it visits. By imposing additional restrictions on f, these authors investigate the issue of the existence of solutions more thoroughly than in the literature from computer science. They also generalise the approach to other partitions of the state space than by signs, called qualitative frames. However, the approach is more restrictive in that only single ordinary differential equations (ODEs) are considered. This is interesting in itself, but not sufficient for many applications of QDEs where uncertainties have to be taken into account.

Another issue which seems unrelated at the first glance is the necessary existence of spurious behaviour (Say & Akin 2002). A spurious behaviour doesn't match the quantitative solution of an ODE covered by the simulation input. Such behaviour is traced back (i) to the impossibility to represent certain types of irrationals, (ii) to the well-known ambiguities of sign algebra, and (iii) generally to the incompleteness of the information about a system that is modelled as a QDE. It is yet unclear what kind of application-oriented models could bring about such paths in a relevant way. Answering such questions requires a precise notion of spurious behaviour that can ideally be linked to established mathematical theory.

This paper formalizes basic ideas about "incomplete knowledge" in a precise sense to clarify the discussion by introducing the general framework of *model ensembles* which includes ODEs, QDEs, differential inclusions, causal loop diagrams and further methods as special cases. Thus the relation between QDEs and differential inclusions can be clarified. Technically, a model ensemble is a (possibly infinite) set $\mathcal{M}$ of functions, where each $f \in \mathcal{M}$ constitutes an ordinary differential equation $\dot{x} = f(x, t)$. By considering not a single model but a whole ensemble of models, a variety of possible system configurations which we can think of under uncertainty is covered. Although not systematized as here, such a style of reasoning is also common, e.g. for param-

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eter variation (e.g. Stainforth *et al.* 2005), model comparison (e.g. Gregory *et al.* 2005), and scenario development (e.g. Nakićenović *et al.* 2000; Millennium Ecosystem Assessement 2005; Swart, Raskin, & Robinson 2004). These basic ingredients are not new, but to my knowledge were never published. Based on this it is shown for a large set of qualitative models that no path of length 2 in an envisionment graph is spurious. At the same time, the definitions contribute to specify new hybrid qualitative-quantitative approaches.

Although a theoretical paper with a broader scope, this work is motivated by applications from sustainability science. The research about sustainable development aims to meet current human needs while maintaining the environment and natural resources for future generations (WCED 1987). In this domain uncertainties about dynamic social-ecological systems pose major challenges, and typologies of such systems and to understand so-called syndromes of global environmental change (Schellnhuber, Lüdeke, & Petschel-Held 2002; Petschel-Held 2005) are an important research field. In this context QDEs are a very valuable tool to analyse causal loop diagrams (Forrester 1968; Sterman 2000) and to deal with uncertainty, generality and non-quantitative knowledge (Petschel-Held *et al.* 1999; Stave 2002).

In the next section, the framework of model ensembles if formally introduced and illustrated with some more conventional examples. Then, basic defintions of QDEs are reformulated using graph theoretical concepts and the framework, including a discussion of spurious behaviour. Subsequently, differential inclusions are recalled and formulated as model ensembles, allowing for a thorough comparison with QDEs. Before concluding, further implications are discussed.

Model Ensembles

A **model ensemble** $\mathcal{M}$ is defined as a set of functions $f : X \times \mathbb{R}_+ \to \mathbb{R}^n$ on a **state space** $X \subseteq \mathbb{R}^n$. These functions are called **models**. In the case of uncertainties, each describes a possible configuration of a real-world system which must be considered. The set $\mathcal{E}$ contains functions $x(\cdot) : \mathbb{R}_+ \to X$, being the space of **admissible trajectories** of the systems, e.g. $\mathcal{E} = C^1(\mathbb{R}_+, X)$. Each model $f \in \mathcal{M}$ defines a family of initial value problems

$$\dot{x} = f(x,t),$$
$$x(0) = x_0,$$

with $x_0 \in X$. It is also possible to consider model ensembles which only contain autonomous models.

Of course, the systems of the model ensemble have (in general) different solutions. Thus, a *set* of trajectories must be assigned to each initial value x_0. The set-valued **solution operator** $\mathcal{S}_\mathcal{M}(\cdot) : X \to \mathcal{P}(\mathcal{E})$ (of a model ensemble $\mathcal{M}$ with respect to a state space X and admissible trajectories $\mathcal{E}$), assigning to an initial state a subset of $\mathcal{E}$, is defined by

$$\begin{aligned}\mathcal{S}_\mathcal{M}(x_0) := \{x(\cdot) \in \mathcal{E} \mid \\ x(0) = x_0, \\ \exists f \in \mathcal{M} \forall t \in \mathbb{R}_+ : \dot{x}(t) = f(x(t),t)\}.\end{aligned}$$

Depending on $\mathcal{E}$ it may be sufficient that the ODE only holds almost everywhere. The solution operator is written with the initial state x_0 as argument to investigate how properties of the solutions change in different subsets of the state space (see the section on further applications below and Eisenack (2006)). If we are interested in all possible initial states, we take the whole state space X as argument and call the elements of $\mathcal{S}_\mathcal{M}(X)$ the **solutions of the model ensemble** $\mathcal{M}$. The solution operator and the way we denote it is also resembles the concept of an evolutionary system as defined by Aubin (2001). The main challenge in reasoning with model ensembles is to find relevant structure in $\mathcal{S}_\mathcal{M}(X)$. This includes

1. representing a model ensemble in a way which is adequate to the modeller and allows for a formal treatment,
2. efficient algorithms to determine $\mathcal{S}_\mathcal{M}(X)$ from a (possibly infinite) model ensemble $\mathcal{M}$,
3. detecting structural features of the solutions of the model ensemble.

We now provide some examples for model ensembles.

EXAMPLE 1: Let $\mathcal{M}$ contain only one function $f : X \times \mathbb{R}_+ \to \mathbb{R}^n$ which is Lipschitz on X, and let the admissible trajectories be $\mathcal{E} = C^1(\mathbb{R}_+, X)$. Then, $\mathcal{S}_\mathcal{M}(x_0)$ contains the usual solutions of the initial value problem with $x(0) = x_0$ which exist on $\mathbb{R}_+$. □

EXAMPLE 2: Given a function $f' : X \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n, (x,t;p) \mapsto f'(x,t;p)$, depending on a parameter vector p, and a finite set P of possible parameterisations, define the finite model ensemble

$$\mathcal{M} := \{f \in C(X \times \mathbb{R}_+, \mathbb{R}^n) \mid f(x,t) = f'(x,t;p), p \in P\}.$$

Then, the solution operator with respect to a set of admissible trajectories provides all "scenario runs" for the different parameterisations. □

EXAMPLE 3: A causal loop diagram (in its simplest form) is a directed graph with marked edges. Each vertex represents a variable, and each edge an influence of the source variable on the target variable which can be marked as positive or as negative. In traditional systems dynamics modelling (Forrester 1968; Sterman 2000), the causal loop diagram is a starting point to develop a quantitative model, usually in the form of an ordinary differential equation (ODE). Since the diagram only contains qualitative information, there is an infinite number of such ODEs for a given diagram. For example (cf. Richardson 1986), $\mathcal{M}$ can be defined as the set of all ODEs $\dot{x} = f(x)$ with state vector $x \in \mathbb{R}^n$ for which the signs of the partial derivatives $[D_j f_i(x)]$ correspond to the signs of the edges ($D_j f_i$ denoting the partial derivative of the ith component of f with respect to x_j). If there is no edge between two variables, the partial derivative vanishes. □

In the next sections, QDEs and DIs are introduced as model ensembles.

QDEs as Model Ensembles

For the sake of simplicity I introduce a simplified version of QDEs that only considers monotonic influences on the change of variables. For this model class only the velocity, but not the state space needs to be investigated. The extension to complete QDEs with landmarks is straightforward but very technical insight (see Eisenack 2006 for a treatment). There is one other difference to the original work of Kuipers (1994): the focus is on time-interval states, while time-point states are not represented explicitly. This has the advantage that solutions of QDEs can be displayed in a much more accessible form (Eisenack & Petschel-Held 2002).

At first we specify the kind of model ensemble which constitutes a QDE. By $\mathcal{A} := \{[+], 0, [-]\}$ we denote the domain of signs, and by $\mathcal{A}_* := \{[+], 0, [-], [?]\}$ the domain of extended signs. Qualitative equality is denoted by $\approx$. We will aditionally use tuples and matrices of (extended) signs, and extend the sign operator $[\cdot]$ and qualitative equivalence component wise. Now a model ensemble can be defined:

DEFINITION 1: *For a given $n \times n$ matrix of signs $\Sigma = (\sigma_{i,j})_{i,j=1,\ldots,n}$, $\sigma_{i,j} \in \mathcal{A}_*$, and a state space $X \subseteq \mathbb{R}^n$ we define the* ***monotonic ensemble***

$$\mathcal{M}(\Sigma) := \{f \in C^1(X, \mathbb{R}^n) \mid \forall x \in X : \big[\mathcal{J}(f)(x)\big] \approx \Sigma\},$$

where $\mathcal{J}(f)$ denotes the Jacobian of f. We call a function $x(\cdot) \in C^1([0,T], \mathbb{R}^n)$, possibly $T = \infty$, ***reasonable*** *under the usual conditions, and define the space of admissible trajectories $\mathcal{E}$ by all reasonable functions with values in X. We call the systems of the model ensemble $\mathcal{M}(\Sigma)$ a* ***QDE****.*

A monotonic ensemble $\mathcal{M}(\Sigma)$ is a model ensemble which only contains autonomous models. Although a set of ODE systems is not an equation we use this designation in analogy to Kuipers (1994). One reason of the original terminology might be that a QDE can be "solved" by considering a constraint satisfaction problem, i.e. a relational equation over a finite set.

Based on DEF. 1, a set-valued solution operator $\mathcal{S}_{\mathcal{M}(\Sigma)}(\cdot)$ is defined. The set of solutions of the monotonic ensemble $\mathcal{S}_{\mathcal{M}(\Sigma)}(X)$ contains all reasonable solutions of all ODE systems contained in the QDE. It should be noted that the properties of the monotonic ensemble are not sufficient to guarantee a global solution for every $f \in \mathcal{M}(\Sigma)$.

Admissible trajectories are discretized as usual by tracking the sign vectors $[\dot{x}(t)]$ for each solution:

DEFINITION 2: *For a given reasonable function $x(\cdot)$ on $[0,T]$ we have an ordered sequence of sign jump points (t_j) with $t_0 = 0$ which subsequently contains all boundary points of the closures of all sets $\{t \in [0,T] | [\dot{x}(t)] = v\}$ with $v \in \{[-], [+]\}^n$. We construct a sequence of sign vectors $\tilde{x} = (\tilde{x}_j) := \big(\big[\dot{x}(\tau_j)\big]\big)$, where we arbitrarily choose $\tau_j \in (t_j, t_{j+1})$. If the sequence (t_j) is finite with m elements, we choose $\tau_m \in (t_m, T)$. The sequence $\tilde{x}$ is called* ***abstraction*** *of $x(\cdot)$.*

The slight difference compared to the standard definitons is that, e.g. times were trajectories go through a saddle point are ignored. Note that the abstraction $\tilde{x}$ does not depend the concrete values $\tau_j \in (t_j, t_{j+1}), j \in \mathbb{N}$, since the sign vector $\big[\dot{x}(t)\big]$ is constant on any interval (t_j, t_{j+1}). The set of the abstractions of all solutions of a monotonic ensemble are entailed by a finite graph in the following way:

DEFINITION 3: *Let $\mathcal{M}(\Sigma)$ be a monotonic ensemble, $\mathcal{E}$ the set of reasonable trajectories and $\mathcal{S}_{\mathcal{M}(\Sigma)}(\cdot)$ the corresponding solution operator. We denote the set of the abstractions of the solutions by*

$$\begin{aligned}\tilde{\mathcal{S}}_{\mathcal{M}(\Sigma)}(v_0) := \{\tilde{x} \mid & \exists x_0 \in X \text{ with } [x_0] \approx v_0, \\ & \exists x(\cdot) \in \mathcal{S}_{\mathcal{M}(\Sigma)}(x_0) : \\ & \tilde{x} \text{ is the abstraction of } x(\cdot)\}.\end{aligned}$$

Then, the directed ***state-transition graph*** *G of the monotonic ensemble is defined by the vertices*

$$V(G) := \{v \in \mathcal{A}^n \mid \exists\, \tilde{x} \in \tilde{\mathcal{S}}_{\mathcal{M}(\Sigma)}(\mathcal{A}^n), j \in \mathbb{N} :\ \tilde{x}_j = v\},$$

called ***qualitative states****, and the edges*

$$\begin{aligned}E(G) := \{(v,w) \mid & \exists\, \tilde{x} \in \tilde{\mathcal{S}}_{\mathcal{M}(\Sigma)}(\mathcal{A}^n), j \in \mathbb{N} : \\ & \tilde{x}_j = v \text{ and } \tilde{x}_{j+1} = w\},\end{aligned}$$

called ***qualitative transitions****.*

For convenience, the state-transition graph of a monotonic ensemble is also called the state-transition graph of a QDE. Thus, we have defined a directed graph G such that all sequences of $\tilde{\mathcal{S}}_{\mathcal{M}(\Sigma)}(\mathcal{A}^n)$ describe a path in G, i.e. the graph completely covers all reasonable solutions of initial value problems $\dot{x} = f(x), x(0) = x_0$ with $f \in \mathcal{M}(\Sigma)$. Note that G is loop free since subsequent coefficients of the abstraction of a reasonable function are different. Note further that formalizing the state-transition graph in that way does not require a completeness proof, since it is complete by definition. Completeness can only be shown for an algorithm that computes the graph (e.g. the QSIM algorithm). In our framework this requires to prove that at least a supergraph, but definitively not a subgraph is determined. Within this framework spuriousity is defined as follows:

DEFINITION 4: *Let G be the state-transition graph of the monotonic ensemble $\mathcal{M}(\Sigma)$. In G a path $v_1, \ldots, v_n$ of lenght n is called* ***spurious*** *if there is no model $f \in \mathcal{M}(\Sigma)$ and no initial velocity $\dot{x}_0$ with $[\dot{x}_0] = v_1$ such that the solution $x(\cdot)$ to the initial value problem $\dot{x} = f(x), \dot{x}(0) = \dot{x}_0$ has $v_1, \ldots, v_n$ as the first n coefficients of its abstraction $\tilde{x}$.*

We discuss the occurence of spurious behaviour in the exact state-transition graph below.

Even without running the QSIM algorithm, some features can be shown directly. Which vertices occur in a state-transition graph? Most basically, $\{[-], [+]\}^n \subseteq V(G)$ due to the following reasons: by chain rule $\ddot{x} = \mathcal{J}(f)(x) \cdot \dot{x}$, such that for assumptions about the sign matrix $\big[\mathcal{J}(f)(x)\big]$ not all sign vectors $[\dot{x}]$ are consistent with all sign vectors $[\ddot{x}]$. However, since no claims about $[\ddot{x}]$ are made, no $[\dot{x}] \in \{[-], [+]\}^n$ can be excluded from being a vertex. The situation is more complicated if some $\dot{x}_i \equiv 0$ on (t_j, t_{j+1}), which implies that also $\ddot{x}_i \equiv 0$ on the same interval.

I now present a necessary criterion for such a vertex to exist (see Eisenack 2006 for a proof). For this, we need the set $Z_0(v) := \{i = 1, \ldots, n | v_i = 0\}$, which assigns to a sign vector $v \in \mathcal{A}^n$ the indices of vanishing components.

PROPOSITION 1: *If $v \in V(G)$, then for all $i \in Z_0(v)$*

$$\exists j, k \notin Z_0(v), j \neq k : 0 \neq \sigma_{i,j} v_j \approx -\sigma_{i,k} v_k \neq 0$$

or $\forall j \notin Z_0(v) : \sigma_{i,j} = 0$.

Additionally, every state-transition graph contains the vertex 0, representing the equilibria of systems of the monotonic ensemble.

Now I will show a characterisation for the existence of edges in the state-transition graph G. It is simplified by considering only vertices with non-vanishing components. When two qualitative states v, w differ only in one component i, there must be a solution of the monotonic ensemble $x(\cdot)$, defined by a model f, which transgresses the main isocline $f_i(x) = 0$ at some time, because this isocline separates the regions of the phase space where $[f(x)] = v$ and $[f(x)] = w$, respectively. A necessary condition for such a transgression is an appropriate sign of $\ddot{x}_i$ on the main isocline, e.g. if $v_i = [-]$ and $w_i = [+]$, then $[\ddot{x}_i] \approx [+]$ is needed. We define the **intermediate state** $v \wedge w$ for $v, w \in \mathcal{A}^n$ by

$$(v \wedge w)_i := \begin{cases} v_i & \text{if } v_i = w_i, \\ 0 & \text{if } v_i \neq w_i. \end{cases}$$

Thus, $Z_0(v \wedge w)$ are the indices of the components which change from v to w (or which are constant in one or both states).

PROPOSITION 2: *Let $v, w \in V(G)$, $v \neq w$, and $Z_0(v) = Z_0(w) = \varnothing$. Then, $(v, w) \in E(G)$ iff*

$$\forall i \in Z_0(v \wedge w)\, \exists j \notin Z_0(v \wedge w) : w_i \cdot (v \wedge w)_j \approx \sigma_{i,j}.$$

For a detailed proof see Eisenack (2006). Here it is important to notice that the proposition is a full characterisation. It is not only a criterion for determining the edges of the state-transition graph, but also shows that every edge (v, w) in the graph actually corrresponds to at least one model in $\mathcal{M}(\Sigma)$ which visits the qualitative states v and w in that temporal order. The main part of the proof is thus to construct an appropriate model and to show that it is an element of the monotonic ensemble. The consequence is that (at least for QDEs described by a monotonic ensemble) subsequent time-interval states computed by the QSIM algorithm are never spurious.

Problems arise, of course, for paths of length 3 or more. However, if we extend the definition of monotonic ensemble towards

$$\mathcal{M}(\Sigma) := \{f \in C^1(X, \mathbb{R}_+, \mathbb{R}^n) \mid \forall x \in X, t \in \mathbb{R}_+ : \\ \big[\mathcal{J}(f)(x,t)\big] \approx \Sigma\},$$

it is expected that the situation changes dramatically. Since the non-autonomous system can, in principle, switch between the models constructed in the above proof at every qualitative state, *every path* of *arbitrary length* corresponds to at least one model. In that sense, there are no spurious behaviours.

It may be questioned whether these results still hold when full QDEs and not only monotonic ensembles are considered. There are no proofs yet, but the extension seems straightforward – although a lot of cases need to be distinguished. For illustration, an example for a model ensemble containing a landmark (λ) and an algebraic constraint ($+$) is (see Eisenack *et al.* 2007 for further examples)

$$\begin{aligned}\mathcal{M}(\Sigma_1, \Sigma_2) := \{f \in & C^1(\mathbb{R}^2 \times \mathbb{R}_+, \mathbb{R}^2) \mid \\ & \exists \lambda \in \mathbb{R} \forall x \in \mathbb{R}^2, t \in \mathbb{R}_+ \\ & \text{with } x_1 \leq \lambda : \big[\mathcal{J}(f)(x,t)\big] \approx \Sigma_1 \\ & \text{and with } x_1 > \lambda : \big[\mathcal{J}(f)(x,t)\big] \approx \Sigma_2 \\ & \text{and } f_1(x,t) = x_1 + x_2.\}\end{aligned}$$

This example also illustrates the need for precise definitions to be clear about what is meant by a spurious behaviour. Here, addition of real numbers is used in the definition of $\mathcal{M}(\Sigma_1, \Sigma_2)$ for the algebraic constraint. The model ensemble would be much larger if in the last line of the definition, addition is used in the qualitative sense, i.e.

$$\forall x \in \mathbb{R}^2, t \in \mathbb{R}_+ : [Df_1(x,t)] \approx [+\,+], \qquad (1)$$

which would, depending on Σ_1 and Σ_2, be either contradictory or redundant.

The Relation between QDEs and Differential Inclusions

Differential inclusions (DIs) are a generalisation of ordinary differential equations. While an ODE assigns a single velocity to points in the state space, for differential inclusions *multiple* velocities can be assigned. We map a state x to a set of possible velocities $F(x)$, and admit a trajectory $x(\cdot)$ as a solution, if $\dot{x}(t)$ is always an element of $F(x(t))$. As in the case of QDEs we cannot generally expect to obtain unique solutions in such a setting, yielding a set-valued solution operator. The first ideas to this approach arose in the 30s of the last century (Zaremba 1936; Marchaud 1934). A broad overview to the fundamentals and subsequent development of the theory is provided by Aubin (1984). Differential inclusions are applied to problems from, e.g. population dynamics (Křivan & Colombo 1998; Guo, Xue, & Li 2003), physics (Maisse & Pousin 1997), climate change (Chahma 2003), differential games (Chodun 1989; Ivanov & Polovinkin 1995) and natural resource management (Bene, Doyen, & Gabay 2001; Cury *et al.* 2005; Eisenack, Scheffran, & Kropp 2006).

One basic motivation – similar to QDEs – is to consider uncertainties which cannot be expressed in a probabilistic way. We may have an ODE $\dot{x} = f(x, t; u)$, depending on a parameter or a control u. If we do not know u exactly but can restrict the value, say, to an interval J such that $u \in J$, we obtain a set of possible values $F(x,t) := \{f(x,t;u) \mid u \in J\}$. We can formulate this as an infinite monotonic ensemble in the following way. For a given autonomous measurable function $f' : X \times U \to \mathbb{R}^n, (x, u) \mapsto f'(x, u)$, where $U \subseteq \mathbb{R}$ is a given interval of control values, set

$$\mathcal{M} := \{f : X \times \mathbb{R}_+ \to \mathbb{R}^n \text{ measurable } \mid \\ f(x,t) = f'(x, u(t)), u(t) \in U\}.$$

Taking absolutely continuous functions as admissible trajectories, the solution operator $\mathcal{S}_{\mathcal{M}}(x_0)$ describes all trajectories starting from x_0 which result from any measurable open-loop control $u(\cdot) : \mathbb{R}_+ \to U$.

In the set-valued standard definition, for a given set-valued map $F : X \to \mathcal{P}(Y)$ (where Y is the velocity space and $\mathcal{P}()$ denotes the power set), an "equation" of the form

$$\dot{x} \in F(x),$$
$$x(0) = x_0,$$

is called a differential inclusion. In most cases an absolutely continuous function $x(\cdot) : I \to X$ on an interval $I = [0, T]$, possibly $T = \infty$ is called a **solution** if $x(0) = x_0$ and $\dot{x}(t) \in F(x(t))$ almost everywhere on I. There are various theorems on the existence of solutions to a differential inclusion (see e.g. Aubin 1991).

From this general perspective, a set-valued map $F : X \to \mathcal{P}(\mathbb{R}^n)$ defines a model ensemble by

$$\mathcal{M} := \{f : X \times \mathbb{R}_+ \to \mathbb{R}^n \mid f(x,t) \text{ measurable with respect to } t \text{ and } \forall t \in \mathbb{R}_+ : f(x,t) \in F(x)\}.$$

Taking the set of absolutely continuous functions on intervals $I = [0, T]$ as space of admissible trajectories $\mathcal{E}$, we obtain a set-valued solution operator

$$\mathcal{S}_F(x_0) := \{x(\cdot) \in \mathcal{E} \mid x(0) = x_0, \exists f \in \mathcal{M} : \dot{x}(t) = f(x(t), t) \text{ almost everywhere } \}.$$

Can a QDE be "simulated" by a DI? To find all possible trajectories which can be brought about by a simple QDE (we stick to the case without landmarks again), we change the perspective from the state space to the velocity space. We could define a set-valued map by $F(x) := \{f(x) \mid f \in \mathcal{M}\}$ such that the solutions of the differential inclusion describe all trajectories. However, if the QDE is specified by a sign matrix $\Sigma = (\sigma_{i,j}) \in \mathcal{A}_*^{n\times n}$, we run into trouble, as the following shows:

Suppose that $f \in \mathcal{M}(\Sigma)$. Since it follows from $\dot{x} = f(x)$ that $\ddot{x} = \mathcal{J}(f)(x) \cdot \dot{x}$, we obtain a second order differential inclusion in the joint state and velocity space:

$$\ddot{x} \in F(\dot{x}, x),$$
$$F : (\dot{x}, x) \mapsto \{\mathcal{J}(f)(x) \cdot \dot{x} \mid f \in \mathcal{M}(\Sigma)\}.$$

This can be simplified to

$$\ddot{x} \in \hat{F}(\dot{x}) := \{M\dot{x} \mid [M] \approx \Sigma\},$$

where M denotes $n \times n$ matrices over the real numbers. We observe that the components $i = 1, \ldots, n$ of $\hat{F}(\dot{x})$ evaluate to

$$\hat{F}_i(\dot{x}) = \begin{cases} 0 & \text{if } \forall j = 1, \ldots, n : \\ & \quad \dot{x}_j \cdot \sigma_{i,j} = 0, \\ \mathbb{R}_+ \setminus \{0\} & \text{else if } \forall j = 1, \ldots, n : \\ & \quad [\dot{x}_j] = \sigma_{i,j} \neq 0 \text{ or } \dot{x}_j \cdot \sigma_{i,j} = 0, \\ \mathbb{R}_- \setminus \{0\} & \text{else if } \forall j = 1, \ldots, n : \\ & \quad -[\dot{x}_j] = \sigma_{i,j} \neq 0 \text{ or } \dot{x}_j \cdot \sigma_{i,j} = 0, \\ \mathbb{R} & \text{otherwise .} \end{cases}$$

Except the trivial case, this unbounded set-valued map is very irregular and allows for a very broad solution set. This simple approach doesn't provide valuable results.

One way to overcome this are linear-interval differential inclusions, which restrict a monotonic ensemble $\mathcal{M}(\Sigma)$ to models for which prescribed interval constraints, given by set-valued maps, hold for the components of the Jacobian.

DEFINITION 5: *Let U be a matrix of compact intervals $(u_{i,j})_{i,j=1,\ldots,n}$, where each interval either vanishes or does not contain 0. A set-valued map $F : X \to \mathcal{P}(\mathbb{R}^n), F(x) := Ux$, where the latter denotes interval-valued multiplication, is called a* ***linear-interval map****.*

Interval-valued multiplication is defined in the usual way by $Ux := \{Mx \mid M \in U\}$, where a matrix $M = (m_{i,j})_{i,j=1,\ldots,n} \in U$ if and only if $\forall i, j = 1, \ldots, n : m_{i,j} \in u_{i,j}$. We regard singletons as intervals. DEF. 5 guarantees that every coefficient of U has a unique sign (which can be related to the coefficients of Σ). Note that a linear-interval map F defines a model ensemble which includes nonlinear models f such that $\forall x \in X : f(x) \in F(x)$.

We saw above that it is not possible to investigate a QDE by considering a differential inclusion $\ddot{x} \in \hat{F}(\dot{x})$. However, if intervals $u_{i,j}$ are known such that $\forall x \in X : D_j f_i(x) \in u_{i,j}$, the linear-interval differential inclusion

$$\ddot{x} \in F(\dot{x}) = U\dot{x},$$

can be set-up. It is very regular and simulates the monotonic ensemble $\mathcal{M}(\Sigma)$ in the following sense. Define the restricted model ensemble

$$\mathcal{M}'(\Sigma, U) := \{f \in M(\Sigma) \mid \forall x \in X : \mathcal{J}(f)(x) \in U\} \subseteq \mathcal{M}(\Sigma).$$

with the solution operator $\mathcal{S}_{\mathcal{M}'(\Sigma,U)}(\cdot)$. Then $\forall x_0 \in X, x(\cdot) \in \mathcal{S}_{\mathcal{M}'(\Sigma,U)}(x_0) : \dot{x}(\cdot) \in \mathcal{S}_F(\dot{x}(0))$. On the other hand, the differential inclusion also covers solutions of non-autonomous ODEs $\dot{x} = f(x, t)$ with $\mathcal{J}(f)(x, t) \in U$ for all $t \in \mathbb{R}_+$. Linear-interval differential inclusions are more general than QDEs in the sense that they also include non-autonomous models, and are more specific in the sense that they only include bounded models. In contrast, qualitative differential equations are deterministic but subsume a broad set of possible configurations.

Discussion and Further Applications

In many applications of qualitative reasoning the discussion of spurious behaviour is mixed with the existence of qualitative trajectories of a QDE which contradict knowledge about the system available to the modeller that is not expressed by qualitative constraints. If this impression is true, one explanation are the roots of the method in qualitative *physics*, where we construct problem-driven models of physical systems. They are perceived as being ontologically unambigous, completely numerically specified and time-invariant. From this viewpoint, the main reason for qualitative modelling are epistimic limitations, i.e. missing knowledge about the "objective" pysical system ("uncertainties"), or efficiency considerations when it is not needed to have access to the complete "objective" system for solving a particular task. By interpreting a model ensemble as uncerainty, it covers all cases that could be potentially considered as being valid due to pragmatic or epistemic limitations.

However, the formalization of QDEs as model ensemble illustrates a further interpretation which is highly relevant in the domain of sustainablity science that motivated this work: it may be that there are multiple non-indentical systems, e.g. social-ecological systems like fisheries, agricultural systems or bioreserves which re-appear in many instances on the world. Although every such case may be different, it often appears that some of them share crucial properties and exhibit certain patterns (e.g. qualitative behaviour with typical temporal logic properties). Then, a model ensemble is the collection of all cases which have to be analysed. In that sense, QDEs do not only represent uncertainty, but also *generality*. Such models are not only meant to provide insights for single applications, but should also apply to a broader set of cases with general features in common. In other words, while resolving uncertainty would in principle lead to narrowing down a QDE until it would be refined to an ODE, for representing generality we do not aim at refining to that point, so that the model ensemble still subsumes a broad range of systems. Such generalised ensembles can be so-called "archetypes" of global environmental change (Eisenack, Lüdeke, & Kropp 2006).

Within the domain of sustainability science it is also unappropriate only to consider autonomous ODEs as constituents of the model ensembles, since social-ecological systems are usually influenced by exogeneous environmental and cultural factors which are not constant in time. Therefore, the extended definition of QDEs above may be more appropriate for these kind of applications, at the same time resolving the problem of spurious behaviour. I expect that this interpretation can also be valuable for other domains where the analysis or design of whole classes of systems is needed.

Once we adopt this viewpoint, further questions can be posed within the framework of model ensembles and some established tractability methods can be described. If it is not possible to find relevant features common to all solutions of a model ensemble $\mathcal{M}$ we can try to identify subsets $\mathcal{M}' \subseteq \mathcal{M}$ for which such robust properties exist. The characterisation of $\mathcal{M}'$ is associated with the discovery of structural features which e.g. bring about problematic or desirable system behaviour. In other terms, conditions under which certain (sub)pattern evolve are found. If $\mathcal{M}$ is partially determined by certain control measures imposed on the system, and $\mathcal{M}'$ by alternative control measures, the differences between the solution operators $\mathcal{S}_{\mathcal{M}}(\cdot)$ and $\mathcal{S}_{\mathcal{M}'}(\cdot)$ are of interest.

There are cases where solutions of a model ensemble are artifacts from the assumptions the modeller made. Then it is important to restrict the analysis so that the artifacts are eliminated. Generally, a **restriction** means a restriction of the model ensemble to some $\mathcal{M}' \subseteq \mathcal{M}$, of the admissible trajectories to some $\mathcal{E}' \subseteq \mathcal{E}$, or of the state space to some $X' \subseteq X$. Very "unlikely" or "irrelevant" cases which cannot be refuted on base of the original model ensemble are further reasons to restrict $\mathcal{M}$ or even $\mathcal{E}$. The analytical function constraint and phase plane constraints are examples for the latter, while filtering marginal cases (Eisenack & Petschel-Held 2002; Bouwer & Bredeweg 2002) for the former.

The concept of restricting a model ensemble can also be seen as a formalization of finding the best level of abstraction for practical engineering problems. Qualitative modelling can start with a very general model ensemble which is then successively restricted only up to the level where it becomes concrete enough to achieve its intended task.

Finally, the perspective of model ensembles opens the view for established as well as potential future hybrid or semi-qualitative modelling techniques. One basic motivation for such hybrid methods is to include more than just monotonicity assumptions about a system, if they are available. For example, NSIM restricts $\mathcal{E}$ by introducing envelopes on the solutions (Kay & Kuipers 1993), while Q2 restricts $\mathcal{M}$ to those models where landmarks are (constant) within prescribed quantitative intervals and monotonic functions within monotonic envelopes (Kuipers 1994). Q3 basically remains within this specification but developes more powerful reasoning techniques to determine solutions (Berleant & Kuipers 1997). All these methods, although deterministic in their core, are close to the ideas of differential inclusions. They may be improved by using more results from the respective numerical analysis.

As a new semi-qualitative example I outline a technique which is based on the considerations of the last section (see Eisenack 2006 for details). After solving a QDE with sign matrix Σ, quantitative bounds are considered by setting up a linear-interval differential inclusion $\ddot{x} \in F(\dot{x}) = U\dot{x}$ where the signs of the intervals coefficients of U correspond to the signs of Σ. If we want to identify conditions for a given successor state w to be reached from a state v, we define – in the velocity space – the cones $K(v) := \{\dot{x} \in \mathbb{R}^n \mid [\dot{x}] = v\}$ for $v \in \mathcal{A}^n$. For the linear-interval differential inclusion, the so-called absorption basin $\mathrm{Abs}_F(\bar{K}(v), \bar{K}(v) \cap \bar{K}(w))$ of the closure of such cones contains all initial velocities $\dot{x}_0$ such that for all solutions $\dot{x}(\cdot) \in \mathcal{S}_F(\dot{x}_0)$ with $[\dot{x}_0] = v$ there exists a $T > 0$ with $[\dot{x}(T)] = w$. Such absorption basins can be computed using the viability algorithm (Saint-Pierre 1994; Cardaliaguet, Quincampoix, & Saint-Pierre 1994).

All these methods share the idea to complement qualitative knowledge in the sense of monotonicities and landmarks with quantitative knowledge in the sense of ODEs or set-valued maps. However, we may think of further possibilities. Often more knowledge than about trends and thresholds seems to be available, while it is very difficult to come up with quantitative estimates. This may be due to very poor data conditions (e.g. agricultural yield in developing countries, fish catches in international waters) or due to difficulties in operationalizing variables (e.g. political power or poverty). *It would therefore be of high value to refine model ensembles without resorting to quantities*, raising the question whether there is some relevant type of non-quantitative knowledge that cannot be represented as a QDE. Ordner of magnitudes may be a candidate, but established formalizations still refer to magnitudes on the real line (Travé-Massuyès, Ironi, & Dague 2004).

A further candidate are ordinal assumptions. These relate to the strength of influences in causal loop diagrams. If the influence of one variable on another is stronger than the influence of a third, this can be interpreted as partial order on the partial derivatives of models f of a monotonic ensemble $\mathcal{M}(\Sigma)$ of the form

$$\forall x \in X : D_k f_i(x) > D_l f_j(x),$$

for a set of tuples (i, j, k, l). The restricted model ensemble $\mathcal{M}' \subseteq \mathcal{M}(\Sigma)$ contains only those models that respect a prescribed partial order of this kind. Eisenack et al. (2006; 2007) present some methods to exploit such kind of knowledge. However, it appears that not all implications that can be made from ordinal assumptions are exploited yet. Although some theoretical results exists, well working algorithms are not established yet. Finally, ordinal assumptions obviously boil down to statements about the sign $\left[d_{k,l}^{i,j}\right]$, where $d_{k,l}^{i,j} := D_l f_j \cdot D_k f_i - D_k f_j \cdot D_l f_i$. We can thus re-state the question of whether there is some kind of information "between signs and numbers".

Conclusion

In this paper I presented a formalisation of QDEs within the new framework of model ensembles which appears to embed also differential inclusions and further established quantiative and semi-qualitative methods.

Their particular similarities and differences become visible. While QDEs are deterministic and autonomous, differential inclusions also include non-autonomous dynamics. On the other hand, QDEs are less restrictive in the sense that they do not need to be explicitly quantitatively constrained by set-valued maps. Therefore, neither of these approaches can be reduced to the other.

As a by-product the notion of spurious behaviour can be further clarified. It is shown that a qualitative behaviour consisting of two subsequent time-interval states is never spurious. Furthermore, there are indications that an extension of standard QDEs to non-autonomous models of a certain kind may completely resolve this problem.

The framework of model ensembles can be used to specify the notion of uncertainty typically used in qualitative reasoning and extends it to the notion of generality in a certain sense that is highly relevant for the design of whole classes of systems which only share some common features.

Finally, the framework of model ensembles allows for defining various extensions of QDEs in a consistent way, opening the field for further qualitative and semi-qualitative methods.

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