Reasoning with Qualitative Velocity

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Abstract

The concept of qualitative velocity, together with qualitative distance and orientation, are very important in order to represent spatial reasoning for moving objects, such as robots. We consider the propositional dynamic logic which deals with qualitative velocity and enables us to represent some reasoning tasks about qualitative properties. The use of logic provides a general framework which improves the capacity of reasoning. This way, we can infer additional information by using axioms and the logic apparatus. In this paper we present sound and complete relational dual tableau that can be used for verification of validity of formulas of the logic in question.

1 Introduction

Qualitative reasoning, QR, tries to simulate the way of humans think in almost all situations. For example, we do not need to know the exact value of velocity and position of a car in order to drive it. As said in [Delafontaine *et al.*, 2011], when raising or answering questions about moving objects, both qualitative and quantitative responses are possible. However, human beings are more likely to prefer to communicate in qualitative categories, supporting their intuition, rather than using quantitative measures. On the other hand, representing and reasoning with qualitative information can overcome information overload, that is, more information has to be handled than can be processed.

A form of QR is order of magnitude reasoning, where the values are represented by different qualitative classes. For example, talking about velocity we may consider *slow*, *normal*, and *quick* as qualitative classes.

The use of logic in QR, as in other areas of AI, provides a general framework which improves the capacity of solving problems and, as we will see in this paper, allows us to deal with the reasoning problem. This way, we can infer additional information by using axioms and the logic apparatus. There are several applications of logics for QR (see e.g., [Bennett *et al.*, 2002; Duckham *et al.*, 2006]) and many of them concern spatial reasoning. As an example of logic for order of magnitude reasoning, see [Burrieza *et al.*, 2010]; a theorem prover for one of these logics can be seen in [Golińska-Pilarek and Muñoz-Velasco, 2009], implemented in [Golińska-Pilarek *et al.*, 2008].

The concept of qualitative velocity [Escrig and Toledo, 2002; Stolzenburg *et al.*, 2002], together with qualitative distance and orientation, are very important in order to represent spatial reasoning for moving objects, such as robots. Recent papers [Cohn and Renz, 2007; Liu *et al.*, 2009; 2008] try to make progress in the development of qualitative kinematics models, as given in [Forbus *et al.*, 1987; Nielsen, 1988; Faltings, 1992]. The problem of the relative movement of one physical object with respect to another can been treated by the Region Connection Calculus [Randell *et al.*, 1992] and the Qualitative Trajectory Calculus [Van de Weghe *et al.*, 2005; Delafontaine *et al.*, 2011]. However, as far as we know, the first paper which proposes a logic framework for qualitative velocity is [Burrieza *et al.*, 2011], where the Propositional Dynamic Logic for order of magnitude qualitative to deal with the concept of qualitative velocity is proposed. The main advantages of this approach are: the possibility of constructing complex relations from simpler ones; the flexibility for using different levels of granularity; its possible extension by adding other spatial components, such as position, distance, cardinal directions, etc.; the use of a language close to programming languages; and, above all, the strong support of logic in spatial reasoning. Following [Escrig and Toledo, 2002], velocity of an object B with respect to another object A is represented by two components: module and orientation, each one given by a qualitative class. If we consider a velocity of B with respect to A, and another velocity of C with respect to B, the composition of these two velocities consists of obtaining the velocity of C with respect to A. For example, if (Q,l) represents a *quick* velocity towards the *left* orientation of B with respect to A, and (N,r) is a *normal* velocity towards the *right* of C with respect to B, the composition is a velocity of C with respect to A, that could be either (Q,l) or (N,l), that is, a *quick* or *normal* velocity towards the left orientation. The results of these compositions could depend on the specific problem we are dealing with. In the following section, we consider the logic QV where some assumptions about these compositions are posed in its models.

In this paper we present sound and complete relational dual tableau for the Propositional Dynamic Logic of qualitative velocity introduced in [Burrieza *et al.*, 2011], which can be used to verification of validity of its formulas. The system is based on Rasiowa-Sikorski diagrams for first-order logic [Rasiowa and Sikorski, 1960]. The common language of most of relational dual tableaux is the logic of binary relations, which is a logical counterpart to the class RRA of (representable) relation algebras introduced by [Tarski, 1941]. The formulas of the classical logic of binary relations are intended to represent statements saying that two objects are related. Relations are specified in the form of relational terms. Terms are built from relational variables and/or relational constants with relational operations of union, intersection, complement, composition, and converse.

Relational dual tableaux are powerful tools for verification of validity as well as for proving entailment, model checking (i.e., verification of truth of a statement in a particular fixed finite model) and satisfaction (i.e., verification that a statement is satisfied by some fixed objects of a finite model). A comprehensive survey on applications of dual tableaux methodology to various theories and logics can be found in [Orłowska and Golińska-Pilarek, 2011]. The main advantage of relational methodology is the possibility of representation within a uniform formalism the three basic components of formal systems: syntax, semantics, and deduction apparatus. Hence, the relational approach provides a general framework for representation, investigation and implementation of theories with different languages and/or semantics.

The paper is organized as follows. In Section 2 we present the Propositional Dynamic Logic of qualitative velocity, QV, its syntax and semantics. Relational formalization of the logic is presented in Section 3. In Section 4 we present the relational dual tableau for this logic, and we prove its soundness and completeness; moreover, we show an example of the relational proof of validity of a formula. Conclusions and final remarks are discussed in Section 5.

2 Logic QV for reasoning with qualitative velocity

In this section we present the syntax and semantics of the logic QV for order of magnitude qualitative reasoning to deal with the concept of qualitative velocity. We consider the set of qualitative velocities $L_1 = \{\mathsf{z}, \mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3\}$, where $\mathsf{z}, \mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$ represent zero, slow, normal, and quick, respectively; and the set of qualitative orientations $L_2 = \{\mathsf{n}, \mathsf{o}_1, \mathsf{o}_2, \mathsf{o}_3, \mathsf{o}_4\}$ representing none, front, right, back, and left orientations, respectively. Thus, we consider four qualitative classes for the module of the velocities, and five qualitative classes for the orientation of the velocity. Orientations o_j and $\mathsf{o}_{\mathsf{j}+2}$, for $\mathsf{j} \in \{1, 2\}$, are interpreted as *opposite*. Furthermore, orientations o_j and $\mathsf{o}_{\mathsf{j}+1}$, for $\mathsf{j} \in \{1, 2, 3\}$, are interpreted as *perpendicular*.

The logic QV is an extension of propositional dynamic logic PDL which is a framework for specification and verification of dynamic properties of systems. It is a multimodal logic with the modal operations of necessity and possibility determined by binary relations understood as state transition relations or input-output relations associated with computer programs. The vocabulary of the language of QV consists of symbols from the following pairwise disjoint sets: $\mathbb{V}$ - a countably infinite set of propositional variables; $\mathbb{C} = L_1 \times L_2$ - the set of constants representing labels from the set $L_1 \times L_2$; $\mathbb{SP} = \{\otimes_\star | \star \in \mathbb{C}\}$ - the set of relational constants representing specific programs; $\{\cup, ;, ?, ^*\}$ - the set of relational operations, where $\cup$ is interpreted as a nondeterministic choice, ; is interpreted as a sequential composition of programs, ? is the test operation, and * is interpreted as a nondeterministic iteration; $\{\neg, \vee, \wedge, \rightarrow, [\,], \langle\rangle\}$ - the set of propositional operations of negation, disjunction, conjunction, implication, necessity, and possibility, respectively.

The set of QV-relational terms interpreted as compound programs and the set of QV-formulas are the smallest sets containing $\mathbb{SP}$ and $\mathbb{V} \cup \{\bot\} \cup \mathbb{C}$, respectively, and satisfying the following conditions:

- If S and T are QV-relational terms, then so are $S \cup T$, $S\,; T$, and T^*.
- If φ is a QV-formula, then $\varphi?$ is a QV-relational term.
- If φ and ψ are QV-formulas, then so are $\neg\varphi$, $\varphi\vee\psi$, $\varphi\wedge\psi$, and $\varphi \rightarrow \psi$.
- If φ is a QV-formula and T is a QV-relational term, then $[T]\varphi$ and $\langle T\rangle\varphi$ are QV-formulas.

Given a binary relation R on a set W and $X \subseteq W$, we define:

$$R(X) \stackrel{\text{df}}{=} \{w \in W \mid \exists x \in X,\ (x, w) \in R\}.$$

Fact 1 *For every binary relation R on a set W and for all $X, Y \subseteq W$:*

$$R(X) \subseteq Y \textit{ iff } (R^{-1}\,; (X \times W)) \subseteq (Y \times W).$$

A QV-*model* is a structure $\mathcal{M} = (W, m)$, where W is a nonempty set of states and m is a meaning function satisfying the following conditions:

- $W = \bigcup_{\star\in\mathbb{C}} \star$ where all $\star$'s are pairwise disjoint subsets of states understood as states of objects affected by a qualitative velocity
- $m(p) \subseteq W$ for every $p \in \mathbb{V}$
- $m(\star) = \star$, for every $\star \in \mathbb{C}$
- $m(\otimes_\star) \subseteq W \times W$, for every $\otimes_\star \in \mathbb{SP}$, and for all $\mathsf{v}, \mathsf{v}_\mathsf{r}, \mathsf{v}_\mathsf{s} \in L_1$ and for all $\mathsf{o}, \mathsf{o}_\mathsf{j}, \mathsf{o}_{\mathsf{j}+1}, \mathsf{o}_{\mathsf{j}+2} \in L_2$, the following hold:
  - (S1) $m(\otimes_{(\mathsf{v},\mathsf{o})})\,; m(\otimes_{(\mathsf{z},\mathsf{n})}) = m(\otimes_{(\mathsf{v},\mathsf{o})})$
  - (S2) $m(\otimes_{(\mathsf{v},\mathsf{o}_\mathsf{j})})\,; m(\otimes_{(\mathsf{v},\mathsf{o}_{\mathsf{j}+2})}) = m(\otimes_{(\mathsf{z},\mathsf{n})})$, for $\mathsf{j} \in \{1, 2\}$
  - (S3) $m(\otimes_{(\mathsf{v},\mathsf{o}_{\mathsf{j}+1})})(m(\mathsf{v}, \mathsf{o}_\mathsf{j})) \subseteq m(\mathsf{v}, \mathsf{o}_\mathsf{j}) \cup m(\mathsf{v}, \mathsf{o}_{\mathsf{j}+1})$, for $\mathsf{j} \in \{1, 2, 3\}$
  - (S4) $m(\otimes_{(\mathsf{v}_\mathsf{s},\mathsf{o}_{\mathsf{j}+1})})(m(\mathsf{v}_\mathsf{r}, \mathsf{o}_\mathsf{j})) \subseteq m(\mathsf{v}_\mathsf{s}, \mathsf{o}_{\mathsf{j}+1})$, for $\mathsf{j} \in \{1, 2, 3\}$ and $\mathsf{r} < \mathsf{s}$
  - (S5) $m(\otimes_{(\mathsf{v}_\mathsf{s},\mathsf{o})})(m(\mathsf{v}_\mathsf{r}, \mathsf{o})) \subseteq m(\mathsf{v}_\mathsf{s}, \mathsf{o}) \cup m(\mathsf{v}_3, \mathsf{o})$, for $\mathsf{s} \in \{2, 3\}$ and $\mathsf{r} < \mathsf{s}$
  - (S6) $m(\otimes_{(\mathsf{v}_\mathsf{s},\mathsf{o}_{\mathsf{j}+2})})(m(\mathsf{v}_\mathsf{r}, \mathsf{o}_\mathsf{j})) \subseteq m(\mathsf{v}_\mathsf{s}, \mathsf{o}_{\mathsf{j}+2}) \cup m(\mathsf{v}_{\mathsf{s}-1}, \mathsf{o}_{\mathsf{j}+2})$, for $\mathsf{j} \in \{1, 2\}$, $\mathsf{s} \in \{2, 3\}$, and $\mathsf{r} < \mathsf{s}$

m extends to all the compound QV-relational terms and formulas:

- $m(T^*) = m(T)^* = \bigcup_{i \geq 0} m(T^i)$, where T^0 is the identity relation on W and $T^{i+1} \stackrel{\text{df}}{=} (T^i \,; T)$
- $m(S \cup T) = m(S) \cup m(T)$
- $m(S \,; T) = m(S) \,; m(T)$
- $m(\varphi?) = \{(s, s) \in W \times W : s \in m(\varphi)\}$
- $m(\neg\varphi) = W \setminus m(\varphi)$
- $m(\varphi \vee \psi) = m(\varphi) \cup m(\psi)$
- $m(\varphi \wedge \psi) = m(\varphi) \cap m(\psi)$
- $m(\varphi \rightarrow \psi) = m(\neg\varphi) \cup m(\psi)$
- $m([T]\varphi) = \{s \in W |$ for all $t \in W$, if $(s, t) \in m(T)$, then $t \in m(\varphi)\}$
- $m(\langle T \rangle\varphi) = \{s \in W |$ exists $t \in W$ such that $(s, t) \in m(T)$ and $t \in m(\varphi)\}$

Given a QV-model $\mathcal{M} = (W, m)$, a QV-formula φ is said to be *satisfied in* $\mathcal{M}$ *by* $s \in W$, $\mathcal{M}, s \models \varphi$ for short, whenever $s \in m(\varphi)$. As usual, a formula is true in a model whenever it is satisfied in all states of the model and it is QV-valid if it is true in all QV-models.

Intuitively, $(s, s') \in m(T)$ means that there exists a computation of program T starting in the state s and terminating in the state s'. Program $S \cup T$ performs S or T nondeterministically; program $S \,; T$ performs first S and then T. Expression $\varphi?$ is a command to continue if φ is true, and fail otherwise. Program T^* performs T zero or more times sequentially. For example, the formula $\langle(\mathsf{v}_1, \mathsf{o}_4)?\rangle\varphi$ is satisfied in s whenever s is a slow velocity towards the left orientation and φ is satisfied in s; the formula $[\otimes^*_{(\mathsf{v}_3,\mathsf{o}_2)}]\varphi$ is satisfied in s iff for every velocity s' obtained by the repetition of the composition of s with a quick velocity towards the right orientation a nondeterministically chosen finite number of times, φ is satisfied in s'; the formula $[\otimes_{(\mathsf{v}_1,\mathsf{o}_4)} \,; \otimes_{(\mathsf{v}_2,\mathsf{o}_2)}]\varphi$ is satisfied in s iff for every velocity s' obtained by composing s with a slow velocity towards the left orientation followed by a normal velocity towards the right orientation, φ is satisfied in s'.

3 Relational representation of logic QV

In this section we present the relational formalization of logic QV providing a framework for deduction in logic QV. First, we define the relational logic $\mathsf{RL_{QV}}$ appropriate for expressing QV-formulas. Then, we translate all QV-formulas into relational terms and we show the equivalence of validity between a modal formula and its corresponding relational formula. The vocabulary of the language of the relational logic $\mathsf{RL_{QV}}$ consists of symbols from the following pairwise disjoint sets: $\mathbb{OV} = \{x, y, z, \ldots\}$ - a countably infinite set of object variables; $\mathbb{RV} = \{P, Q, \ldots\}$ - a countably infinite set of binary relational variables; $\mathbb{RC} = \{1, 1'\} \cup \{R_\star, \Psi_\star | \star \in \mathbb{C}\}$ - the set of relational constants, where $\mathbb{C}$ is defined as in QV-models; $\mathbb{OP} = \{-, \cup, \cap, ;, {}^{-1}, {}^*\}$ - the set of relational operation symbols representing the usual operations on relations (complement $(-)$, union $(\cup)$, intersection $(\cap)$, composition $(;)$, and converse $({}^{-1})$) and the specific operation of iteration of a relation $({}^*)$. The intuitive meaning of the relational representation of the symbols of logic QV is as follows: propositional variables are represented by relational variables; constants from $\mathbb{C}$ are represented by relational constants $\Psi_\star$ interpreted as right ideal binary relations; relational constants $R_\star$ correspond to specific programs $\otimes_\star$; the relational constants 1 (the universal relation), $1'$ (the identity relation), and relational operations are used to represent compound QV-formulas.

The set of $\mathsf{RL_{QV}}$-terms is the smallest set containing relational variables and relational constants and closed on all the relational operations. $\mathsf{RL_{QV}}$-formulas are of the form xTy, where T is an $\mathsf{RL_{QV}}$-relational term and x, y are object variables. An $\mathsf{RL_{QV}}$-model is a structure $\mathcal{M} = (W, m)$ where W is defined as in QV-models and m is the meaning function that satisfies:

- $m(P) \subseteq W \times W$, for every $P \in \mathbb{RV} \cup \{R_\star | \star \in \mathbb{C}\}$
- $m(\Psi_\star) = \star \times W$, for every $\star \in \mathbb{C}$
- $m(1')$ is an equivalence relation on W
- $m(1') \,; m(P) = m(P) \,; m(1') = m(P)$, for every $P \in \mathbb{RV} \cup \mathbb{RC}$ (the extensionality property)
- $m(1) = W \times W$
- For all $\mathsf{v}, \mathsf{v_r}, \mathsf{v_s} \in L_1$ and for all $\mathsf{o}, \mathsf{o_j}, \mathsf{o_{j+1}}, \mathsf{o_{j+2}} \in L_2$, the following hold:

(RS1) $m(R_{(\mathsf{v},\mathsf{o})}) \,; m(R_{(\mathsf{z},\mathsf{n})}) = m(R_{(\mathsf{v},\mathsf{o})})$

(RS2) $m(R_{(\mathsf{v},\mathsf{o_j})}) \,; m(R_{(\mathsf{v},\mathsf{o_{j+2}})}) = m(R_{(\mathsf{z},\mathsf{n})})$, for $\mathsf{j} \in \{1, 2\}$

(RS3) $m(R_{(\mathsf{v},\mathsf{o_{j+1}})})^{-1} \,; m(\Psi_{(\mathsf{v},\mathsf{o_j})}) \subseteq m(\Psi_{(\mathsf{v},\mathsf{o_j})}) \cup m(\Psi_{(\mathsf{v},\mathsf{o_{j+1}})})$, for $\mathsf{j} \in \{1, 2, 3\}$

(RS4) $m(R_{(\mathsf{v_s},\mathsf{o_{j+1}})})^{-1} \,; m(\Psi_{(\mathsf{v_r},\mathsf{o_j})}) \subseteq m(\Psi_{(\mathsf{v_s},\mathsf{o_{j+1}})})$, for $\mathsf{j} \in \{1, 2, 3\}$ and $\mathsf{r} < \mathsf{s}$

(RS5) $m(R_{(\mathsf{v_s},\mathsf{o})})^{-1} \,; m(\Psi_{(\mathsf{v_r},\mathsf{o})}) \subseteq m(\Psi_{(\mathsf{v_s},\mathsf{o})}) \cup m(\Psi_{(\mathsf{v_3},\mathsf{o})})$, for $\mathsf{s} \in \{2, 3\}$ and $\mathsf{r} < \mathsf{s}$

(RS6) $m(R_{(\mathsf{v_s},\mathsf{o_{j+2}})})^{-1} \,; m(\Psi_{(\mathsf{v_r},\mathsf{o_j})}) \subseteq m(\Psi_{(\mathsf{v_s},\mathsf{o_{j+2}})}) \cup m(\Psi_{(\mathsf{v_{s-1}},\mathsf{o_{j+2}})})$, for $\mathsf{j} \in \{1, 2\}$, $\mathsf{s} \in \{2, 3\}$, and $\mathsf{r} < \mathsf{s}$

- m extends to all the compound relational terms as follows:

$$m(-T) = m(1) \cap -m(T),$$
$$m(S \cup T) = m(S) \cup m(T),$$
$$m(S \cap T) = m(S) \cap m(T),$$
$$m(T^{-1}) = m(T)^{-1},$$
$$m(S \,; T) = m(S) \,; m(T),$$
$$m(T^*) = m(T)^*.$$

Symbols $-$, $\cup$, $\cap$, ${}^{-1}$, and $;$ occurring at the right sides of equalities above denote the usual operations on relations of complement, union, intersection, converse, and composition, respectively.

Observe that the conditions (RS1), ..., (RS6) are relational counterparts of the conditions (S1), ..., (S6) assumed in QV-models. An $\mathsf{RL_{QV}}$-model $\mathcal{M}$ in which $1'$ is interpreted as the identity is said to be *standard*. Let $v \colon \mathbb{OV} \rightarrow W$ be a valuation in an $\mathsf{RL_{QV}}$-model $\mathcal{M}$. An $\mathsf{RL_{QV}}$-formula xTy is said to be satisfied in $\mathcal{M}$ by v whenever $(v(x), v(y)) \in m(T)$. A formula φ is true in $\mathcal{M}$ if it is satisfied in $\mathcal{M}$ by all the valuations and it is $\mathsf{RL_{QV}}$-valid whenever it is true in all $\mathsf{RL_{QV}}$-models.

Now, we define the translation τ of QV-terms and QV-formulas into $\mathsf{RL_{QV}}$-relational terms. Let τ' be a one-to-one mapping that assigns relational variables to propositional variables. The translation τ is defined as follows:

- $\tau(p) = (\tau'(p)\,;1)$, for every $p \in \mathbb{V}$
- $\tau(\star) = \Psi_\star$, for every $\star \in \mathbb{C}$
- $\tau(\otimes_\star) = R_\star$, for every $\otimes_\star \in \mathbb{SP}$

For all relational terms T and S:

- $\tau(T^*) = \tau(T)^*$
- $\tau(S \cup T) = \tau(S) \cup \tau(T)$
- $\tau(S\,;T) = \tau(S)\,;\tau(T)$
- $\tau(\neg\varphi) = -\tau(\varphi)$
- $\tau(\varphi?) = 1' \cap \tau(\varphi)$
- $\tau(\varphi \vee \psi) = \tau(\varphi) \cup \tau(\psi)$
- $\tau(\varphi \wedge \psi) = \tau(\varphi) \cap \tau(\psi)$
- $\tau(\varphi \to \psi) = \tau(\neg\varphi \vee \psi)$
- $\tau(\langle T\rangle\varphi) = \tau(T)\,;\tau(\varphi)$
- $\tau([T]\varphi) = -(\tau(T)\,;-\tau(\varphi))$.

Relational terms obtained from formulas of logic QV include both declarative information and procedural information provided by these formulas. The declarative part which represents static facts about a domain is represented by means of a Boolean reduct of algebras of relations, and the procedural part, which is intended to model dynamics of the domain, requires the relational operations. In the relational terms which represent the formulas after the translation, the two types of information receive a uniform representation and the process of reasoning about both statics and dynamics, and about relationships between them can be performed within the framework of a single uniform formalism.

Theorem 1
For every QV-formula φ and for all object variables x and y, the following conditions are equivalent:

1. *φ is QV-valid.*
2. *$x\tau(\varphi)y$ is $\mathsf{RL_{QV}}$-valid.*

4 Relational dual tableau for QV

In this section we present a dual tableau for the logic $\mathsf{RL_{QV}}$ that can be used for verification of validity of QV-formulas. Relational dual tableaux are determined by the axiomatic sets of formulas and rules which apply to finite sets of relational formulas. The axiomatic sets take the place of axioms. The rules are intended to reflect properties of relational operations and constants. There are two groups of rules: decomposition rules and specific rules. Although most often the rules of dual tableaux are finitary, the dual tableau system for logic QV includes an infinitary rule reflecting the behaviour of an iteration operation. Given a formula, the decomposition rules of the system enable us to transform it into simpler formulas, or the specific rules enable us to replace a formula by some other formulas. The rules have the following general form:

$$\text{(rule)} \qquad \frac{\Phi(\overline{x})}{\Phi_1(\overline{x}_1, \overline{u}_1, \overline{w}_1) \mid \ldots \mid \Phi_n(\overline{x}_n, \overline{u}_n, \overline{w}_n) \mid \ldots}$$

where $n \in J$, for some (possibly infinite) set J, $\Phi(\overline{x})$ is a finite (possibly empty) set of formulas whose object variables are among the elements of set$(\overline{x})$, where $\overline{x}$ is a finite sequence of object variables and set$(\overline{x})$ is a set of elements of sequence $\overline{x}$; every $\Phi_j(\overline{x}_j, \overline{u}_j, \overline{w}_j)$, $j \in J$, is a finite non-empty set of formulas, whose object variables are among the elements of set$(\overline{x}_j)$ $\cup$ set$(\overline{u}_j)$ $\cup$ set$(\overline{w}_j)$, where $\overline{x}_j, \overline{u}_j, \overline{w}_j$ are finite sequences of object variables such that set$(\overline{x}_j) \subseteq$ set$(\overline{x})$, set$(\overline{u}_j)$ consists of the object variables that may be instantiated to arbitrary object variables when the rule is applied (usually to the object variables that appear in the set to which the rule is being applied), set$(\overline{w}_j)$ consists of the object variables that must be instantiated to pairwise distinct new variables (not appearing in the set to which the rule is being applied) and distinct from any variable of sequence $\overline{u}_j$. A rule of the form (rule) is *applicable* to a finite set X of formulas whenever $\Phi(\overline{x}) \subseteq X$. As a result of an application of a rule of the form (rule) to set X, we obtain the sets $(X \setminus \Phi(\overline{x})) \cup \Phi_j(\overline{x}_j, \overline{u}_j, \overline{w}_j)$, for every $j \in J$. A set to which a rule is applied is called the *premise* of the rule, and the sets obtained by the application of the rule are called its *conclusions*. If the set J is finite, then a rule of the form (rule) is said to be *finitary*, otherwise it is referred to as *infinitary*. Thus, if J has n elements, then the rule of the form (rule) has n conclusions.

A finite set $\{\varphi_1, \ldots, \varphi_n\}$ of $\mathsf{RL_{QV}}$-formulas is said to be an $\mathsf{RL_{QV}}$-*set* whenever for every $\mathsf{RL_{QV}}$-model $\mathcal{M}$ and for every valuation v in $\mathcal{M}$ there exists $i \in \{1, \ldots, n\}$ such that φ_i is satisfied by v in $\mathcal{M}$. It follows that the first-order disjunction of all the formulas from an $\mathsf{RL_{QV}}$-set is valid in the first-order logic. A rule of the form (rule) is $\mathsf{RL_{QV}}$-*correct* whenever for every finite set X of $\mathsf{RL_{QV}}$-formulas, $X \cup \Phi(\overline{x})$ is an $\mathsf{RL_{QV}}$-set if and only if $X \cup \Phi_j(\overline{x}_j, \overline{u}_j, \overline{w}_j)$ is an $\mathsf{RL_{QV}}$-set, for every $j \in J$, i.e., the rule preserves and reflects validity. It follows that ',' (comma) in the rules is interpreted as disjunction and '|' (branching) is interpreted as conjunction.

$\mathsf{RL_{QV}}$-dual tableau includes decomposition rules of the following forms, for any object variables x and y and for any relational terms S and T:

$$(\cup)\ \frac{x(S \cup T)y}{xSy, xTy} \qquad (-\cup)\ \frac{x{-}(S \cup T)y}{x{-}Sy \mid x{-}Ty}$$

$$(\cap)\ \frac{x(S \cap T)y}{xSy \mid xTy} \qquad (-\cap)\ \frac{x{-}(S \cap T)y}{x{-}Sy, x{-}Ty}$$

$$(-)\ \frac{x{-}{-}Ty}{xTy}$$

$$(^{-1})\ \frac{xT^{-1}y}{yTx} \qquad (-^{-1})\ \frac{x{-}T^{-1}y}{y{-}Tx}$$

$$(;)\quad \frac{x(S\,;T)y}{xSz, x(S\,;T)y \mid zTy, x(S\,;T)y}$$
for any object variable z

$$(-;)\quad \frac{x-(S\,;T)y}{x-Sz, z-Ty} \qquad (^*)\quad \frac{xT^*y}{xT^iy, xT^*y}$$
for a new object variable z

$$(-^*)\quad \frac{x-(T^*)y}{x-(T^0)y \mid \ldots \mid x-(T^i)y \mid \ldots}$$
for any $i \geq 0$ where $T^0 = 1'$, $T^{i+1} = T\,;T^i$

Below we list the specific rules of $\mathsf{RL_{QV}}$-dual tableau.

For all object variables x, y, z and for every relational term $T \in \mathbb{RC}$:

$$(1'1)\quad \frac{xTy}{xTz, xTy \mid y1'z, xTy}$$
$$(1'2)\quad \frac{xTy}{x1'z, xTy \mid zTy, xTy}$$

For every $\star \in \mathbb{C}$ and for all object variables x and y:

$$(\text{right})\quad \frac{x\Psi_\star y}{x\Psi_\star z, x\Psi_\star y}$$
for any object variable z

For every $T \in \{R_{(\mathsf{z},\mathsf{n})}\} \cup \{R_{(\mathsf{v_i},\mathsf{o_j})} \mid 1 \leq \mathsf{i} \leq 3, 1 \leq \mathsf{j} \leq 4\} \cup \{\Psi_\star | \star \in \mathbb{C}\}$ and for all object variables x and y:

$$(\text{cut})\quad \frac{}{xTy \mid x-Ty}$$

For all $\mathsf{v}, \mathsf{v_r}, \mathsf{v_s} \in L_1$, $\mathsf{o}, \mathsf{o_j}, \mathsf{o_{j+1}}, \mathsf{o_{j+2}} \in L_2$, and for all object variables x and y:

$$(r1 \subseteq)\quad \frac{xR_{(\mathsf{v},\mathsf{o})}y}{xR_{(\mathsf{v},\mathsf{o})}z, xR_{(\mathsf{v},\mathsf{o})}y \mid zR_{(\mathsf{z},\mathsf{n})}y, xR_{(\mathsf{v},\mathsf{o})}y}$$
for any object variable z

$$(r1 \supseteq)\quad \frac{x-R_{(\mathsf{v},\mathsf{o})}y}{x-R_{(\mathsf{v},\mathsf{o})}z, z-R_{(\mathsf{z},\mathsf{n})}y, x-R_{(\mathsf{v},\mathsf{o})}y}$$
for a new object variable z

$$(r2 \subseteq)\quad \frac{xR_{(\mathsf{z},\mathsf{n})}y}{xR_{(\mathsf{v},\mathsf{o_j})}z, xR_{(\mathsf{z},\mathsf{n})}y \mid zR_{(\mathsf{v},\mathsf{o_{j+2}})}y, xR_{(\mathsf{z},\mathsf{n})}y}$$
for any object variable z and $\mathsf{j} \in \{1, 2\}$

$$(r2 \supseteq)\quad \frac{x-R_{(\mathsf{z},\mathsf{n})}y}{x-R_{(\mathsf{v},\mathsf{o_j})}z, z-R_{(\mathsf{v},\mathsf{o_{j+2}})}y, x-R_{(\mathsf{z},\mathsf{n})}y}$$
for a new object variable z and $\mathsf{j} \in \{1, 2\}$

$$(r3)\quad \frac{x\Psi_{(\mathsf{v},\mathsf{o_j})}y, x\Psi_{(\mathsf{v},\mathsf{o_{j+1}})}y}{zR_{(\mathsf{v},\mathsf{o_{j+1}})}x, K \mid z\Psi_{(\mathsf{v},\mathsf{o_j})}y, K}$$
for any object variable z

$\mathsf{j} \in \{1, 2, 3\}$ and $K = x\Psi_{(\mathsf{v},\mathsf{o_j})}y, x\Psi_{(\mathsf{v},\mathsf{o_{j+1}})}y$

$$(r4)\quad \frac{x\Psi_{(\mathsf{v_s},\mathsf{o_{j+1}})}y}{zR_{(\mathsf{v_s},\mathsf{o_{j+1}})}x, x\Psi_{(\mathsf{v_s},\mathsf{o_{j+1}})}y \mid z\Psi_{(\mathsf{v_r},\mathsf{o_j})}y, x\Psi_{(\mathsf{v_s},\mathsf{o_{j+1}})}y}$$
for any object variable z and $\mathsf{j} \in \{1, 2, 3\}$ and $\mathsf{r} < \mathsf{s}$

$$(r5)\quad \frac{x\Psi_{(\mathsf{v_s},\mathsf{o})}y, x\Psi_{(\mathsf{v_3},\mathsf{o})}y}{zR_{(\mathsf{v_s},\mathsf{o})}x, K \mid z\Psi_{(\mathsf{v_r},\mathsf{o})}y, K}$$
for any object variable z

$\mathsf{s} \in \{2, 3\}$ and $\mathsf{r} < \mathsf{s}$ and $K = x\Psi_{(\mathsf{v_s},\mathsf{o})}y, x\Psi_{(\mathsf{v_3},\mathsf{o})}y$

$$(r6)\quad \frac{x\Psi_{(\mathsf{v_s},\mathsf{o_{j+2}})}y, x\Psi_{(\mathsf{v_{s-1}},\mathsf{o_{j+2}})}y}{zR_{(\mathsf{v_s},\mathsf{o_{j+2}})}x, K \mid z\Psi_{(\mathsf{v_r},\mathsf{o_j})}y, K}$$
for any object variable z and $\mathsf{j} \in \{1, 2\}$, $\mathsf{s} \in \{2, 3\}$,

$\mathsf{r} < \mathsf{s}$, and $K = x\Psi_{(\mathsf{v_s},\mathsf{o_{j+2}})}y, x\Psi_{(\mathsf{v_{s-1}},\mathsf{o_{j+2}})}y$

A set of $\mathsf{RL_{QV}}$-formulas is said to be an $\mathsf{RL_{QV}}$-*axiomatic set* whenever it includes a subset of either of the following forms, for all object variables x, y for every relational term T, for any $\star \in \mathbb{C}$, and for any $\# \in \mathbb{C} \setminus \{\star\}$:

(Ax1) $\{x1'x\}$

(Ax2) $\{x1y\}$

(Ax3) $\{xTy, x-Ty\}$

(Ax4) $\bigcup_{\star\in\mathbb{C}}\{x\Psi_\star y\}$

(Ax5) $\{x-\Psi_\star y, x-\Psi_\# y\}$

Let φ be an $\mathsf{RL_{QV}}$-formula. An $\mathsf{RL_{QV}}$-*proof tree for* φ is a tree with the following properties:

- The formula φ is at the root of this tree.
- Each node except the root is obtained by an application of an $\mathsf{RL_{QV}}$-rule to its predecessor node.
- A node does not have successors whenever its set of formulas is an $\mathsf{RL_{QV}}$-axiomatic set or none of the rules is applicable to its set of formulas.

Observe that the proof trees are constructed in the top-down manner, and hence every node has a single predecessor node.

A branch of an $\mathsf{RL_{QV}}$-proof tree is said to be *closed* whenever it contains a node with an $\mathsf{RL_{QV}}$-axiomatic set of formulas. A tree is *closed* iff all of its branches are closed. An $\mathsf{RL_{QV}}$-formula φ is $\mathsf{RL_{QV}}$-*provable* whenever there is a closed $\mathsf{RL_{QV}}$-proof tree for it which is then refereed to as its $\mathsf{RL_{QV}}$-*proof.*

4.1 Soundness

In order to prove that an $\mathsf{RL_{QV}}$-provable formula is $\mathsf{RL_{QV}}$-valid it suffices to show that all the axiomatic sets are $\mathsf{RL_{QV}}$-valid and all the rules of an $\mathsf{RL_{QV}}$-dual tableau preserve and reflect validity of sets which are their premisses and conclusions.

Proposition 1

1. *The* $\mathsf{RL_{QV}}$*-rules are* $\mathsf{RL_{QV}}$*-correct.*
2. *The* $\mathsf{RL_{QV}}$*-axiomatic sets are* $\mathsf{RL_{QV}}$*-sets.*

Due to Proposition 1, we obtain:

Theorem 2 (Soundness)
Let φ *be an* $\mathsf{RL_{QV}}$*-formula. If* φ *is* $\mathsf{RL_{QV}}$*-provable, then it is* $\mathsf{RL_{QV}}$*-valid.*

Proof
Let φ be an $\mathsf{RL_{QV}}$-provable formula. Then, there exists an $\mathsf{RL_{QV}}$-proof tree of φ such that each of its branches is closed, that is it ends with an $\mathsf{RL_{QV}}$-axiomatic set of formulas. Thus, by Proposition 1, going from the bottom to the top of the tree, we conclude that the set of formulas at the root of the tree is $\mathsf{RL_{QV}}$-valid. □

4.2 Completeness

In order to prove that an $\mathsf{RL_{QV}}$-valid formula has an $\mathsf{RL_{QV}}$-proof, we suppose that the formula does not have any $\mathsf{RL_{QV}}$-proof and we construct a model falsifying a formula in question.

A branch b of an $\mathsf{RL_{QV}}$-proof tree is said to be *complete* whenever it is closed or it satisfies the following $\mathsf{RL_{QV}}$-completion conditions:

For all object variables x and y and for all relational terms S and T:

Cpl($\cup$) (resp. Cpl($-\cap$)) If $x(S \cup T)y \in b$ (resp. $x-(S \cap T)y \in b$), then both $xSy \in b$ (resp. $x-Sy \in b$) and $xTy \in b$ (resp. $x-Ty \in b$), obtained by an application of the rule ($\cup$) (resp. ($-\cap$)).
Cpl($\cap$) (resp. Cpl($-\cup$)) If $x(S \cap T)y \in b$ (resp. $x-(S \cup T)y \in b$), then either $xSy \in b$ (resp. $x-Sy \in b$) or $xTy \in b$ (resp. $x-Ty \in b$), obtained by an application of the rule ($\cap$) (resp. ($-\cup$)).
Cpl($-$) If $x(--T)y \in b$, then $xTy \in b$, obtained by an application of the rule ($-$).
Cpl($^{-1}$) If $xT^{-1}y \in b$, then $yTx \in b$, obtained by an application of the rule ($^{-1}$).
Cpl($-^{-1}$) If $x-T^{-1}y \in b$, then $y-Tx \in b$, obtained by an application of the rule ($-^{-1}$).
Cpl($;$) If $x(S\,;T)y \in b$, then for every object variable z, either $xSz \in b$ or $zTy \in b$, obtained by an application of the rule ($;$).
Cpl($-;$) If $x-(S\,;T)y \in b$, then for some object variable z, both $x-Sz \in b$ and $z-Ty \in b$, obtained by an application of the rule ($-;$).
Cpl(*) If $xT^*y \in b$, then for every $i \geq 0$, $xT^iy \in b$, obtained by an application of the rule (*);
Cpl($-^*$) If $x-(T^*)y \in b$, then for some $i \geq 0$, $x-(T^i)y \in b$, obtained by an application of the rule ($-^*$).

For all object variables x and y and for every relational term $T \in \mathbb{RC}$:

Cpl($1'1$) If $xTy \in b$, then for every object variable z, either $xTz \in b$ or $y1'z \in b$, obtained by an application of the rule ($1'1$). Cpl($1'2$) If $xTy \in b$, then for every object variable z, either $x1'z \in b$ or $zTy \in b$, obtained by an application of the rule ($1'1$).

For every $\star \in \mathbb{C}$ and for all object variables x and y:

Cpl(right) If $x\Psi_\star y \in b$, then for every object variable z, $x\Psi_\star z \in b$, obtained by an application of the rule (right).

For every $T \in \{R_{(\mathsf{v},\mathsf{o})}, R_{(\mathsf{z},\mathsf{n})}\} \cup \{\Psi_\star | \star \in \mathbb{C}\}$ and for all object variables x and y:

Cpl(cut) Either $xTy \in b$ or $x-Ty \in b$, obtained by an application of the rule (cut).

For all $\mathsf{v}, \mathsf{v_r}, \mathsf{v_s} \in L_1$, $\mathsf{o}, \mathsf{o_j}, \mathsf{o_{j+1}}, \mathsf{o_{j+2}} \in L_2$, and for all object variables x and y:

Cpl($r1 \subseteq$) If $xR_{(\mathsf{v},\mathsf{o})}y \in b$, then for every object variable z either $xR_{(\mathsf{v},\mathsf{o})}z \in b$ or $zR_{(\mathsf{z},\mathsf{n})}y \in b$, obtained by an application of the rule ($r1 \subseteq$).
Cpl($r1 \supseteq$) If $x-R_{(\mathsf{v},\mathsf{o})}y \in b$, then for some object variable z both $x-R_{(\mathsf{v},\mathsf{o})}z \in b$ and $z-R_{(\mathsf{z},\mathsf{n})}y \in b$, obtained by an application of the rule ($r1 \supseteq$).
Cpl($r2 \subseteq$) If $xR_{(\mathsf{z},\mathsf{n})}y \in b$, then for every object variable z either $xR_{(\mathsf{v},\mathsf{o_j})}z \in b$ or $zR_{(\mathsf{v},\mathsf{o_{j+2}})}y \in b$, for $\mathsf{j} \in \{1,2\}$, obtained by an application of the rule ($r2 \subseteq$).
Cpl($r2 \supseteq$) If $x-R_{(\mathsf{z},\mathsf{n})}y \in b$, then for some object variable z both $x-R_{(\mathsf{v},\mathsf{o_j})}z \in b$ and $z-R_{(\mathsf{v},\mathsf{o_{j+2}})}y \in b$, for $\mathsf{j} \in \{1,2\}$, obtained by an application of the rule ($r2 \supseteq$).
Cpl($r3$) If $\mathsf{j} \in \{1,2,3\}$ and both $x\Psi_{(\mathsf{v},\mathsf{o_j})}y \in b$ and $x\Psi_{(\mathsf{v},\mathsf{o_{j+1}})}y \in b$, then for every object variable z either $zR_{(\mathsf{v},\mathsf{o_{j+1}})}x \in b$ or $z\Psi_{(\mathsf{v},\mathsf{o_j})}y \in b$, obtained by an application of the rule ($r3$).
Cpl($r4$) If $\mathsf{j} \in \{1,2,3\}, r < s$ and $x\Psi_{(\mathsf{v_s},\mathsf{o_{j+1}})}y \in b$, then for every object variable z either $zR_{(\mathsf{v_s},\mathsf{o_{j+1}})}x \in b$ or $z\Psi_{(\mathsf{v_r},\mathsf{o_j})}y \in b$, obtained by an application of the rule ($r4$).
Cpl($r5$) If $\mathsf{s} \in \{2,3\}, r < s$ and both $x\Psi_{(\mathsf{v_s},\mathsf{o})}y \in b$ and $x\Psi_{(\mathsf{v_3},\mathsf{o})}y \in b$, then for every object variable z either $zR_{(\mathsf{v_s},\mathsf{o})}x \in b$ or $z\Psi_{(\mathsf{v_r},\mathsf{o})}y \in b$, obtained by an application of the rule ($r5$).
Cpl($r6$) If $\mathsf{j} \in \{1,2\}, \mathsf{s} \in \{2,3\}, r < s$ and both $x\Psi_{(\mathsf{v_s},\mathsf{o_{j+2}})}y \in b$ and $x\Psi_{(\mathsf{v_{s-1}},\mathsf{o_{j+2}})}y \in b$, then for every object variable z either $zR_{(\mathsf{v_s},\mathsf{o_{j+2}})}x \in b$ or $z\Psi_{(\mathsf{v_r},\mathsf{o_j})}y \in b$, obtained by an application of the rule ($r6$).

An $\mathsf{RL_{QV}}$-proof tree is said to be *complete* if and only if all of its branches are complete. A complete non-closed branch of an $\mathsf{RL_{QV}}$-proof tree is said to be *open*.

Note that every $\mathsf{RL_{QV}}$-proof tree can be extended to a complete $\mathsf{RL_{QV}}$-proof tree, i.e., for every $\mathsf{RL_{QV}}$-formula φ there exists a complete $\mathsf{RL_{QV}}$-proof tree for φ.

Due to the forms of $\mathsf{RL_{QV}}$-rules, we obtain:

Fact 2
If a node of an $\mathsf{RL_{QV}}$-proof tree contains an $\mathsf{RL_{QV}}$-formula xTy or $x{-}Ty$, for a relational term $T \in \mathbb{RV} \cup \mathbb{RC}$, then all of its successors contain this formula as well.

The above property enable us to have the following result.

Proposition 2
For every complete branch b of an $\mathsf{RL_{QV}}$-proof tree and for all object variables x and y, the following hold:

1. *If there is a relational term T such that $xTy \in b$ and $x{-}Ty \in b$, then b is closed.*
2. *If for every $\star \in \mathbb{C}$ there exists an object variable z such that $x\Psi_\star z \in b$, then b is closed.*
3. *If $x{-}\Psi_\star y \in b$ and $x{-}\Psi_\# y \in b$, for some $\star, \# \in \mathbb{C}$ such that $\star \neq \#$, then b is closed.*

In order to prove completeness of $\mathsf{RL_{QV}}$-dual tableau, first, we construct a branch structure $\mathcal{M}^b$ determined by an open branch b of a complete $\mathsf{RL_{QV}}$-proof tree.

The branch structure is of the form $\mathcal{M}^b = (W^b, m^b)$, where:

- $W^b = \bigcup_{\star \in \mathbb{C}} \star^b$, where $\star^b = \{x \in \mathbb{OV} \mid x\Psi_\star y \notin b$, for some $y \in \mathbb{OV}\}$
- $m^b(T) = \{(x,y) \in W^b \times W^b \mid xTy \notin b\}$, for every $T \in \mathbb{RV} \cup \mathbb{RC}$
- m^b extends to all the compound relational terms as in the $\mathsf{RL_{QV}}$-models.

Proposition 3 (Branch Model Property) *For every open branch b of an $\mathsf{RL_{QV}}$-proof tree, $\mathcal{M}^b$ is an $\mathsf{RL_{QV}}$-model.*

Let $v^b\colon \mathbb{OV} \to W^b$ be a valuation in $\mathcal{M}^b$ such that $v^b(x) = x$, for every $x \in \mathbb{OV}$. Since $W^b = \mathbb{OV}$, the valuation v^b is well defined.

Proposition 4
For every open branch b of an $\mathsf{RL_{QV}}$-proof tree and for every $\mathsf{RL_{QV}}$-formula φ, if $\mathcal{M}^b, v^b \models \varphi$, then $\varphi \notin b$.

Given an $\mathsf{RL_{QV}}$-branch model $\mathcal{M}^b$, since $m^b(1')$ is an equivalence relation on W^b, we may define the quotient model $\mathcal{M}^b_q = (W^b_q, m^b_q)$ as:

- $W^b_q = \{\|x\| | x \in W^b\}$, where $\|x\|$ is the equivalence class of $m^b(1')$ generated by x
- $m^b_q(T) = \{(\|x\|, \|y\|)) \in W^b_q \times W^b_q | (x,y) \in m^b(T)\}$, for every $T \in \mathbb{RV} \cup \mathbb{RC}$
- m^b_q extends to all the compound relational terms as in the $\mathsf{RL_{QV}}$-models.

Since a branch model satisfies the extensionality property, the definition of $m^b_q(T)$ is correct.

Let v^b_q be a valuation in $\mathcal{M}^b_q$ such that $v^b_q(x) = \|x\|$, for every object variable x.

Proposition 5

1. *The model $\mathcal{M}^b_q$ is a standard $\mathsf{RL_{QV}}$-model.*
2. *For every $\mathsf{RL_{QV}}$-formula φ, $\mathcal{M}^b, v^b \models \varphi$ if and only if $\mathcal{M}^b_q, v^b_q \models \varphi$.*

Thus, we obtain:

Theorem 3 (Completeness)
Let φ be an $\mathsf{RL_{QV}}$-formula. If φ is true in all standard $\mathsf{RL_{QV}}$-models, then φ is $\mathsf{RL_{QV}}$-provable.

Proof
Assume φ is true in all standard $\mathsf{RL_{QV}}$-models. Suppose there is no any closed $\mathsf{RL_{QV}}$-proof tree for φ. Then there exists a complete $\mathsf{RL_{QV}}$-proof tree for φ with an open branch, say b. Since $\varphi \in b$, by Proposition 4, φ is not satisfied by v^b in the branch model $\mathcal{M}^b$. By Proposition 5(2), φ is not satisfied by v^b_q in the quotient model $\mathcal{M}^b_q$. Since $\mathcal{M}^b_q$ is a standard $\mathsf{RL_{QV}}$-model, φ is not true in all standard $\mathsf{RL_{QV}}$-models, a contradiction. □

Theorems 1, 2, and 3, imply:

Theorem 4 (Relational Soundness and Completeness)
For every QV-formula φ and for all object variables x and y, the following conditions are equivalent:

1. *φ is QV-valid.*
2. *$x\tau(\varphi)y$ is $\mathsf{RL_{QV}}$-provable.*

Example 1 Let φ be a QV-formula of the following form:

$$\varphi = (\mathsf{v}, \mathsf{o}_1) \to [\otimes_{(\mathsf{v},\mathsf{o}_2)}]((\mathsf{v}, \mathsf{o}_1) \vee (\mathsf{v}, \mathsf{o}_2)).$$

The translation of φ into $\mathsf{RL_{QV}}$-term is:

$$\tau(\varphi) = -\Psi_{(\mathsf{v},\mathsf{o}_1)} \cup -(R_{(\mathsf{v},\mathsf{o}_2)} \,;\, -(\Psi_{(\mathsf{v},\mathsf{o}_1)} \cup \Psi_{(\mathsf{v},\mathsf{o}_2)})).$$

Figure 1 shows $\mathsf{RL_{QV}}$-proof of the formula $x\tau(\varphi)y$, which by Theorem 4 proves QV-validity of φ. In each node of the tree presented in the example we underline the formulas which determine the rule that has been applied during the construction of the tree and we indicate which rule has been applied. If a rule introduces a variable, then we write how the variable has been instantiated. Furthermore, in each node we write only those formulas which are essential for the application of a rule and the succession of these formulas in the node is usually motivated by the reasons of formatting.

5 Conclusions and future work

We presented sound and complete relational dual tableau for verification of validity of QV-formulas. This system is a first step in order to provide a general framework for improving the capacity of reasoning about moving objects. The direction of our future work is twofold. First of all, we will focus on the extension of the logic by considering other spatial components (relative position, closeness, etc.). On the other hand, it would be needed a prover which is a decision procedure based on the dual tableau presented in this paper.

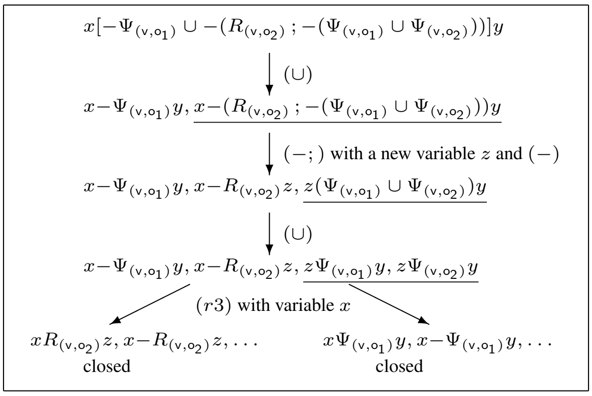

Figure 1: $\mathsf{RL_{QV}}$-proof of QV-validity of the formula $(\mathsf{v}, \mathsf{o}_1) \to [\otimes_{(\mathsf{v},\mathsf{o}_2)}]((\mathsf{v}, \mathsf{o}_1) \vee (\mathsf{v}, \mathsf{o}_2))$.

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References

[Bennett *et al.*, 2002] B. Bennett, A.G. Cohn, F. Wolter, and M. Zakharyaschev. Multi-dimensional modal logic as a framework for spatio-temporal reasoning. *Applied Intelligence*, 17(3):239–251, 2002.

[Burrieza *et al.*, 2010] A. Burrieza, E. Muñoz Velasco, and M. Ojeda-Aciego. Closeness and distance in order of magnitude qualitative reasoning via PDL. *Lecture Notes in Artificial Intelligence*, 5988:71–80, 2010.

[Burrieza *et al.*, 2011] A. Burrieza, E. Muñoz-Velasco, and M. Ojeda-Aciego. A PDL approach for qualitative velocity. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 19(01):11–26, 2011.

[Cohn and Renz, 2007] A.G. Cohn and J. Renz. *Handbook of Knowledge Representation*. Elsevier, 2007.

[Delafontaine *et al.*, 2011] M. Delafontaine, P. Bogaert, A. G. Cohn, F. Witlox, P. De Maeyer, and N. Van de Weghe. Inferring additional knowledge from $\mathsf{QTC_N}$ relations. *Information Sciences*, DOI: 10.1016/j.ins.2010.12.021, 2011.

[Duckham *et al.*, 2006] M. Duckham, J. Lingham, K. Mason, and M. Worboys. Qualitative reasoning about consistency in geographic information. *Information Sciences*, 176(6 6, 22):601–627, 2006.

[Escrig and Toledo, 2002] M. T. Escrig and F. Toledo. Qualitative velocity. *Lecture Notes in Artificial Intelligence*, 2504:28–39, 2002.

[Faltings, 1992] B. Faltings. A symbolic approach to qualitative kinematics. *Artificial Intelligence*, 56(2-3):139–170, 1992.

[Forbus *et al.*, 1987] K.D. Forbus, P. Nielsen, and B. Faltings. Qualitative kinematics: A framework. *Proceedings of the Int. Joint Conference on Artificial Intelligence*, pages 430–437, 1987.

[Golińska-Pilarek and Muñoz-Velasco, 2009] J. Golińska-Pilarek and E. Muñoz-Velasco. Relational approach for a logic for order of magnitude qualitative reasoning with negligibility, non-closeness and distance. *Logic Journal of IGPL*, 17(4):375–394, 2009.

[Golińska-Pilarek *et al.*, 2008] J. Golińska-Pilarek, A. Mora, and E. Muñoz-Velasco. An ATP of a relational proof system for order of magnitude reasoning with negligibility, non-closeness and distance. *Lecture Notes in Artificial Intelligence*, 5351:128–139, 2008.

[Liu *et al.*, 2008] H. Liu, D. J. Brown, and G. M. Coghill. Fuzzy qualitative robot kinematics. *IEEE Transactions on Fuzzy Systems*, 16(3):808–822, 2008.

[Liu *et al.*, 2009] W. Liu, S. Li, and J. Renz. Combining RCC-8 with qualitative direction calculi: Algorithms and complexity. *Proceedings of IJCAI-09*, pages 854–859,, 2009.

[Nielsen, 1988] P.E. Nielsen. A qualitative approach to rigid body mechanics. *University of Illinois at Urbana-Champaign, PhD thesis*, 1988.

[Orłowska and Golińska-Pilarek, 2011] E. Orłowska and J. Golińska-Pilarek. *Dual Tableaux: Foundations, Methodology, Case Studies*, volume 36 of *Trends in Logic*. Springer Science, 2011.

[Randell *et al.*, 1992] D Randell, Z Cui, and A.G. Cohn. A spatial logic based on regions and connection. *Proceedings of KR*, pages 165–176, 1992.

[Rasiowa and Sikorski, 1960] H. Rasiowa and R. Sikorski. On gentzen theorem. *Fundamenta Mathematicae*, 48:57–69, 1960.

[Stolzenburg *et al.*, 2002] F. Stolzenburg, O. Obst, and J. Murray. Qualitative velocity and ball interception. *Lecture Notes in Artificial Intelligence*, 2479:283–298, 2002.

[Tarski, 1941] A. Tarski. On the calculus of relations. *Journal of Symbolic Logic*, 6:73–89, 1941.

[Van de Weghe *et al.*, 2005] N. Van de Weghe, B Kuijpers, P. Bogaert, and P. De Maeyer. A qualitative trajectory calculus and the composition of its relations. *Lecture Notes in Computer Science*, 3799:60–76, 2005.