A General Qualitative Spatio-Temporal Model Based on Intervals

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Abstract

Naming qualitative models allow humans to express spatio-temporal concepts such as "The cinemas are *far away* from my house". In colloquial terms, naming concepts are called relative. In this paper we introduce a general way to define naming qualitative models consisting of: (1) a representation magnitude, (2) the basic step of inference process and (3) the complete inference process. We present a general procedure to solve the representation magnitude and the basic step of inference process of qualitative models based on intervals. The general method is based on the definition of two algorithms: the qualitative sum and the qualitative difference.

1 Introduction

The most widely used way to model commonsense reasoning in the spatial domain is by means of qualitative models. Actually, qualitative reasoning may help to express poorly defined problem situations, support the solution process and lead to a better interpretation of the final results [Werthner, 1994]. In fact, most of the knowledge about time and space is qualitative in nature, that is, it is not necessary to know the exact amount of a spatio-temporal aspect to perform context-dependent comparisons. A clear example is humans who are not good at determining accurate lengths, volumes, etc., whereas they can easily perform context-dependent comparisons and make correct decisions from those comparisons [Hernández, 1994].

Thus, a qualitative representation can be defined as that representation which *makes only as many distinctions as necessary to identify objects, events, situations, etc., in a given context* [Hernández, 1994]. Note that the way to define those distinctions depends on two different aspects. The first one is the level of *granularity*. In this context, granularity refers to a matter of precision in the sense of the amount of information which is included in the representation. Therefore, a fine level of granularity will provide a more detailed information than a coarse level. This is the case of distances, for example, where the required accuracy level depends on the proximity to the place to locate. So, larger distances can be managed with partial and more imprecise knowledge than closer ones.

The second aspect corresponds to the distinction between *comparing* magnitudes and *naming* magnitudes [Clementini *et al.*, 1997]. This distinction refers to the usual comparison between *absolute* and *relative*. From a spatial point of view, that controversy corresponds to the way the relationships among objects in the world are represented. As pointed out by [Levinson, 2003], *absolute* defines an object's location in terms of arbitrary bearings such as, for instance, cardinal directions (North, South, East, West), by resulting in binary relationships. Instead, *relative* leads to ternary relationships. Consequently, for *comparing* magnitudes, an object b is any compared relationship to another object a from the same Point of View (PV). The comparison depends on the orientation of both objects *with respect to* (wrt) the PV, since objects a and b can be at any orientation wrt the PV. An example is the qualitative treatment of compared distances [Escrig and Toledo, 2001] (see Figure 1). In this case, only two extreme orientations are considered: (1) both objects a and b are at the same orientation wrt the PV, represented by $b[Rel]^S_{PV}a$ (i.e., the compared distance PV to a and PV to b) and (2) objects a and b are in the opposite orientation wrt the PV ($b[Rel]^O_{PV}a$).

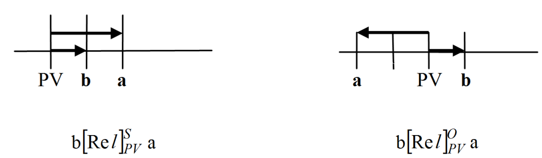


Figure 1: An example of the *compared* distances to be represented [Escrig and Toledo, 2001]

On the other hand, *naming* magnitudes divides the magnitude of any concept into intervals (sharply or overlapped separated, depending on the context (see Figure 2)) such that qualitative labels are assigned to each interval. Note that the result of reasoning with regions of this kind can provide imprecision. This imprecision will be solved by providing disjunction in the result. That is, if an object can be found in several qualitative regions, q_i or q_{i+1} or ... or q_n, then all possibilities are listed as follows $\{q_i, q_{i+1}, \ldots, q_n\}$ by indicating this situation.

Although qualitative models based on comparing magni-

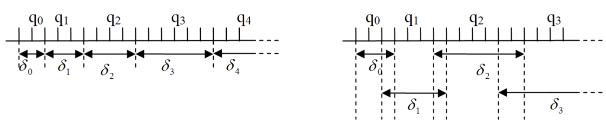
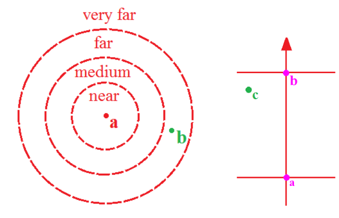


Figure 2: An example of structure relations where: acceptance areas are sharply separated (left) and acceptance areas are overlapped (right)



(a) **(b)**

Figure 3: Relations between objects: (a) Binary: object *a* is the reference and object *b* is referred. The space is divided into a four-distance system centred in the reference object *a* [Jong, 1994] (b) Ternary: objects *a* and *b* define a reference system (in red) with respect to object *c* is referenced. In this case, the space is divided into nine qualitative regions determined by the oriented path from object *a* to object *b* [Freksa and Zimmermann, 1992]

tudes and qualitative naming models based on intervals have been studied, some models have not been solved up to now. Table 1 presents some of the qualitative models developed for dealing with certain spatial concepts. Note that only some of the developed models are illustrated, since it is not possible to depict all of them by lack of space.

So, the aim of this paper is to present a general systemic algorithm that integrates and solves the reasoning process of all qualitative models based on intervals. From the starting point that the development of any qualitative model consists of a representation of the magnitude at hand and the reasoning process, the structure of this paper is as follows: Section 2 describes the representation of a magnitude. The Reasoning Process is introduced in Section 3, and discussed in Section 4

2 Magnitude Representation

In qualitative spatial reasoning, it is common to consider a particular aspect of the physical world, that is, a magnitude such as topology or distance, and to develop a system of qualitative relationships between entities which cover that aspect of the world to some degree. Therefore, the first issue to be solved refers to the way to represent the magnitude to be modelled.

Focusing on qualitative naming models based on intervals, any magnitude is represented by the following three elements:

1. The **number of objects** implied in each relation (i.e. *arity*). A relationship is *binary* when there are only two objects implied (object b wrt object a, i.e. *b wrt a*). So, an object acts as reference (a) and the other one is referred (b). For instance, how far an object is wrt another object is a binary relationship as defined by [Jong, 1994] (see Figure 3a). In this example, the two-dimensional space is divided into several tracks centred in the reference object a. Each track is associated to a unique qualitative value (e.g. *near, medium, far, very far*). So, the relationship between objects a and b will be determined by the track of the interval-based system in which the object b is. Therefore, in the shown example, *b wrt a* is *far*, in other words, b is far from a.

   On the contrary, a relationship is *ternary* when three objects are implied (*c wrt ab*) such that two objects form the reference system (ab) and the other object is referenced wrt such reference system (c). For example, qualitative orientation information in the way [Freksa and Zimmermann, 1992] presented, is represented by an *orientation grid*. This grid is aligned to the orientation determined by two points in space, a and b, unlike the previous example. Thus, the space is divided into nine qualitative regions (*front-left, front, front-right, left, identical-front, right, back-left, back and back-right*). In this way, the orientation of the object *c wrt ab* will correspond to any of these qualitative regions. In particular, in this example, *c wrt ab* is *back-left*.

2. The **set of relations between objects**. It depends on the considered level of granularity. In a formal way, this set of relations between objects is expressed by means of the definition of a *Reference System (RS)*. A RS will contain, at least, a couple of components:

   - A *set of qualitative symbols* in increasing order represented by $Q = \{q_0, q_1, ..., q_n\}$, where q_0 is the qualitative symbol closest to the *Reference Object (RO)* and q_n is the one furthest away, going to infinity. In the two examples depicted in Figure 2, q_n corresponds to q_4 in the case of sharply intervals, while q_3 is the q_n qualitative symbol in the overlapped interval sample. Here, by cognitive considerations, the acceptance areas have been chosen in increasing size. Note that this set defines the different areas in which the workspace is divided and the number of areas will depend on the *granularity* of the task, as introduced above
   - The *structure relations*, $\Delta r = \{\delta_0, \delta_1, ..., \delta_n\}$, describe the acceptance areas for each qualitative symbol q_i. So, δ_0 corresponds to the acceptance area of qualitative symbol q_0; δ_1 to the acceptance area of symbol q_1 and so on. These acceptance areas are quantitatively defined by means of a set of closed or open intervals delimited by two extreme points: the initial point of the interval j, δ^i_j, and the ending point of the interval j, δ^e_j. Thus, for instance, when open intervals are considered, the structure relations are rewritten by:

     $$\Delta r = \{\,]\delta^i_0, \delta^e_0[\, , \,]\delta^i_1, \delta^e_1[\, , \ldots , \,]\delta^i_n, \delta^e_n[\, \}$$

     whereas, for closed intervals, it would be:

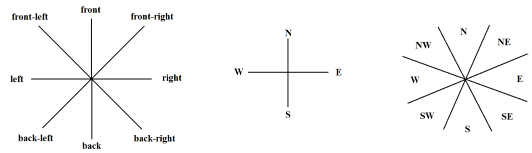
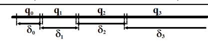
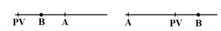
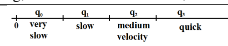
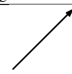
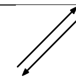
Magnitude	Naming models (based on intervals)	Comparing models
Orientation	 [Hernández, 1992] [Hernández, 1994] [Frank, 1996] [Pacheco *et al.*, 2006] [Ligozat, 1998] [Renz and Mitra, 2004]	
Distance	 [Zimmermann, 1993] [Jong, 1994] [Clementini *et al.*, 1997] [Escrig and Toledo, 2000]	 [Escrig and Toledo, 2001]
Velocity	 [Escrig and Toledo, 2002]	
Trajectories	 [de Weghe *et al.*, 2005a] [de Weghe *et al.*, 2005b] [Gottfried, 2008]	 [Liu and Goghill, 2005]

Table 1: Qualitative naming models versus qualitative comparing models

$$\Delta r = \left\{ \left[\delta_0^i, \delta_0^e\right], \left[\delta_1^i, \delta_1^e\right], \dots, \left[\delta_n^i, \delta_n^e\right] \right\}$$

As a consequence, the acceptance area of a particular entity of a magnitude, $AcAr(entity)$, is δ_j if entity value is between the initial and ending points of δ_j, that is, $\delta_j^i \leq value\,(entity) \leq \delta_j^e$

3. The **operations**. The number of operations associated to a representation corresponds to the possible change in the point of view. For instance, if the relationship is binary (*b wrt a*), only one operation can be defined: inverse (*a wrt b*). Nevertheless, it is possible to define five different operations when the relationship between objects is ternary (*c wrt ab*) [Freksa and Zimmermann, 1992]: inverse (*c wrt ba*), homing (*a wrt bc*), homing-inverse (*a wrt cb*), shortcut (*b wrt ac*) and shortcut-inverse (*b wrt ca*). An iconic representation of the obtained relationships from these operations is depicted in Figure 4.

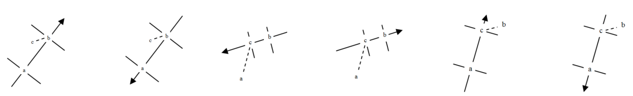


Figure 4: Iconic representation of the relationship *c wrt ab* and the result of applying the five operations to the original relationship

3 The Reasoning Process

The *reasoning process* is divided into two parts:

- **The Basic Step of the Inference Process (BSIP)**. It can be defined as: "given two relationships, (1) the object *b wrt a reference system RS, RS1*, and (2) the object *c wrt another reference system RS, RS2*, such that the object *b* is included into the second reference system, the BSIP obtains the relationship *c wrt RS1*". Figure 5 shows the general BSIP for orientation and positional models not based on projections (*a*) as well as two particular examples of the BSIP: (b) when binary relationships are considered and (c) when ternary relationships are used. In Spatial Reasoning, the BSIP is usually represented by *composition tables*. These tables can be obtained either by hand or automatically by means of algorithms, if they exist.

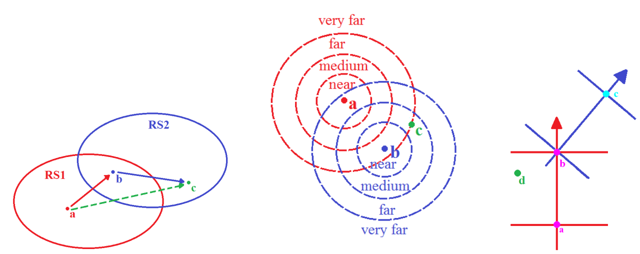


Figure 5: The general BSIP for orientation and positional models not based on projections (a); (b) presents an example of the BSIP for binary relationships based on the named distances system (Jong, 1994) such that RS1 (in red) has as reference object *a*, while RS2 (in blue) is defined from object *b*. So, from the relations *b wrt a* and *c wrt b*, *c wrt a* might be inferred. Finally, (c) shows an example of the BSIP when ternary relationships are used. Actually, a sample of the model presented by (Freksa & Zimmermann, 1992) is depicted. Note that objects *a* and *b* describe the RS1 (in red), whereas the RS2 (in blue) is defined from objects *b* and *c*. In this case, *d wrt ab* has to be inferred from *c wrt ab* and *d wrt bc*

- **The Complete Inference Process (CIP)**. It is necessary when more than two objects (in binary relationships) or three objects (in ternary relations) are involved in the reasoning mechanism. Mainly, it consists of repeating the BSIP as many times as possible with the initial information and the information provided by some BSIP until no more information can be inferred

3.1 The Basic Step of the Inference Process

Basically, the BSIP is defined as the process of inferring the relationship between two (or three) entities of a magnitude from the knowledge of two other relationships such that there is an object in common in both relationships. The way to infer the new relationship depends on the considered magnitude. However, all qualitative models based on intervals define the magnitude in the same way, as abovementioned. For that reason, an abstraction can be done by resulting in a general algorithm. Here, we propose a general algorithm based on qualitative sums and differences that solves the inference process for all models based on intervals.

The General Algorithm

As previously introduced, magnitudes are represented by three different elements: the number of objects implied in each relationship, the set of relationships between entities and the operations that can be defined. Note that there is a difference between concepts of commonsense knowledge. Thus, for example, *time* is a scalar magnitude, while *space* is much more complex mainly due to its inherent multi-dimensionality. This inherent feature leads to a higher degree of freedom and an increased possibility of describing entities and relationships between entities. Because of the richness of space, its multi-dimensionality and its multiple aspects, most work in qualitative reasoning has focused on single aspects of space such as, for example, topology, orientation or distance. Nevertheless, as pointed out in [Freksa, 1992], relationships between entities can be seen as movements in the space or spatial deformations in physical space. As a result, when the relationships between entities are considered as directed vectors and using as reference orientation *ab*, three different situations can take place (see Figure 6):

- relationships between entities are in the same orientation
- relationships between entities are in the opposite orientation
- relationships between entities are at any orientation

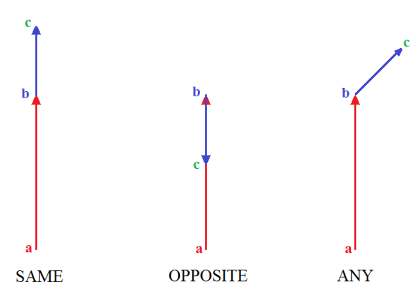


Figure 6: Representation of relationships between entities in terms of orientation by using as reference orientation *ab*

Therefore, the inferred relationship will be composed of all possible relationships between the entities by considering the three possible orientations. According to a deeper analysis of the possible orientations, it is clear that the extreme cases are obtained when the implied objects are in the same orientation and when they are in the opposite one. Consequently, if both extreme cases are solved, the result will be built as a disjunction of qualitative symbols from the inferred area closest to the RO to the furthest one. With the aim of automatically solving these extreme cases, the qualitative sum of intervals as well as the qualitative difference of intervals are defined.

The Qualitative Sum

Let q_i be the qualitative symbol which represents a relationship *b wrt a reference system RS1*, and let q_j be the qualitative symbol referred to the relationship *c wrt another reference system RS2*, such that *b* is included into the second reference system.

Supposing that the two relationships are binary, we would have a situation similar to the one illustrated in Figure 7. In this example, from the knowledge *b wrt a* = q_3 and *c wrt b* = q_2, *c wrt a* will be inferred. Graphically, after locating both entities b and c in any place in their corresponding qualitative areas, q_i and q_j respectively (extreme cases are depicted in Figure 8), it is clear that the possible resulting relationships are $\{q_3, q_4\}$. However, it is possible to achieve the same solution from a mathematical point of view. The development of such a method has several advantages. It does not require to represent the relationships for any composition. It is important specially when the dimensionality of the magnitude is high. Moreover, it can be applied to all the models based on intervals since the reasoning mechanism is the same in all of them.

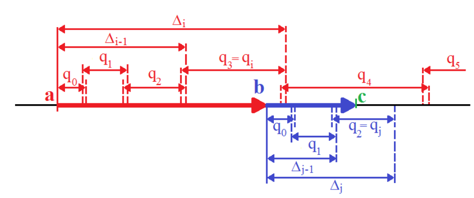


Figure 7: Example of qualitative sum when structure relations with overlapped acceptance areas are used

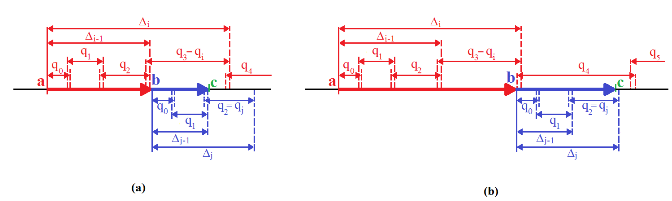


Figure 8: Extreme positions where entities *b* and *c* can be located by keeping the relationships *b wrt a reference system RS1* and *c wrt another reference system RS2*. So, (a) refers to the worst case: both entities are located in the initial points of their acceptance areas (leads to LB); while (b) represents the best case: both entities are situated in the ending points of their respective acceptance areas (leads to UB)

Therefore, the qualitative sum of the two corresponding intervals δ_i and δ_j, results in a range of qualitative symbols given by:

$$AcAr\left(\Delta_{i-1} + \Delta_{j-1}\right) \ldots AcAr\left(\Delta_i + \Delta_j\right) \quad (1)$$

where Δ_k represents the distance from the origin to δ_k, i.e. the sum of consecutive intervals from the origin to δ_k. This concept can be mathematically defined by assuming that δ_+ is the origin of positive values and δ_- is the corresponding origin of negative values, as follows:

$$\forall k = 0, 1, \ldots, n \; \Delta_k = \begin{cases} \delta_k^e - \delta_+^i \text{ if } \delta_k^e \geq 0 \\ |\delta_k^i - \delta_-^e| \text{ otherwise} \end{cases} \quad (2)$$

Algorithm 1 Qualitative Sum

Input: q_i : relationship *b wrt a RS, RS1*
q_j : relationship *c wrt another RS, RS2* (*b* is included into the RS2)
Δr : structure relations
Output: $Result$: disjunction of qualitative symbols for relationship *c wrt RS1*

```
BEGIN
if Δ_j ≪ δ_i then
    UB ← q_i
else if i == max then
    UB ← q_i
else
    Find_UB_qualitative_sum (Δ_j, δ_{i+1}, Δr, i+1, UB)
end if
Find_LB_qualitative_sum (Δ_{j−1}, δ_i, Δr, i, LB)
Build_Result (LB, UB, Result)
END
```

The developed method proposed to solve the qualitative sum of intervals, sketched in Algorithm 1, is divided into three steps:

1. **Obtaining the Upper Bound (UB) of the result** (see Algorithm 2). With the aim to obtain the upper bound of the disjunct of qualitative symbols for the relationship *c wrt a*, the case in which entities *b* and *c* are equivalent to the ending points of their respective acceptance areas is studied (see Figure 8b). Under this hypothesis, three different cases can occur:
   - The distance from the origin of qualitative areas (δ_+ or δ_-) to δ_j, Δ_j, is much lower than the sum of acceptance areas to δ_i, Δ_i. In this case, the absorption rule is applied. This rule, stated in [Clementini *et al.*, 1995], means that whether an interval δ_i is k times greater than other, δ_j, then it can be assumed, without loss of information, that the sum or difference of them is δ_i. Mathematically, this generality is expressed as follows:

   $$(\delta_i \gg \delta_j) \Leftrightarrow \delta_i \geq k * \delta_j \Rightarrow \delta_i \pm \delta_j \cong \delta_i \quad (3)$$

     where k is a constant which depends on the context. Thus, if that rule is applied, the interval δ_j will be disregarded wrt δ_i.
   - δ_i corresponds to the last defined qualitative area. This fact leads δ_i to be the upper bound of the result.
   - Otherwise, an iterative procedure has been defined in order to recursively look for the minimum qualitative area, δ_k, which satisfies:

   $$\begin{aligned} \Delta_j &\leq \delta_{i+1} + \delta_{i+2} + \ldots + \delta_k \Leftrightarrow \Delta_j \leq \Delta_{(i+1)..k} \Leftrightarrow \\ &\Leftrightarrow \begin{cases} \Delta_j \leq \left(\delta_k^e - \delta_{i+1}^i\right) \text{ if } \delta_{i+1}^i \geq 0 \\ \Delta_j \leq \left(|\delta_k^i - \delta_{i+1}^e|\right) \text{ otherwise} \end{cases} \end{aligned} \quad (4)$$

     It stops when it comes to the last qualitative defined region or when the sum of acceptance areas from the origin to δ_j, i.e. Δ_j, is less or equal than the sum of acceptance areas starting from δ_{i+1} to δ_k with $k > i$.

   Going back to the example shown in Figure 7, suppose that the acceptance areas have been defined such as $\Delta r = \{[0,4], [3,8], [7,15], [13,25], [22,37], [34,\infty[\}$.
   So, $\Delta_j = \Delta_2 = \delta_2^e - \delta_+^i = 15 - 0 = 15$. Therefore, the algorithm looks for that δ_k that satisfies Equation 4. In this case, $k = 4$ since $\Delta_{(i+1)..(i+1)} = \left(\delta_{i+1}^e - \delta_{i+1}^i\right) = \left(\delta_4^e - \delta_4^i\right) = 37 - 22 = 15$ which is equal to Δ_2. As a result, the UB of the qualitative sum is q_4.

2. **Obtaining the Lower Bound (LB) of the result**. With this purpose, another recursive function has been implemented (Algorithm 3). Note that, unlike the previous case, values of entities *b* and *c* are supposed to be equivalent to the initial points of their respective acceptance areas (see Figure 8a). Thus, the expression to be satisfied in this case is:

   $$\Delta_{j-1} \leq \delta_i + \delta_{i+1} + \ldots + \delta_k \quad (5)$$

   It will stop when it comes to the last qualitative region of the structure relations or when the distance from the origin to the qualitative area previous to δ_j, that is, Δ_{j-1}, is less or equal than the sum of acceptance areas starting from δ_j to δ_k with $k \leq i$. Again, consider the example depicted in Figure 7. Now, $\Delta_{j-1} = \Delta_1 = \delta_1^e - \delta_+^i = 8 - 0 = 8$ is required. And the searched qualitative symbol (q_k) is provided by the Equation 5. In this case, $k = 3$ given that $\Delta_{i..i} = \left(\delta_i^e - \delta_i^i\right) = \left(\delta_3^e - \delta_3^i\right) = 25 - 13 = 12$, that is greater than Δ_1. Consequently, the LB of the qualitative sum for this example is q_3

3. **Building the result** (see Algorithm 4). Basically, the implemented procedure provides the list of qualitative regions from the lower bound (LB) to the upper one (UB). Again, based on the illustrated example, the resulting disjunct of qualitative symbols expressing the relationship *c wrt RS1* would be $\{q_3, q_4\}$

Algorithm 2 Find UB Qualitative Sum

Input: $\Delta_j : \left(\delta_j^e - \delta_+^i\right)$ or $|\delta_j^i - \delta_-^e|$ if $\delta_j^i \geq 0$ or not, respectively
$\Delta_{inc} : \delta_{i+1} + \delta_{i+2} + \ldots + \delta_k$
Δr : structure relations
k : index of the qualitative area under study (initially $i+1$)
Output: $Result$: upper bound of the disjunction of qualitative symbols for the relationship *c wrt RS1*
BEGIN
if $k == max$ **then**
  $UB \leftarrow q_k$
else if $\Delta_j \leq \Delta_{inc}$ **then**
  $UB \leftarrow q_k$
else
  $Find\_UB\_qualitative\_sum\left(\Delta_j, \Delta_{inc} + \delta_{k+1}, \Delta r, k+1, UB\right)$
end if
END

Algorithm 3 Find LB Qualitative Sum

Input: $\Delta_{j-1} : \left(\delta_{j-1}^e - \delta_+^i\right)$ or $|\delta_{j-1}^i - \delta_-^e|$ if $\delta_{j-1}^i \geq 0$ or not, respectively
$\Delta_{inc} : \delta_i + \delta_{i+1} + \ldots + \delta_k$
Δr : structure relations
k : index of the qualitative area under study (initially i)
Output: $Result$: lower bound of the disjunction of qualitative symbols for the relationship *c wrt RS1*
BEGIN
if $k == max$ **then**
  $LB \leftarrow q_k$
else if $\Delta_{j-1} \leq \Delta_{inc}$ **then**
  $LB \leftarrow q_k$
else
  $Find\_LB\_qualitative\_sum\left(\Delta_{j-1}, \Delta_{inc} + \delta_{k+1}, \Delta r, k+1, LB\right)$
end if
END

The Qualitative Difference

When the given relationships are opposite directed, as the example shown in Figure 9, the qualitative difference of intervals must be solved. With this aim, a new method has been designed. With a similar reasoning mechanism to the qualitative sum, the qualitative difference of two intervals δ_i and δ_j is given by:

$$AcAr\left(\Delta_i - \Delta_j\right) \ldots AcAr\left(\Delta_{i-1} - \Delta_{j-1}\right) \qquad (6)$$

Nevertheless, a new problem arises: a bigger amount can be subtracted of a lower one (i.e. obtaining $\Delta_i - \Delta_j$ when $\Delta_j > \Delta_i$). For solving that, the advantage of the commutative property is used. Therefore, two definitions of the resulting range of acceptance areas are distinguished by depending on the relationship between the two amounts Δ_i and Δ_j:

$$\begin{cases} AcAr\left(\Delta_i - \Delta_j\right) \ldots AcAr\left(\Delta_{i-1} - \Delta_{j-1}\right) & \text{when } \Delta_i \geq \Delta_j \\ AcAr\left(\Delta_j - \Delta_i\right) \ldots AcAr\left(\Delta_{j-1} - \Delta_{i-1}\right) & \text{otherwise} \end{cases} \qquad (7)$$

From that definition, the process to obtain the qualitative difference consists of the following three steps:

Algorithm 4 Build Result

Input: LB : qualitative symbol of the lower bound of the result
UB : qualitative symbol of the upper bound of the result
Output: $Result$: disjunction of qualitative symbols for the relationship *c wrt RS1*
BEGIN
$Result \leftarrow \{\}$
for $q_k = LB$ TO UB **do**
  $Result \leftarrow Result \cup q_k$
end for
END

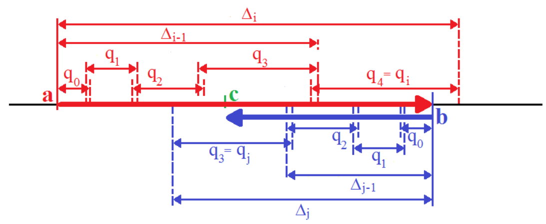


Figure 9: Example of qualitative difference when structure relations with overlapped acceptance areas are used

1. **Obtaining the upper bound**. The upper bound of the range of acceptance areas is computed considering the best case for the given relationships, that is, when the entity values are equivalent to the initial points of their acceptance areas (see Figure 10b). Under this hypothesis, a recursively function that searches the minimum acceptance area, δ_k, that satisfies the comparison $\Delta_{j-1} \leq \delta_i + \delta_{i-1} + \ldots + \delta_k$, has been implemented. Therefore, it will stop when it comes to consider the first region of the relation structure or when the sum of acceptance areas from the origin to δ_j (without including δ_j), Δ_{j-1}, is less or equal than the sum of acceptance areas starting from δ_i to δ_j with $k \leq i$.

   As an example, suppose that the acceptance areas have been defined such as $\Delta r = \{[0,4], [3,8], [7,15], [13,25], [22,37], [34,\infty[\}$. So, from the knowledge *b wrt a* $= q_4$ and *c wrt b* $= q_3$, *c wrt a* must be inferred. We know $\Delta_{j-1} = \Delta_2 = \delta_2^e - \delta_+^i = 15 - 0 = 15$. Thus, the algorithm looks for that δ_k that satisfies $\Delta_{j-1} \leq \delta_i + \delta_{i-1} + \ldots + \delta_k$. In this example, $k = 3$ since $\Delta_{i-1..i} = \Delta_{3..4} = \left(\delta_4^e - \delta_3^i\right) = 37 - 13 = 24$ is greater than Δ_2, whereas $\Delta_{i..i} = \Delta_{4..4} = \left(\delta_4^e - \delta_4^i\right) = 37 - 22 = 15$ is less than that amount. Therefore, the upper bound of the resulting disjunct of relationships corresponding to *c wrt a* is q_3. Graphically, it can be observed in Figure 10b that the entity c is in the area where the acceptance areas δ_2 and δ_3 overlap. Consequently, as we are searching the upper bound, the resulting acceptance area for this case is δ_3.

2. **Obtaining the lower bound**. The lower bound is computed supposing the worst case for both relationships: the entity values are equivalent to the ending points of their acceptance areas (see Figure 10a). So, as in the

case of the upper bound of the qualitative sum, three cases can occur:

- Whether the absorption rule is satisfied, the lower bound will be δ_i or δ_j by depending on $\Delta_i \geq \Delta_j$ or $\Delta_j > \Delta_i$ respectively
- If δ_i, or δ_j when $\Delta_j > \Delta_i$, is the first defined acceptance area, then δ_i, or δ_j respectively, is the lower bound because there is no any previous area to be considered
- Otherwise, a recursive backward search among the defined qualitative areas is applied. Its aim is to find the qualitative area δ_k that satisfies the relationship $\Delta_j \leq \delta_{i-1} + \delta_{i-2} + \ldots + \delta_k$ ($\Delta_i \leq \delta_{j-1} + \delta_{j-2} + \ldots + \delta_k$).

Considering again the illustrated example, we have that $\Delta_j = \Delta_3 = \delta_3^e - \delta_+^i = 25 - 0 = 25$ and the searched k is equal to 2, as depicted in Figure 10a, given that $\Delta_{i-1..i-1} = \Delta_{3..3} = \delta_3^e - \delta_3^i = 25 - 13 = 12 < \Delta_3$ and $\delta_{i-1} + \delta_{i-2} = \Delta_{i-2..i-1} = \Delta_{2..3} = \delta_3^e - \delta_2^i = 25 - 7 = 18$ which is greater than Δ_3. As a result, the lower bound for our example is q_2

3. **Building the result** (Algorithm 4). The same procedure used to build the list of qualitative symbols from the LB to the UB in the qualitative sum is applied to obtain the desired result for this operation. In the shown example, the output of this procedure would be $\{q_2, q_3\}$

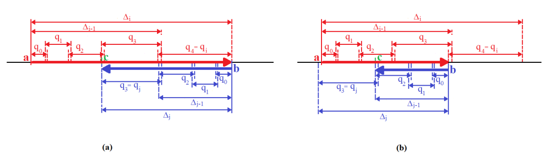


Figure 10: Extreme positions where entities b and c can be located by keeping the relationships *b wrt a reference system* and *c wrt another reference system*. So, (a) refers to the worst case: both entities are located in the ending points of their acceptance areas (leads to LB); while (b) represents the best case: both entities are situated in the initial points of their respective acceptance areas (leads to UB)

The remaining issue is to know which orientation of the relationships corresponds to the lower (upper) bound of the result. That is, determining if the upper bound of the resulting list of qualitative areas corresponds to the upper bound of the qualitative sum (in which case *Find UB Qualitative Sum* would be called), to its lower bound (*Find LB Qualitative Sum* would be performed), to the upper bound of the qualitative difference (in which case *Find UB Qualitative Difference* will be invoked) or, on the contrary, to its lower bound (*Find LB Qualitative Difference* would be run). And the same occurs with the lower bound of the disjunction of the qualitative areas for the inferred relationship. This issue depends on two different aspects:

- possible values for the magnitude are positive and/or negatives
- the definition of the BSIP for each magnitude

Hence, the resulting disjunction of qualitative areas for the inferred relationship will be built from different ways to obtain the lower and upper bounds.

Regarding the CIP, it can be formalized as a Constraint Satisfaction Problem (CSP) since knowledge about relationships between entities is often given in the form of constraints. Note that CSP is *consistent* if it has a *solution*. Moreover, a CSP can be represented by a *constraint network* where each node is labelled by a variable X_i or by the variable index i, and each directed edge is labelled by the relationship between the variables it links. Consequently, a path consistency algorithm can be used as a heuristic test for whether the defined constraint network is *consistent* [Allen, 1983], and, therefore, if the CSP has a *solution*. Thus, a number of algorithms for path consistency has been developed from its definition: a constraint graph is *path consistent* if for pairs of nodes (i, j) and all paths $i - i_1 - i_2 - \ldots - i_n - j$ between them, the direct constraint $c_{i,j}$ is tighter than the indirect constraint along the path, i.e. the composition of constraint $c_{i,i_1} \otimes \ldots \otimes c_{i_n,j}$ [Frühwirth, 1994a] [Frühwirth, 1994b].

A straight-forward way to enforce path-consistency on a CSP is to strengthen relationships by successively applying the following operation until a fixed point is reached:

$$c_{ij} := c_{ij} \oplus c_{ik} \otimes c_{kj} \tag{8}$$

where the part ($c_{ik} \otimes c_{kj}$) of the formula computes composition and it obtains the constraint c_{ij}. This result is intersected ($\oplus$) with the preceding computed or used-defined constraints (if they exist). The complexity of such an algorithm is $O\left(n^3\right)$ where n is the number of nodes in the constraint graph [Mackworth and Freuder, 1985].

It is worth noting that, given that the BSIP is different from each instance of the general qualitative model and the CIP is the repetition of the BSIP, a different CSP will be will be defined for solving each CIP in terms of qualitative sums and differences, although the structure of the program is kept.

4 Conclusions

Physical space and its properties play essential roles in all sorts of actions and decisions. As a consequence, the ability to reason in and about space is crucial for system involved in theses actions and decisions. So, given that qualitative representations are one source of flexibility in commonsense reasoning, since they are more stable than quantitative representations, and provide a level of description that can be more easily matched and reasoned with. Although positional information has been the starting point, the spatial reasoning can imply different physical properties by requiring high dimensional descriptions. For that reason, both magnitude representation and reasoning process have been analysed by leading to a general algorithm which solves the representation and the inference process of any qualitative model based on intervals in n dimensions. It has been instanced to two different qualitative models: (1) naming distance, and (2) qualitative velocity by obtaining the same results to the presented ones in [Escrig and Toledo, 1998] [Escrig and Toledo, 2002], respectively.

Acknowledgements

The authors want to thank the support received from the Robotic Intelligence Lab at Universitat Jaume I, funded by Fundació Caixa-Castelló-Bancaixa (P1-1B2008-51)

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