The Qualitative Difference Resolution Rule

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Abstract

Consolidation is inferring the behavioral description of a device by composing the behavioral descriptions of its components, e.g., deriving the qualitative differential equations (QDEs) of a device from those of its components. In previous work, Dormoy and Raiman described the qualitative resolution rule, which is a general rule for deriving QDEs of combinations of components. However, the qualitative resolution rule is intractable in general. As a step toward understanding tractable qualitative reasoning, I present a new QDE resolution rule, the qualitative difference resolution rule, that supports the tractable consolidation of components in which direction of flow is dependent on the signs of pressure differences. Pipes and containers are general types of components that match this rule. The pressure regulator example also matches this rule.

Introduction

The task of consolidation is to infer the behavioral description of a device from the behavioral descriptions of its components [Bylander and Chandrasekaran, 1985; Bylander, 1991]. For example, if the components of a device are described by qualitative differential equations (QDEs), then the output of consolidation are the QDEs for the device. Consolidation differs from qualitative simulation and envisioning [de Kleer and Brown, 1984; Forbus, 1984; Kuipers, 1986] in that consolidation results in the global laws of the device rather than sequences of device states. These global laws correspond to a kind of device understanding and have the potential for making qualitative simulation more efficient [Dormoy and Raiman, 1988].

In previous work, I proposed and implemented a conceptual representation and reasoning method for performing consolidation [Bylander and Chandrasekaran, 1985; Bylander, 1991]. This approach is primarily based on predicating paths within the components with their conceptual behavior (e.g., allow, expel, pump, move) and inferring the conceptual behavior of path combinations. The method is tractable if few components have behavioral modes; however, the relationship between conceptual behaviors and QDEs was not specified.

In work on QDEs, Dormoy and Raiman [1988] discovered the qualitative resolution rule (QR rule), which can be used to derive the QDEs that follow from a given set of QDEs. Dormoy [1988] showed how the QR rule can be used to perform consolidation, and he provided a heuristic method for using the QR rule. Also, de Kleer [1991] has developed a general method for deriving prime implicates from a set of QDEs. However, these approaches are intractable in general; thus, they leave open the question of when tractable consolidation of QDEs can be performed.

As a step toward answering this question, I present the *qualitative difference resolution rule* (QDR rule). In terms of my previous work on consolidation, this rule corresponds to the inferences of allow, pump, and move behaviors from allow and expel behaviors. A more precise characterization is that the QDR rule supports the tractable consolidation of components in which direction of flow is dependent on the signs of pressure differences. In particular, this rule "resolves" variables corresponding to connections between components. The remaining variables in the final set of QDEs correspond to the external ports of a device and the internal parameters of the components.

I also show how the QDR rule applies to general QDE descriptions for pipes and containers so that any configuration of pipes and containers can be tractably consolidated. Finally, the QDR rule is applied to the pressure regulator example.

Before these results are described, I briefly review Q1 [Williams, 1988], the qualitative algebra used to describe the QDR rule and the other results. Q1 permits the mixture of qualitative (sign) expressions with quantitative (real) expressions, e.g., the signs of pressure differences.

Q1 provides an operator [ ] to convert quantitative expressions to qualitative ones. If e is a quantitative expression, then $[e]$ can be $[+]$, $[0]$, or $[-]$, i.e., positive, zero, or negative.

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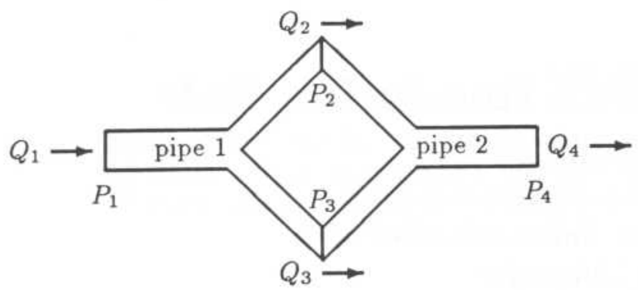

Figure 1: Two Three-Ended Pipes Connected Together

Q1 also provides the sign operators $\oplus$, $\ominus$, $\otimes$, and $\oslash$ with the traditional definitions. For example, $[+] \oplus [0] = [+]$, $[+] \oplus [-] = [?]$ ($[?]$ denotes an unknown sign), $[+] \otimes [0] = [0]$, $[+] \otimes [-] = [-]$, and so on.

I vary from the notation of Q1 as follows. $\approx$ denotes "qualitative equality"; given two signs s_1 and s_2, $s_1 \approx s_2$ iff $s_1 = s_2$ or $s_1 = [?]$ or $s_2 = [?]$. Another variation is that ∂x is used instead of $d/dt(x)$. Finally, to express conditional behaviors, conditions such as $x > 0$ are permitted. If c is a condition then:

$$[c] = \begin{cases} [+] & \text{if } c \text{ is true} \\ [0] & \text{if } c \text{ is false} \end{cases}$$

A nice property of conditions is that $([c_1] \otimes [e]) \oplus ([c_2] \otimes [e]) = [c_1 \vee c_2] \otimes [e]$.

Pressure Differences

To understand the QDR rule, it is important to understand the need for using pressure differences (e.g., $[P_1 - P_2]$) instead of sign subtraction of pressures (e.g., $[P_1] \ominus [P_2]$). This also applies to pressure derivatives as well (e.g., $[\partial P_1 - \partial P_2]$ instead of $[\partial P_1] \ominus [\partial P_2]$). Using pressure differences is not a new idea; however, I show that this modeling technique has special properties that can be exploited.

The reason for using signs of pressure differences is that using the signs of pressures makes it difficult to infer direction of flow. With $[Q] \approx [P_1] \ominus [P_2]$, $[Q]$ cannot be determined if both $[P_1]$ and $[P_2]$ have the same sign. However, $[Q] \approx [P_1 - P_2]$ does not have this defect.

To show where this difference matters, consider the situation in Figure 1 in which two three-ended pipes have two connections between them. In this situation, it is desirable to infer that the two connected pipes behave like a single two-ended pipe. QDEs 1–8 model the relationships among the flows and pressures based on their signs.

$$\begin{aligned}
[Q_1] &\approx [P_1] \ominus [P_2] \ominus [P_3] && (1) \\
[-Q_2] &\approx [P_2] \ominus [P_1] \ominus [P_3] && (2) \\
[-Q_3] &\approx [P_3] \ominus [P_1] \ominus [P_2] && (3) \\
[Q_1] &\approx [Q_2] \oplus [Q_3] && (4) \\
[Q_2] &\approx [P_2] \ominus [P_3] \ominus [P_4] && (5) \\
[Q_3] &\approx [P_3] \ominus [P_2] \ominus [P_4] && (6) \\
[-Q_4] &\approx [P_4] \ominus [P_2] \ominus [P_3] && (7) \\
[Q_4] &\approx [Q_2] \oplus [Q_3] && (8)
\end{aligned}$$

QDEs 1–4 model pipe 1, QDEs 5–8 model pipe 2, and each P_i is a pressure, and Q_i, a rate of flow, at the points indicated in Figure 1. For example, QDE 1 states that if P_1 is positive, and P_2 and P_3 are negative, then Q_1 is positive.[1]

One would expect that QDEs 9 and 10 would model the behavior of the device:

$$\begin{aligned}
[Q_1] &\approx [P_1] \ominus [P_4] && (9) \\
[Q_1] &\approx [Q_4] && (10)
\end{aligned}$$

Unfortunately, neither QDE follows from QDEs 1–8. For example, assigning $[-]$ to Q_1, Q_2, and P_4 and $[+]$ to the other variables satisfies QDEs 1–8, but violates QDE 9 and 10.

The problem is the inadequacy of using signs of pressures. Consider using the signs of their differences to model Figure 1, as in QDEs 11–18:

$$\begin{aligned}
[Q_1] &\approx [P_1 - P_2] \oplus [P_1 - P_3] && (11) \\
[-Q_2] &\approx [P_2 - P_1] \oplus [P_2 - P_3] && (12) \\
[-Q_3] &\approx [P_3 - P_1] \oplus [P_3 - P_2] && (13) \\
[Q_1] &\approx [Q_2] \oplus [Q_3] && (14) \\
[Q_2] &\approx [P_2 - P_3] \oplus [P_2 - P_4] && (15) \\
[Q_3] &\approx [P_3 - P_2] \oplus [P_3 - P_4] && (16) \\
[-Q_4] &\approx [P_4 - P_2] \oplus [P_4 - P_3] && (17) \\
[Q_4] &\approx [Q_2] \oplus [Q_3] && (18)
\end{aligned}$$

For example, QDE 11 states that if P_1 is greater than P_2 and P_3, then Q_1 is positive.

Now QDEs 19 and 20, the description of the device:

$$\begin{aligned}
[Q_1] &\approx [P_1 - P_4] && (19) \\
[Q_1] &\approx [Q_4] && (20)
\end{aligned}$$

can be demonstrated.

To do this, I use a modified version of the qualitative resolution (QR) rule [Dormoy and Raiman, 1988]. If x is a real-valued variable, and if e_1 and e_2 are qualitative expressions, then the QR rule can be stated as:

$$[x] \approx e_1 \text{ and } [-x] \approx e_2 \text{ imply } [0] \approx e_1 \oplus e_2$$

For example, QDE 12 and QDE 15 imply $[0] \approx [P_2 - P_1] \oplus [P_2 - P_3] \oplus [P_2 - P_4]$.

In addition, the following theorem shall be useful:

Theorem 1 (Qualitative Compatibility Rule) *If $x_1, x_2, \ldots, x_n$ are real-valued variables, then* $[x_1 - x_n] \approx \bigoplus_{i=1}^{n-1} [x_i - x_{i+1}]$.

The QDE in the theorem is satisfied no matter how the variables are ordered. This theorem is so named because, in the case of pressure variables, it leads to constraints like QDE 21, which enforce the compatibility condition of system dynamics [Shearer *et al.*, 1971]:

$$[P_i - P_k] \approx [P_i - P_j] \oplus [P_j - P_k] \qquad (21)$$

[1] If negative pressure seems too bizarre, consider the same QDEs using flow and pressure derivatives.

An advantage of QDE 21 over previous qualitative formulations of the compatibility condition [de Kleer and Brown, 1984; Williams, 1984] is that QDE 21 follows from the Q1 algebra; thus asserting additional QDEs is not logically necessary.

Now deriving QDE 19 can proceed as follows ("QR m n" denotes a derivation using the QR rule on QDEs m and n, and "Th. 1" denotes that the correspond QDE follows from Theorem 1):

$$[0] \approx [P_2 - P_1] \oplus [P_2 - P_3] \oplus [P_2 - P_4] \quad (22) \quad \text{QR 12 15}$$
$$[0] \approx [P_2 - P_1] \oplus [P_1 - P_3] \oplus [P_3 - P_2] \quad (23) \quad \text{Th. 1}$$
$$[0] \approx [P_2 - P_1] \oplus [P_1 - P_3] \oplus [P_2 - P_4] \quad (24) \quad \text{QR 22 23}$$
$$[0] \approx [P_2 - P_1] \oplus [P_1 - P_4] \oplus [P_4 - P_2] \quad (25) \quad \text{Th. 1}$$
$$[0] \approx [P_2 - P_1] \oplus [P_1 - P_3] \oplus [P_1 - P_4] \quad (26) \quad \text{QR 24 25}$$
$$[Q_1] \approx [P_1 - P_3] \oplus [P_1 - P_4] \quad (27) \quad \text{QR 11 26}$$
$$[0] \approx [P_3 - P_1] \oplus [P_1 - P_2] \oplus [P_2 - P_3] \quad (28) \quad \text{Th. 1}$$
$$[0] \approx [P_3 - P_1] \oplus [P_2 - P_3] \oplus [P_2 - P_4] \quad (29) \quad \text{QR 22 28}$$
$$[0] \approx [P_3 - P_4] \oplus [P_4 - P_2] \oplus [P_2 - P_3] \quad (30) \quad \text{Th. 1}$$
$$[0] \approx [P_3 - P_1] \oplus [P_2 - P_3] \oplus [P_3 - P_4] \quad (31) \quad \text{QR 29 30}$$
$$[-Q_3] \approx [P_3 - P_1] \oplus [P_3 - P_4] \quad (32) \quad \text{QR 13 31}$$
$$[Q_3] \approx [P_3 - P_1] \oplus [P_3 - P_4] \quad (33) \quad \text{QR 16 31}$$
$$[0] \approx [P_3 - P_1] \oplus [P_3 - P_4] \quad (34) \quad \text{QR 32 33}$$
$$[0] \approx [P_3 - P_1] \oplus [P_1 - P_4] \oplus [P_4 - P_3] \quad (35) \quad \text{Th. 1}$$
$$[0] \approx [P_3 - P_1] \oplus [P_1 - P_4] \quad (36) \quad \text{QR 34 35}$$
$$[Q_1] \approx [P_1 - P_4] \quad (19) \quad \text{QR 27 36}$$

$[Q_4] \approx [P_1 - P_4]$ can be similarly derived, from which QDE 20 follows.

Although the QR rule can be used to derive QDEs 19 and 20 from QDEs 11–18, it is clearly tedious to do so. Fortunately, there is another resolution rule considerably shortens the length of the derivation, and, more importantly, generalizes the derivation and leads to a tractable application. Before the qualitative difference resolution rule (QDR rule) is described, some useful definitions are provided.

Conditional Difference Systems

Let $\boldsymbol{Y}_n$ denote n variables $y_1, y_2, \ldots, y_n$, and let $\boldsymbol{X}_{nn}$ denote n^2 variables $x_{1,1}, x_{1,2}, \ldots, x_{1,n}, \ldots, x_{n,1}, x_{n,2}, \ldots, x_{nn}$. I shall say that the variables $\boldsymbol{Y}_n$ *are dependent on the differences* $\boldsymbol{X}_{nn}$ if:

$$\begin{array}{ll} [y_i] \approx \bigoplus_{j=1}^{n} [x_{ij}], & 1 \leq i \leq n \\ [x_{ik}] \approx [x_{ij}] \oplus [x_{jk}], & 1 \leq i,j,k \leq n \\ x_{ii} = 0, & 1 \leq i \leq n \end{array}$$

The idea is that each y_i is a "flow" variable and each x_{ij} is a "pressure difference" variable. $[x_{ik}] \approx [x_{ij}] \oplus [x_{jk}]$ and $x_{ii} = 0$ are "compatibility" constraints. For example, QDEs 11-13 satisfy this definition in the following way:

$$\begin{array}{lll} y_1 = Q_1 & x_1 = P_1 & x_{ij} = x_i - x_j \\ y_2 = -Q_2 & x_2 = P_2 & \\ y_3 = -Q_3 & x_3 = P_3 & \end{array}$$

Let $\boldsymbol{C}_{nn}$ denote n^2 conditions (refer to the introduction for a definition of conditions). I shall say that the variables $\boldsymbol{Y}_n$ *are conditionally dependent on the differences* $\boldsymbol{X}_{nn}$ *by conditions* $\boldsymbol{C}_{nn}$ if:

$$\begin{array}{ll} [y_i] \approx \bigoplus_{j=1}^{n} ([c_{ij}] \otimes [x_{ij}]), & 1 \leq i \leq n \\ [x_{ik}] \approx [x_{ij}] \oplus [x_{jk}], & 1 \leq i,j,k \leq n \\ x_{ii} = 0, & 1 \leq i \leq n \\ [c_{ij}] \approx [c_{ji}], & 1 \leq i,j \leq n \end{array}$$

I shall call $\boldsymbol{Y}_n$, $\boldsymbol{X}_{nn}$, and $\boldsymbol{C}_{nn}$ a *conditional difference system*.

This extends the idea of dependence on differences so that a flow can be conditionally dependent on pressure differences. For example, QDEs 11–13, 15–17 form the following conditional difference system:

$$\begin{array}{ll} y_1 = Q_1 & x_1 = P_1 \\ y_2 = -Q_2 & x_2 = P_2 \\ y_3 = -Q_3 & x_3 = P_3 \\ y_4 = Q_2 & x_4 = P_2 \\ y_5 = Q_3 & x_5 = P_3 \\ y_6 = -Q_4 & x_6 = P_4 \\ \multicolumn{2}{c}{x_{ij} = x_i - x_j} \end{array} \quad C_{6,6} = \begin{pmatrix} F & T & T & F & F & F \\ T & F & T & F & F & F \\ T & T & F & F & F & F \\ F & F & F & F & T & T \\ F & F & F & T & F & T \\ F & F & F & T & T & F \end{pmatrix}$$

where T and F stand for true and false, respectively. For instance, QDE 11 can be recovered from this information as follows:

$$\begin{aligned} [Q_1] &= [y_1] \\ &\approx ([c_{1,1}] \otimes [x_{1,1}]) \oplus ([c_{1,2}] \otimes [x_{1,2}]) \oplus \\ &\quad ([c_{1,3}] \otimes [x_{1,3}]) \oplus ([c_{1,4}] \otimes [x_{1,4}]) \oplus \\ &\quad ([c_{1,5}] \otimes [x_{1,5}]) \oplus ([c_{1,6}] \otimes [x_{1,6}]) \\ &= ([F] \otimes [P_1 - P_1]) \oplus ([T] \otimes [P_1 - P_2]) \oplus \\ &\quad ([T] \otimes [P_1 - P_3]) \oplus ([F] \otimes [P_1 - P_4]) \oplus \\ &\quad ([F] \otimes [P_1 - P_5]) \oplus ([F] \otimes [P_1 - P_6]) \\ &= ([0] \otimes [P_1 - P_1]) \oplus ([+] \otimes [P_1 - P_2]) \oplus \\ &\quad ([+] \otimes [P_1 - P_3]) \oplus ([0] \otimes [P_1 - P_4]) \oplus \\ &\quad ([0] \otimes [P_1 - P_5]) \oplus ([0] \otimes [P_1 - P_6]) \\ &= ([+] \otimes [P_1 - P_2]) \oplus ([+] \otimes [P_1 - P_3]) \\ &= [P_1 - P_2] \oplus [P_1 - P_3] \end{aligned}$$

Two conditional difference systems can be merged into a single conditional difference system by adding compatibility constraints and lots of F conditions. Thus, if each component in a device is described as a conditional difference system, then the combination of the components with additional compatibility constraints is also a conditional difference system. Often, the compatibility constraints are theorems of qualitative algebra, such as QDE 21.

Note that if two components are connected, then their flows (and flow derivatives) at the connection have opposite signs (assuming some reasonable convention, e.g., flow inward is positive) and their pressures (and pressure derivatives) at the connection are equal. In the example conditional difference system above, $[y_2] = [-y_4]$ and $x_2 = x_4$.

The QDR Rule

Finally, the QDR rule can be specified.

Theorem 2 (Qualitative Difference Resolution Rule) *If $\boldsymbol{Y}_{n+2}$ is conditionally dependent on $\boldsymbol{X}_{n+2,n+2}$ by $\boldsymbol{C}_{n+2,n+2}$, if $[y_{n+1}] \approx [-y_{n+2}]$, and if $x_{n+1,n+2} = x_{n+2,n+1} = 0$, then $\boldsymbol{Y}_n$ is conditionally dependent on $\boldsymbol{X}_{nn}$ by $\boldsymbol{C}'_{nn}$, where $\boldsymbol{C}'_{nn}$ is determined from $\boldsymbol{C}_{n+2,n+2}$ by:*

$$c'_{ij} = c_{ij} \vee ((c_{i,n+1} \vee c_{i,n+2}) \wedge (c_{n+1,j} \vee c_{n+2,j}))$$

The appendix contains the proof of the QDR rule. If the requirements of the QDR rule are satisfied, then the variables y_{n+1}, y_{n+2}, and, for all i, $x_{n+1,i}$, $x_{i,n+1}$, $x_{n+2,i}$, and $x_{i,n+2}$ can be resolved/eliminated from the QDEs as long as the conditions do not refer to these variables. For example, the QDR rule can be applied twice to QDEs 11–13, 15–17. In one instance, $[y_2] = [-y_4]$ and $x_{2,4} = x_{4,2} = 0$. In the second instance, $[y_3] = [-y_5]$ and $x_{3,5} = x_{5,3} = 0$. The successive results of the two applications to the $\boldsymbol{C}_{6,6}$ matrix in the previous column are as follows:

$$\begin{pmatrix} T & / & T & / & T & T \\ / & / & / & / & / & / \\ T & / & T & / & T & T \\ / & / & / & / & / & / \\ T & / & T & / & T & T \\ T & / & T & / & T & T \end{pmatrix} \begin{pmatrix} T & / & / & / & / & T \\ / & / & / & / & / & / \\ / & / & / & / & / & / \\ / & / & / & / & / & / \\ / & / & / & / & / & / \\ T & / & / & / & / & T \end{pmatrix}$$

In the first application, the second and fourth columns and rows are resolved, which is indicated by the /'s. The conditions in the remaining 4×4 matrix are all T, e.g., $c'_{1,5} = c_{1,5} \vee ((c_{1,2} \vee c_{1,4}) \wedge (c_{2,5} \vee c_{4,5})) = F \vee ((T \vee F) \wedge (F \vee T)) = T$.

In the second application, the third and fifth columns and rows are resolved, leaving only $[Q_1] \approx [P_1 - P_1] \oplus [P_1 - P_4] = [P_1 - P_4]$ and $[-Q_4] \approx [P_4 - P_1] \oplus [P_4 - P_4] = [P_4 - P_1]$. QDE 20, $[Q_1] \approx [Q_4]$, follows.

In general, the size of conditions derived using the QDR rule can grow combinatorially. However, if all the conditions are either T or F, then all the conditions derived using the QDR rule will also be either T or F. This leads to the following theorem:

Theorem 3 (QDR Tractability)
If $\boldsymbol{Y}_n$ is conditionally dependent on $\boldsymbol{X}_{nn}$ by $\boldsymbol{C}_{nn}$, if each condition in $\boldsymbol{C}_{nn}$ is either T or F, and if there are m two-element disjoint sets $\{i, j\}$, $1 \leq i, j \leq n$, indicating equalities of the form $[y_i] = [-y_j]$ and $x_{ij} = x_{ji} = 0$, then there is an $O(mn^2)$ algorithm for eliminating all the variables that share a subscript with any of the m sets.

Using the QDR rule, there are $O(n^2)$ updates to be performed for each pair of equalities. Because each condition is either T or F, the size of the conditions do not increase. m pairs of equalities imply $O(mn^2)$ time.

ports:	$port_1, \ldots, port_n$	
variables:	$Q_1, P_1, \ldots, Q_n, P_n$	
constraints:	$[Q_i] \approx \bigoplus_{j=1}^{n} [P_i - P_j]$,	$1 \leq i \leq n$
	$[\partial Q_i] \approx \bigoplus_{j=1}^{n} [\partial P_i - \partial P_j]$,	$1 \leq i \leq n$

Figure 2: Qualitative Model for Pipes

The Qualitative Continuity Rule

Before describing examples of using the QDR rule, it is interesting that QDE 20, the qualitative conservation law for the configuration in Figure 1, can be derived without using QDEs 14 and 18, the qualitative conservation laws for the components. It turns out that QDE 14 can be derived from QDEs 11–13, and QDE 18 can be derived from QDEs 15–17. There is a general rule that underlies these derivations:

Theorem 4 (Qualitative Continuity Rule)
If $\boldsymbol{Y}_n$ is conditionally dependent on $\boldsymbol{X}_{nn}$, then $\bigoplus_{i=1}^{n} [y_i] \approx [0]$.

For example, QDEs 11–13 form a conditional difference system as follows:

$$\begin{array}{ll} y_1 = Q_1 & x_1 = P_1 \\ y_2 = -Q_2 & x_2 = P_2 \\ y_3 = -Q_3 & x_3 = P_3 \\ & x_{ij} = x_i - x_j \end{array} \quad \boldsymbol{C}_{3,3} = \begin{pmatrix} F & T & T \\ T & F & T \\ T & T & F \end{pmatrix}$$

From the qualitative continuity (QC) rule, QDE 37 follows:

$$[Q_1] \oplus [-Q_2] \oplus [-Q_3] \quad \approx \quad [0] \tag{37}$$

which is equivalent to QDE 14, $[Q_1] \approx [Q_2] \oplus [Q_3]$.

Thus, a conditional difference system of flows and pressure differences implies a qualitative version of the continuity condition of system dynamics [Shearer *et al.*, 1971]. Similar to the qualitative compatibility rule, an advantage of the QC rule over previous qualitative formulations of the continuity condition [de Kleer and Brown, 1984; Williams, 1984] is that the QC rule follows from a conditional difference system and is not an additional "law" that must be added to constrain the system.

Pipes

Figure 2 is a qualitative model for pipes with n ports, $n \geq 1$. Q_i is the rate of flow into the pipe through $port_i$; Q_i is negative if flow is outward. P_i is the pressure at $port_i$. Semantics of connection are: Each port can be connected to at most one other port. If $port_i$ is connected to $port_j$, then $Q_i = -Q_j$ and $P_i = P_j$.

Figure 2 defines two sets of QDEs. The first set specifies n QDEs, relating each Q_i to the pressures. The direction of flow for any $port_i$ corresponds to the "sum" of pressure differences (the sign summation of P_i minus other pressures). The second set specifies similar QDEs for the first derivatives.

$$
\begin{array}{rll}
\text{ports:} & port_1, \ldots, port_n & \\
\text{variables:} & Q_1, P_1, \ldots, Q_n, P_n, A, P & \\
\text{constraints:} & [Q_i] \approx [P_i - P], & 1 \le i \le n \\
& [-\partial A] \approx \bigoplus_{i=1}^{n} [P - P_i] & \\
& [\partial Q_i] \approx [\partial P_i - \partial P], & 1 \le i \le n \\
& [-\partial^2 A] \approx \bigoplus_{i=1}^{n} [\partial P - \partial P_i] & \\
& [P] \approx [A] & \\
& [\partial P] \approx [\partial A] &
\end{array}
$$

Figure 3: Qualitative Model for Containers

Theorem 5 (Pipe Continuity Laws)
For a pipe with n ports, $\bigoplus_{i=1}^{n}[Q_i] \approx [0]$ *and* $\bigoplus_{i=1}^{n}[\partial Q_i] \approx [0]$.

The QC rule applies to the QDEs given in Figure 2.

Theorem 6 (Pipe Consolidation Law)
If a pipe with m *ports has* k *connections to a pipe with* n *ports (*$k < m$ *and* $k < n$*), then a pipe with* $m+n-2k$ *ports describes their combined behavior.*

Just as the QDR rule was applied twice for the two connections in Figure 1, it can be applied k times for k connections to obtain the QDEs relating flows and pressures at the external ports and another k times to obtain the QDEs relating flow and pressure derivatives.

Consequently, the consolidation of any configuration of pipes can be done very efficiently. Qualitative conservation laws can also be efficiently inferred.

Containers

Figure 3 is a qualitative model for a container with n ports, $n \ge 1$. In addition to the ports' variables, A is the amount in the container, and P is the pressure inside the container.

The constraints as shown in Figure 3 are: (1) The direction of flow at any port is the sign of the difference between the port's pressure and the container's pressure. (2) Change in the container's amount depends on the qualitative sum of the differences between the container's pressure and the ports' pressures. For example, the container's amount will increase if the container's pressure is lower than the ports' pressures. (3,4) The flow and pressure derivatives and the amount's second derivative have similar constraints. (5,6) The container's pressure depends on the container's amount. In particular, pressure increases or decreases as the amount increases or decreases.

Theorem 7 (Container Continuity Laws)
For a container with n ports, $\bigoplus_{i=1}^{n}[Q_i] \approx [\partial A]$ *and* $\bigoplus_{i=1}^{n}[\partial Q_i] \approx [\partial^2 A]$

The QC rule directly applies to the containers QDEs.

The QDR rule can clearly be applied to connected pipes and containers. However, two connected containers cannot be described as a single container because the consolidated QDEs will have two amount and two pressure variables associated with the two containers. The QDR rule eliminates the variables of the connected ports, but does not eliminate "internal" variables. Of course, $[P] \approx [A]$ and $[\partial P] \approx [\partial A]$ should be kept in any consolidated description.

The container model does not place any restrictions on the ranges of pressures and amounts. To model containers with lower limits of zero for pressures and amounts, one can simply require $A \ge 0$ and $P \ge 0$.

To model a container with maximum capacity A_{max}, $[\partial P] \approx [\partial A]$ can be replaced with $[A < A_{max}] \otimes [\partial P] \approx [\partial A]$. If the container is full, then $[\partial A] = [0]$ and ∂P is no longer constrained by this QDE. Instead, ∂P will be governed by the $-\partial^2 A$ QDE of the container model. Since $\partial^2 A$ must also be zero, the values of both P and ∂P must lie somewhere between the values at the ports.

The Pressure Regulator

In this section, I show how our qualitative models and the QDR rule can be used to consolidate the pressure regulator example [de Kleer and Brown, 1984]. Figure 4 shows the pressure regulator on the left and its division into four components (called α, β, γ, and δ) on the right.

α, γ, and δ are pipes with 2, 3, and 2 ports, respectively. β is a valve, which is modeled as a component with three ports, one of which is "blocked." Although no flow can occur through β's third port, it still is a point of interaction, in this case, with the pipe δ. In particular, the pressure from δ will be the "pressure" to close the valve's position.

Our QDE description of the valve is in Figure 5. The first QDE specifies that, if the valve is open ($V > 0$), then the direction of flow Q_1 corresponds to the sign of the pressure difference $P_1 - P_2$; else Q_1 is zero. The second QDE is a similar constraint for Q_2. The third QDE makes $port_3$ the blocked port.

The ∂Q_1 constraint is somewhat complex because it makes no assumptions about direction of flow or the position of the valve. There are two factors that influence ∂Q_1.

(1) If the valve is open ($V > 0$), the difference between pressure derivatives will be an influence, e.g., an increasing P_1 and a decreasing P_2 will tend to make Q_1 increase.

(2) If the valve is open, then an increasing/decreasing position of the valve will tend to increase/decrease the *magnitude* of the change of rate of flow, e.g., if the valve position is decreasing (the valve is closing) and the direction of flow (the pressure difference) is negative, then the flow tends to go towards zero, i.e., the flow tends to increase.

If one factor is positive and the other negative, then it is unclear whether flow is increasing or decreasing. ∂Q_2 has a similar constraint.

The last constraint models the relationship between the position of the valve V and the pressure at the blocked port P_3. It specifies that the direction of

Figure 4: The Pressure Regulator and Its Components

ports: $port_1, port_2, port_3$
variables: $Q_1, P_1, Q_2, P_2, Q_3, P_3, V$
constraints:
$$[Q_1] \approx [V > 0] \otimes [P_1 - P_2]$$
$$[Q_2] \approx [V > 0] \otimes [P_2 - P_1]$$
$$[Q_3] \approx [0]$$
$$[\partial Q_1] \approx ([V > 0] \otimes [\partial P_1 - \partial P_2]) \oplus ([V > 0] \otimes [\partial V] \otimes [P_1 - P_2])$$
$$[\partial Q_2] \approx ([V > 0] \otimes [\partial P_2 - \partial P_1]) \oplus ([V > 0] \otimes [\partial V] \otimes [P_2 - P_1])$$
$$[\partial V] \approx [V > 0 \wedge V < V_{max}] \otimes [-\partial P_3]$$

Figure 5: Qualitative Model for Pressure Regulator Valve

change of the valve's position will be opposite of the direction of change of the pressure at $port_3$ as long as the valve is not completely closed ($V > 0$) and not completely open ($V < V_{max}$). V_{max} is a positive constant representing the maximum position of the valve.

Using the pipe consolidation law, γ and δ can be consolidated into a three-ended pipe with ports $port_1^\gamma$, $port_2^\gamma$, and $port_1^\delta$ (superscripts indicate the original component).

Although β is not a pipe, its QDEs relating flow and pressure along with those of the pipe α form a conditional difference system as follows:

$$\begin{array}{ll} y_1 = Q_1^\alpha & x_1 = P_1^\alpha \\ y_2 = Q_2^\alpha & x_2 = P_2^\alpha \\ y_3 = Q_1^\beta & x_3 = P_1^\beta \\ y_4 = Q_2^\beta & x_4 = P_2^\beta \end{array} \quad C_{4,4} = \begin{pmatrix} F & T & F & F \\ T & F & F & F \\ F & F & F & V^\beta > 0 \\ F & F & V^\beta > 0 & F \end{pmatrix}$$
$$x_{ij} = x_i - x_j$$

The connection between α and β implies $y_2 = -y_3$ and $x_2 = x_3$, so the QDR rule and a few simplifications lead to QDEs 38 and 39:

$$[Q_1^\alpha] \approx [V^\beta > 0] \otimes [P_1^\alpha - P_2^\beta] \quad (38)$$
$$[Q_2^\beta] \approx [V^\beta > 0] \otimes [P_2^\beta - P_1^\alpha] \quad (39)$$

Of course, $[Q_3^\beta] \approx [0]$ should be retained in $\alpha\beta$'s behavioral description.

However, the constraints on flow and pressure derivatives of β do not conform to the QDR rule (or the QC rule, for that matter). Both the ∂Q_1^β and ∂Q_2^β QDEs have an extra term (their dependence on ∂V) that does not fit into the QDR rule. In such cases, explicit compatibility constraints must be added, as in QDEs 40–44:

$$[\partial Q_1^\beta] \approx [V > 0] \otimes [x_{1,2}^\beta] \quad (40)$$
$$[\partial Q_2^\beta] \approx [V > 0] \otimes [x_{2,1}^\beta] \quad (41)$$
$$[x_{1,2}^\beta] \approx [\partial P_1^\beta - \partial P_2^\beta] \oplus ([\partial V^\beta] \otimes [P_1^\beta - P_2^\beta]) \quad (42)$$
$$[x_{2,1}^\beta] \approx [\partial P_2^\beta - \partial P_1^\beta] \oplus ([\partial V^\beta] \otimes [P_2^\beta - P_1^\beta]) \quad (43)$$
$$[0] \approx [x_{1,2}^\beta] \oplus [x_{2,1}^\beta] \quad (44)$$

Now to consolidate QDEs 40–44 with those of α, additional compatibility constraints are necessary, resulting in the following conditional difference system:

$$\begin{array}{ll} y_1 = \partial Q_1^\alpha & x_1 = P_1^\alpha \\ y_2 = \partial Q_2^\alpha & x_2 = P_2^\alpha \\ y_3 = \partial Q_1^\beta & x_3 = P_1^\beta \\ y_4 = \partial Q_2^\beta & x_4 = P_2^\beta \end{array} \quad C_{4,4} = \begin{pmatrix} F & T & F & F \\ T & F & F & F \\ F & F & F & V^\beta > 0 \\ F & F & V^\beta > 0 & F \end{pmatrix}$$

$$[x_{ij}] \approx \begin{cases} [\partial x_i - \partial x_j], & 1 \le i, j \le 2 \\ [\partial x_i - \partial x_j] \oplus ([\partial V^\beta] \otimes [x_i - x_j]), & \text{otherwise} \end{cases}$$
$$[x_{ik}] \approx [x_{ij}] \oplus [x_{jk}], \qquad 1 \le i, j, k \le 4$$

$y_2 = -y_3$ and $x_{2,3} = x_{3,2} = 0$, so applying the QDR rule results in QDEs 45–49:

$$[\partial Q_1^\alpha] \approx [V > 0] \otimes [x_{1,2}^{\alpha\beta}] \quad (45)$$
$$[\partial Q_2^\beta] \approx [V > 0] \otimes [x_{2,1}^{\alpha\beta}] \quad (46)$$
$$[x_{1,2}^{\alpha\beta}] \approx [\partial P_1^\alpha - \partial P_2^\beta] \oplus ([\partial V^\beta] \otimes [P_1^\alpha - P_2^\beta]) \quad (47)$$
$$[x_{2,1}^{\alpha\beta}] \approx [\partial P_2^\beta - \partial P_1^\alpha] \oplus ([\partial V^\beta] \otimes [P_2^\beta - P_1^\alpha]) \quad (48)$$
$$[0] \approx [x_{1,2}^{\alpha\beta}] \oplus [x_{2,1}^{\alpha\beta}] \quad (49)$$

Of course, $[\partial V^\beta] \approx [V^\beta > 0 \wedge V^\beta < V_{max}^\beta] \otimes [-\partial P_3^\beta]$ should be retained.

Similar reasoning can be used to consolidate $\alpha\beta$ with $\gamma\delta$. The end result is a behavioral description with two ports, $port_1^\alpha$ and $port_2^\gamma$, and the following QDEs:

$$[Q_1^\alpha] \approx [V^\beta > 0] \otimes [P_1^\alpha - P_2^\gamma] \quad (50)$$
$$[Q_2^\gamma] \approx [V^\beta > 0] \otimes [P_2^\gamma - P_1^\alpha] \quad (51)$$
$$[\partial Q_1^\alpha] \approx [V^\beta > 0] \otimes [x_{1,2}] \quad (52)$$
$$[\partial Q_2^\gamma] \approx [V^\beta > 0] \otimes [x_{2,1}] \quad (53)$$

$$[x_{1,2}] \approx [\partial P_1^\alpha - \partial P_2^\gamma] \oplus ([\partial V^\beta] \otimes [P_1^\alpha - P_2^\gamma]) \quad (54)$$
$$[x_{2,1}] \approx [\partial P_2^\gamma - \partial P_1^\alpha] \oplus ([\partial V^\beta] \otimes [P_2^\gamma - P_1^\alpha]) \quad (55)$$
$$[0] \approx [x_{1,2}] \oplus [x_{2,1}] \quad (56)$$
$$[\partial V^\beta] \approx [V^\beta > 0 \wedge V^\beta < V_{max}^\beta] \otimes [-\partial P_3^\beta] \quad (57)$$

The presence of ∂P_3^β in the last QDE is a problem. One possibility is to assume that it is equal to ∂P_2^γ, which is implicitly done in de Kleer and Brown [1984]. It can be shown, though, that:

$$[\partial P_3^\beta] \approx ([V^\beta > 0] \otimes [\partial P_1^\alpha]) \oplus [\partial P_2^\gamma] \quad (58)$$

This, however, requires additional reasoning beyond the QDR rule.

Remarks

The QDR rule can be used to perform tractable consolidation of components for which the direction of flow is dependent on the signs of pressure differences. In this paper, I have shown that pipes and containers can be modeled to fit the QDR rule. With the exception of one QDE, consolidation of the pressure regulator can also be accomplished with the QDR rule. I believe that the QDR rule explains why many examples in the qualitative reasoning literature can be efficiently processed. To the extent that the components in these examples are pipe-like or container-like, efficient reasoning can be guaranteed.

One limitation of the QDR rule is that no variables in the conditions are eliminated. The simplest example of this limitation is a one-way valve, which would have a QDE like $[Q_1] \approx [P_1 > P_2] \otimes [P_1 - P_2]$. If a one-way valve is connected to three-ended pipes, there is no easy solution to eliminating P_1 and P_2 in the condition.

Another limitation is that the QDR rule results in loss of information. For example, if there is one connection between two three-ended pipes, the consolidated QDEs do not enforce the constraint that flow from one pipe to the other can only be in one direction. In this sense, the QDR rule produces *abstractions* of connected components, and not equivalences.

The final, perhaps most important, limitation is that the QDEs of a component must have the appropriate form, i.e., be a conditional difference system. Whether our approach can be extended to additional types of components (e.g., pumps, transformers) and phenomena (e.g., momentum, heights), and, if not, what additional resolution rules are needed, are the subject of further investigation.

Proof of the QDR Rule

The QDEs for a conditional difference system include:

$$[y_{n+1}] \approx \bigoplus_{j=1}^{n+2}([c_{n+1,j}] \otimes [x_{n+1,j}])$$
$$[y_{n+2}] \approx \bigoplus_{j=1}^{n+2}([c_{n+2,j}] \otimes [x_{n+2,j}])$$

Because $x_{n+1,n+2} = 0$ and $[x_{n+1,j}] \approx [x_{n+1,n+2}] \oplus [x_{n+2,j}]$ for all j between 1 and $n+2$, it follows that $[x_{n+1,j}] \approx [x_{n+2,j}]$ for all j between 1 and $n+2$. With $x_{n+2,n+1} = 0$, $x_{n+1,n+1} = 0$, and $x_{n+2,n+2} = 0$, the following QDEs can be derived:

$$[y_{n+1}] \approx \bigoplus_{j=1}^{n}([c_{n+1,j}] \otimes [x_{n+1,j}])$$
$$[y_{n+2}] \approx \bigoplus_{j=1}^{n}([c_{n+2,j}] \otimes [x_{n+1,j}])$$

Because $[y_{n+1}] \approx [-y_{n+2}]$, the QR rule can be applied, leading to:

$$[0] \approx \bigoplus_{j=1}^{n}([c_{n+1,j} \vee c_{n+2,j}] \otimes [x_{n+1,j}])$$

Consider $x_{n+1,1}$. $x_{1,1} = 0$ and $[x_{1,1}] \approx [x_{1,n+1}] \oplus [x_{n+1,1}]$ implies $[x_{n+1,1}] = [-x_{1,n+1}]$, so:

$$[c_{n+1,1} \vee c_{n+2,1}] \otimes [x_{1,n+1}] \approx \bigoplus_{j=2}^{n}([c_{n+1,j} \vee c_{n+2,j}] \otimes [x_{n+1,j}])$$

Assume that $c_{n+1,1} \vee c_{n+2,1}$ is true, i.e.:

$$[x_{1,n+1}] \approx \bigoplus_{j=2}^{n}([c_{n+1,j} \vee c_{n+2,j}] \otimes [x_{n+1,j}])$$

Because $[x_{1,n+1}] \approx [x_{1,2}] \oplus [x_{2,n+1}]$:

$$[c_{n+1,2} \vee c_{n+2,2}] \otimes [x_{1,n+1}] \approx ([c_{n+1,2} \vee c_{n+2,2}] \otimes [x_{1,2}]) \oplus ([c_{n+1,2} \vee c_{n+2,2}] \otimes [x_{2,n+1}])$$

Since $[x_{2,n+1}] \approx [-x_{n+1,2}]$, the QR rule can be applied:

$$[x_{1,n+1}] \oplus ([c_{n+1,2} \vee c_{n+2,2}] \otimes [x_{1,n+1}]) \approx ([c_{n+1,2} \vee c_{n+2,2}] \otimes [x_{1,2}]) \oplus \bigoplus_{j=3}^{n}([c_{n+1,j} \vee c_{n+2,j}] \otimes [x_{n+1,j}])$$

Note that $[x_{1,n+1}] \oplus ([c] \otimes [x_{1,n+1}]) = [x_{1,n+1}]$ for any condition c. Further note that the QR rule can be similarly applied for the remaining j from 3 to n, resulting in:

$$[x_{1,n+1}] \approx \bigoplus_{j=2}^{n}[c_{n+1,j} \vee c_{n+2,j}] \otimes [x_{1,j}])$$

Now consider the QDE for y_1:

$$[y_1] \approx \bigoplus_{j=1}^{n+2}([c_{1,j}] \otimes [x_{1,j}])$$

Because $[x_{1,n+1}] = [x_{1,n+2}]$, it follows that:

$$[y_1] \approx ([c_{1,n+1} \vee c_{1,n+2}] \otimes [x_{1,n+1}]) \oplus \bigoplus_{j=1}^{n}([c_{1,j}] \otimes [x_{1,j}])$$

Recall that $c_{ij} = c_{ji}$, so $c_{n+1,1} \vee c_{n+2,1} = c_{1,n+1} \vee c_{1,n+2}$. Hence, $x_{1,n+1}$ is a factor only if $[c_{n+1,1} \vee c_{n+2,1}]$ is true, so the QDE for $x_{1,n+1}$ derived above under the assumption that $[c_{n+1,1} \vee c_{n+2,1}]$ is true can be used to substitute for $[x_{1,n+1}]$, leading to:

$$[y_1] \approx ([c_{1,n+1} \vee c_{1,n+2}] \otimes (\bigoplus_{j=2}^{n}[c_{n+1,j} \vee c_{n+2,j}] \otimes [x_{1,j}])) \oplus \bigoplus_{j=1}^{n}([c_{1,j}] \otimes [x_{1,j}])$$

which after a few simplifications becomes:

$$[y_1] \approx \bigoplus_{j=1}^{n}([c_{1,j} \vee ((c_{1,n+1} \vee c_{1,n+2}) \wedge (c_{n+1,j} \vee c_{n+2,j}))] \otimes [x_{1,j}])$$

which is the same as:

$$[y_1] \approx \bigoplus_{j=1}^{n}([c'_{1,j}] \otimes [x_{1,j}])$$

The other QDEs for y_2 to y_n can be similarly derived. $c'_{ij} = c'_{ji}$ follows from $c_{ij} = c_{ji}$. QED.

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