Data Fusion and Parameter Estimation Using Qualitative Models: The Qualitative Kalman Filter *

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Abstract: Most sensor based systems employ a large variety of sensors to obtain information. How the information obtained from different sensing devices is combined to form a description of the system is the sensor fusion problem. Most statistical sensor fusion systems use differential equations to inter-relate the values of sensor observations and system parameters. In this paper, we investigate model-based parameter estimation for noisy processes when the process models are incomplete or imprecise. The underlying representation of our models is qualitative in the sense of Qualitative Reasoning (QR) and Qualitative Physics from the Artificial Intelligence literature. We adopt a specific qualitative representation, namely that advocated by Kuipers [13], in which a well defined mathematical description of a qualitative model is given in terms of operations on (possibly unbounded) intervals of the reals.

This paper overviews a theory for fusion of noisy observations of a stochastic system when qualitative models of the system processes and sensor observations are employed. The interested reader is referred elsewhere [18] for a more detailed exposition. We demonstrate our theory using real data from a mobile robot application which utilises sonar and laser time-of-flight and gyroscope information to disseminate surface curvature.

1 INTRODUCTION

The relationships between sensory information in data fusion systems is often in the form of quantitative, algebraic models and in practise parameter estimation systems incorporating such models suffer numerous problems. For example, complex models can be unwieldy, the interpretation of sensor output might require detailed knowledge of the physics of the sensing process [5] or extensive sensor calibration might be required [16]. Further, a sufficient representation of imprecision must be included to cater for unmodelled or non-determinable components such as temperature changes or hysteresis effects in sensor components [3, 11].

The Kalman filter [2] is perhaps the most popular model-based approach to data fusion. It maintains an estimate of the "mean" of a state vector and its "covariance" with respect to the true state. It uses algebraic differential models which are hardly ever *accurate* descriptions of the environment at this level of precision and apart from a few simple cases (for example, when the modelling error is known to be constant [7]) analytical solutions to modelling inaccuracies are cumbersome, massively under confident or mathematically intractable. The alternatives are to mask the deficiencies of the process model by introducing fictitious noise inputs (in effect, blurring the estimates further), over weighting the most recent data (limited memory filtering) or by adaptive noise estimation using observation and estimation residuals [11]. These methods require adjustment for each application by experimentation and are therefore fragile. We believe that the process model itself should reflect the imprecision when information is incomplete and to this end, we explore an alternative modelling representation, the *qualitative model*, in which the *imprecision* of the models and the *uncertainty* due to noisy observations are treated separately. The statistical noise uncertainty description is layered on top of an interval-based systematic error description of the domain model.

A Qualitative representation describes model parameters as intervals on the reals and process and observation models are described by qualitative differential equations (QDE). For example, the linguistic statement "to the left" may be represented as a range of possible bearings $(0, \frac{\pi}{2})$ radians and the statement "monotonic increasing function of y and x" would be the set of functions for y whose derivative with respect to x lies in $(0, \infty)$. Smaller intervals may be used where more detailed information is available. For example, when the zero return from a gyroscope sensor subject to drift is guaranteed to correspond to a true value within $(-0.01, 0.01)$ rads per sec or numeric envelopes bound the functional relationship between x and y.

Ideally, we seek to utilise all the information available whether it is incomplete, imprecise or uncertain while maintaining an accurate estimate of state. In

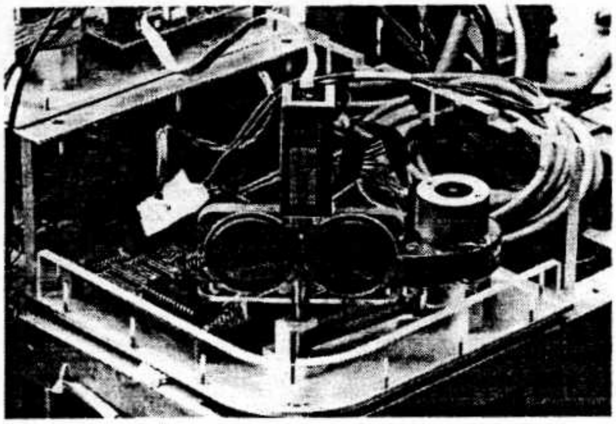


Figure 1: Servo-mounted sonar and gyroscope system.

this paper, we describe the *Qualitative filter* for qualitative estimation and sensor fusion, which uses abstract (as opposed to approximate) process and observation models to maintain a qualitative measure of the parameter values while utilising the abstraction properties of the Kalman filter: namely, its ability to maintain an estimate of mean and variance irrespective of the exact form of the underlying noise probability density distributions.

In Section 2, we introduce the requirements for a qualitative data fusion framework with an example drawn from the robot sensing domain. In Section 3, we describe the conditions for unbiased estimation using ∞-norm opinion pool methods in conjunction with Dempster-Shafer mass values. We introduce mass assignment functions for mean estimators (analogous to the Kalman filter) and median estimators (for robustness). We develop the ∞-norm Dempster-Shafer theory and subsequently derive expressions for the filter prediction and update phases. Finally, in Section 4, we demonstrate the Qualitative filter with real data obtained in the robot sensing domain.

2 THE ROBOT SENSING DOMAIN

The sensors we investigate (sonar, laser time-of-flight) offer a 2-D view of the environment (1-D signal propagation and 1-D rotation of sensor head). Therefore, the robot identifies features through the intersection of a plane with the environment and Leonard [14] shows that feature curvature is a good discriminant in such circumstances. Hoffman [10] contemplates encoding shape by curvature and concludes that a qualitative description of curvature suffices to capture the essential discriminants of the environment. However, since curvature is a second order derivative it is noisy [1].

In this section, we investigate how qualitative information about curvature can be computed and we demonstrate how a mobile robot, equipped with a sonar tracking sensor, a laser time-of-flight range sensor and a gyroscope (see Figure 1) can distinguish feature curvature using qualitative models interrelating its sensor cues [1]. In later sections, we will investigate the problem of noise filtering using qualitative models. In Figure 2, the robot (indicated by

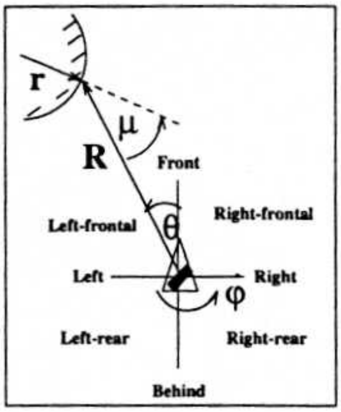


Figure 2: System Geometry.

$\triangle$) passes in front of a curved surface (shaded). For specular (i.e. mirror-like) surfaces the robot sees only the reflection normal to the surface (i.e $\mu = 0$) [12, 14]. The robot tracks the feature by following this normal reflection (see Figure 3). If the sonar (indicated by ■ in Figure 2) turns with rate $\dot{\phi}$ while tracking a beacon with curvature $\frac{1}{r}$ lying a distance R from the robot and at a bearing θ relative to the forward motion of the robot then [19]:

$$\dot{\phi}(r + R) = -\dot{R}\tan\theta. \quad (1)$$

The robot is able to determine the surface curvature r of a feature in the environment by fusing qualitative information about $\tan\theta$, $\dot{R}$ and $\dot{\phi}$. Table 1 shows qualitative behaviours consistent with Equation (1) [2]. To illustrate the interpretation of this table consider the first entry in conjunction with the *convex* surface depicted in Figure 3. When the feature lies *left-frontal* (i.e. Qmag(tan(θ))=+) and the range R is decreasing (i.e. Qdir(R)=-) and the gyro is turning anticlockwise (i.e. Qdir(ϕ)=+) then the feature must be a *convex* surface (i.e. Qmag($\frac{1}{r}$)=+). This table can be constructed from Equation (1) using qualitative knowledge of the quantitative algebraic building blocks [13, 20]. For example, qualitative multiplication **Qmult**: $[Qmag(A) = +] \wedge [Qmag(B) = +] \Rightarrow [Qmult(A, B) = +]$ or qualitative negation **Qminus**: $[Qmag(A) = +] \Rightarrow [Qminus(A) = -]$. As is evident

[1] This section builds on work presented elsewhere [20].

[2] The qualitative magnitude and derivative operators are:

$$\begin{aligned} Qmag(X) &= sign(X) \subseteq (-\infty, \infty) \\ Qdir(X) &= sign(\dot{X}) \subseteq (-\infty, \infty). \end{aligned}$$

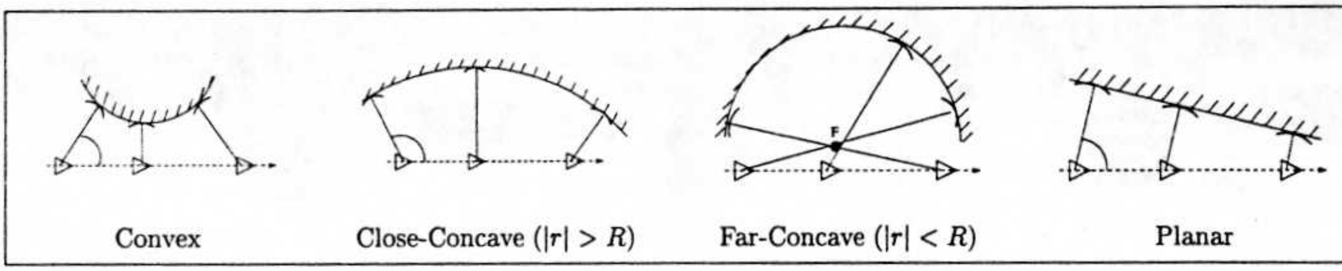


Figure 3: Normal incidence reflection behaviour. *Close-concave* **behaviour is observed when the robot moves between a concave surface and the focal point F of the surface.**

Qmag($\tan\theta$)	Qdir(R)	Qdir(ϕ)	Qmag($\frac{1}{r}$)	Surface Type
+	+	+	−	convex
+	+	−	+/−	concave/convex
+	−	+	+/−	concave/convex
+	−	−	−	convex
−	+	+	+/−	concave/convex
−	+	−	−	convex
−	−	+	−	convex
−	−	−	+/−	concave/convex
+/0/−	+/−	0	0	plane

Table 1: Curvature inferred from sonar and gyroscope cues.

Qmag($\frac{1}{r+R}$)	Qmag($\frac{\partial^2 R}{\partial \phi^2}$)	Qmag($\frac{1}{r}$)	Surface Type
+	+	+	convex
0	+	0	plane
−	+	−	close-concave
+	−	−	far-concave

Table 2: Curvature inferred from sonar, gyroscope and laser cues.

from Table 1, however, the sonar/gyroscope combination is unable to disambiguate curvature types when $R + r > 0$ (i.e. when $\frac{\dot{R}\tan\theta}{\dot{\phi}} < 0$): sonar cues for *far-concave* surfaces exhibit identical qualitative behaviour to *convex* surface cues (see Figure 3). For specular objects the robot is unable to gain any further information by scanning the object using the sonar sensor. However, we may overcome this problem using a laser time-of-flight sensor in conjunction with the sonar sensors. Since the wavelength of laser light (typically 800nm) is significantly shorter than that of ultra-sound (typically 0.5cm), imperfections in the surface cause the laser signal to scatter back towards the sensor for any incident angle μ. The qualitative behaviour of range against bearing for a complete scan of the surface using laser light is able to disambiguate *far-concave* and *convex* surfaces when $R \mp r > 0$. To see this, if μ is the reflectance angle of the laser signal relative to the feature normal (see Figure 2), then it can be shown that:

$$\frac{\partial^2 R}{\partial \phi^2} = R\tan(\mu)^2 + R\left[1+\tan(\mu)^2\right]\left[1+\frac{R}{r\cos\mu}\right].$$

Since for all scenarios R, $\tan^2\mu$ and $\cos\mu$ are positive, then we can see that when $r + R > 0$, $\frac{\partial^2 R}{\partial \phi^2} < 0$ if and only if the surface is *concave* ($r < 0$). Thus, when $R + r > 0$, the laser is able to resolve the ambiguity in the sonar qualitative model. However, using qualitative information alone, the laser is unable to disambiguate *planar* and *convex* surfaces. Hence, we must use the sonar and laser sensors together to determine the surface curvature. Thus, not only does data fusion eliminate noise but it also helps to overcome ambiguity problems in the qualitative models.

In the remainder of this paper, we develop efficient methods for filtering noise in systems employing qualitative representations.

3 THE QUALITATIVE FILTER

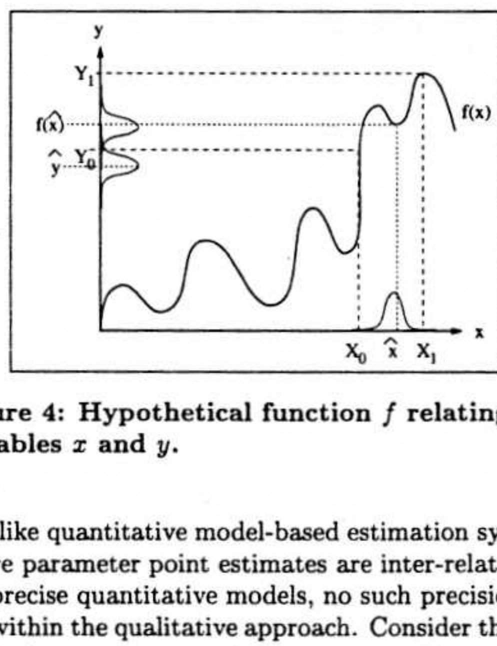


Figure 4: Hypothetical function f relating two variables x and y.

Unlike quantitative model-based estimation systems where parameter point estimates are inter-related using precise quantitative models, no such precision exists within the qualitative approach. Consider the way in which estimates $\hat{x}$ and $\hat{y}$ for two variables x and y are combined using the Kalman filter (see Figure 4). A function f (i.e. process or observation model) transforms $\hat{x}$ into the y-space and the point estimates $f(\hat{x})$ and $\hat{y}$ are combined by weighted averaging. When f is not known precisely we no longer have the requisite point estimates and the best we can say about $f(\hat{x})$ is that it must take some value between Y_0 and Y_1.

In our approach to estimation using qualitative models we transform our point estimate $\hat{x}$ into a Dempster-Shafer mass estimate for the region (X_0, X_1). This mass value is then assigned to the region (Y_0, Y_1) according to our imprecise description of f (that any point in the region (X_0, X_1) maps into (Y_0, Y_1)) and fused with mass estimates generated by $\hat{y}$ using the Dempster rule of combination. Uncertainty in the position of the true state within the quantity-space is represented as a variance measure of the mass function. This approach to filtering is explained in detail in the remainder of this section.

3.1 Notation

At this point, we must introduce the concepts used in the theory developed in later sections.

The *true state* x of the system is a vector $x =< x_0, \ldots, x_n >$ in which the parameter W_i takes the quantitative value x_i.

A *qualitative value* is an interval of the reals. A *state* Q of a set of parameters $\{W\}$ is a vector of qualitative values $Q =< q_0, \ldots, q_n >$, in which the parameter W_i takes the qualitative value q_i. The *quantity space* of a parameter is the set of possible qualitative values the parameter can take.

A Finite State Machine (FSM) takes the role of the process and observation models of the Kalman filter and describes the allowable evolutions of a system. [3] A finite state machine has an associated set $T \subseteq S \times S$ where S is the set of states. Thus, $(Q_i, Q_j) \in T$ iff there exists a single-step transition of the FSM from Q_i to Q_j. The set of *successor states* $R(Q)$ from any state Q is defined:

$$R(Q) = \{Q' | (Q, Q') \in T\}.$$

A *basic path* in the FSM is a sequence of states $Q_0, Q_1, \ldots, Q_{m-1}, Q_m$ such that $\forall i \in 0, \ldots, m-1 \,.\, (Q_i, Q_{i+1}) \in T$. A *compound path* which ends at Q_m is the union of basic paths, each of which end at Q_m. A *path* (denoted $\wp$) is either a basic path or a compound path.

A single observation is denoted z_i and a set of $t+1$ observations indexed a to $a+t$ inclusive is denoted z_a^{a+t}. The set of observations z_a^{a+t} may include observations from different sensors. A set of $t+1$ observations from a single sensor indexed i is denoted $z_{[i,a]}^{[i,a+t]}$.

The mass assigned to a state Q by the set of $t+1$ observations z_0^t along path ρ ending at Q is denoted $m_\rho(Q|z_0^t)$.

The operator E is the expectation over observation ensembles z_0^t.

[3] The FSM is generated from qualitative constraints (*confluences* in the QR literature) by, for example, QSIM [13].

3.2 The P-Norm Estimator Framework

We advocate a weighted information, independent opinion pool paradigm to data fusion [4]. We shall extend the approach to dependent (correlated) information in Section 3.4.

The total evidence assigned to state Q due to the observation set z_0^t is the sum over the evidence offered by each *alternative* path leading to Q. The evidence assigned to each path is *reinforced* by each observation. We introduce a general class of estimators with free parameters which assign weights to paths (i.e. parameter $p \geq 0$) and individual observations (i.e. parameters $K_i \geq 0$). [4] For any state Q and all paths $\wp$ which end at Q the p-norm estimate $m(Q|z_0^t)$ assigned to Q due to observations z_0^t is:

$$m(Q|z_0^t) = k \left[\sum_{\wp} \prod_{i=0}^{t} m_{\wp}^{K_i p}(Q|z_i) \right]^{\frac{1}{p}} \tag{2}$$

where k is a normalisation constant so that $\left[\sum_Q m^p(Q|z_0^t)\right]^{\frac{1}{p}} = 1$. Bayesian methods and the Dempster rule of combination belong to the set of *1-norm* estimators. If for some state Q, $\lim_{t\to\infty} m(Q|z_0^t) = 1$, then m is said to have *converged* to Q. If $m(Q|z_0^t)$ converges then, it must do so to a state Q which contains the true state x otherwise the estimator is deemed to be *biased*.

We will now justify the introduction of the p weights as we derive a sufficient condition for unbiasedness in terms of the expected value of the mass assignment function $m(Q|z)$. Suppose that the mass function $m(Q|z_0^t)$ is formed by fusing two sets of observations z_0^{s-1} and z_s^t. Evidence assigned by z_0^{s-1} and z_s^t is weighted by K_1 and K_2 respectively. For two sets of compound paths through the state network, $\wp \in Qpaths$, and $\wp' \in Q'paths$ such that each member of $Qpaths$ ends at state Q and each member of $Q'paths$ ends at Q', the class of weighted, p-norm estimators satisfies:

$$\begin{aligned}
&\frac{m(Q|z_0^t)}{m(Q'|z_0^t)} \\
&= \left(\frac{\left[\sum_{\wp \in Qpaths} [m_\wp(Q|z_0^{s-1})^{K_1} m_\wp(Q|z_s^t)^{K_2}]^p\right]^{\frac{1}{p}}}{\left[\sum_{\wp' \in Q'paths} [m'_\wp(Q'|z_0^{s-1})^{K_1} m_{\wp'}(Q'|z_s^t)^{K_2}]^p\right]^{\frac{1}{p}}} \right) \\
&\geq \left(\frac{\max_\wp [m_\wp(Q|z_0^{s-1})^{K_1} m_\wp(Q|z_s^t)^{K_2}]}{B^{\frac{1}{p}} \max_{\wp'} [m_{\wp'}(Q'|z_0^{s-1})^{K_1} m_{\wp'}(Q'|z_s^t)^{K_2}]} \right) \\
&= \frac{1}{B^{\frac{1}{p}}} \left(\frac{\max_{\wp_1} m_{\wp_1}(Q|z_0^{s-1})}{\max_{\wp'_1} m_{\wp'_1}(Q'|z_0^{s-1})} \right)^{K_1} \left(\frac{\max_{\wp_2} m_{\wp_2}(Q|z_s^t)}{\max_{\wp'_2} m_{\wp'_2}(Q'|z_s^t)} \right)
\end{aligned}$$

where $\wp = \wp_1 \cup \wp_2$ and $\wp' = \wp'_1 \cup \wp'_2$. The number of paths in the FSM which lead to Q' is $B \geq 1$.

[4] The weights p and K_i must be positive so that the evidence is assigned to the estimate with appropriate polarity.

We now consider the fusion of ensembles of observations from N experiments repeated under the same deterministic conditions, each of which yields $t+1$ observations. For two states Q and Q' within the quantity-space for X such that $Q \cap Q' = \emptyset$ and $x \in Q$ and for observations indexed a to b the likelihood ratio $F(Q, Q', a, b, N)$ is defined to be:

$$F(Q,Q',a,b,N) = \log\left(\prod_{i=0}^{N-1} \frac{\max_{\wp} m_{\wp}(Q|(z_a^b)_i)}{\max_{\wp'} m_{\wp'}(Q'|(z_a^b)_i)}\right).$$

By Equation (3), F satisfies:

$$\begin{aligned} F(Q,Q',0,t,N) \;\geq\; & K_1 F(Q,Q',0,t-s,N) + \\ & K_2 F(Q,Q',s,t,N) - \frac{NB}{p}. \end{aligned}$$

In the limit $N \to \infty$ we obtain:

$$\begin{aligned} \lim_{N\to\infty} F(Q,Q',0,t,N) \;\geq\; & \lim_{N\to\infty}[K_1 F(Q,Q',0,t-s,N) \\ & + K_2 F(Q,Q',s,t,N)] \\ & - \frac{B}{p} \lim_{N\to\infty} N \end{aligned}$$

and by the strong law of large numbers [9] we have:

$$\lim_{N\to\infty} F(Q,Q',0,t,N) =_{a.s.} E(F(Q,Q',0,t,1)) \lim_{N\to\infty} N. \quad (4)$$

Therefore, if we assume that K_1 and K_2 are deterministic, we have:

$$\begin{aligned} E(F(Q,Q',0,t,1)) \;\geq_{a.s.}\; & K_1 E(F(Q,Q',0,s-1,1)) \\ & + K_2 E(F(Q,Q',s,t,1)) \\ & - \frac{B}{p}. \end{aligned} \quad (5)$$

The estimate $m(Q|z_0^t)$ is biased only if $x \in Q$ and the mass converges to some other state Q' say. That is:

$$\exists Q' \,.\, \lim_{N\to\infty} F(Q,Q',0,t,N) = -\infty.$$

By Equation (4), a *sufficient* condition to prevent a biased estimate is for $E(F(Q,Q',0,t,1)) \geq 0$. If we now assign $p \to \infty$ then Equation (5) is independent of B and we can see, by Equation (5), that sufficient conditions for an unbiased estimate are $E(F(Q,Q',0,s-1,1)) \geq 0$ and $E(F(Q,Q',s,t,1)) \geq 0$. We may use Equation (5) to recursively reduce the size of the observation sets to single observations. This leads us to the following theorem. For the class of ∞-norm estimators (i.e. $p \to \infty$), if $\wp$ and $\wp'$ are paths which end at Q and Q' respectively then an estimate $m(Q|z_j)$ is unbiased if for observation j:

$$\exists Q : x \in Q, \wp \,.\, \forall Q', \wp' \,.\, E\left(\log \frac{m_{\wp}(Q|z_j)}{m_{\wp'}(Q'|z_j)}\right) \geq 0.$$

We will assume the limit case $p \to \infty$ for the remainder of this paper. This is the ∞-norm formalism.

Mass Assignment Functions

We initially obtain a general unbiased estimator for binary quantity-spaces $\{Q, \neg Q\}$ and then build estimators for N region spaces from this. For a landmark l separating the decision regions Q and $\neg Q$, the following ***mass assignment function*** is an estimator for the mean of an arbitrary pdf $f(z|x)$ [18]:

$$m_{mean}(Q|z) = \begin{cases} 1 & \iff z \in Q \\ \exp(-|z-l|) & \iff z \in \neg Q. \end{cases} \quad (6)$$

and the following is the mass assignment function for the median of an arbitrary pdf:

$$m_{med}(Q|z) = \begin{cases} 1 & \iff z \in Q \\ \alpha & \iff z \in \neg Q \end{cases} \quad (7)$$

where the constant α satisfies $\alpha < 1$.

One immediate conclusion is that the mean estimator log likelihood ratio $\log \frac{m_{mean}(Q|z_0^t)}{m_{mean}(\neg Q|z_0^t)}$ between binary states Q and $\neg Q$ converges, on average, more slowly the closer x is to the landmark l. The choice of K_i in Equation (2) is therefore crucial to the convergence rate of the estimator and in Section 3.4, we describe the form for K_i which maintains the most certain estimates.

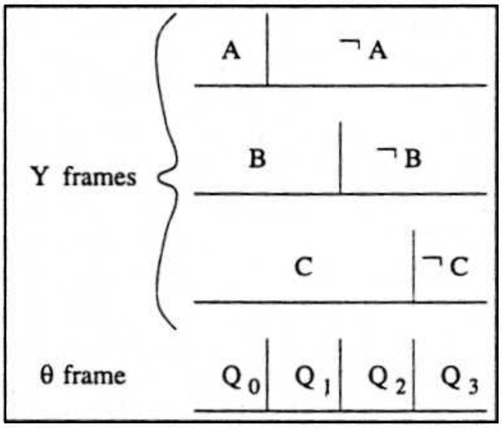


Figure 5: The three binary frames of discernment Υ corresponding to the four region refined frame Θ.

To extend our estimation framework from binary quantity-spaces to N region spaces, we show that filtering in a quantity-space comprising $N-1$ landmarks can be formulated as the conjunction of $N-1$ binary decision problems, each comprising a single landmark. We use the notion of *coarse* frames of discernment from the Dempster-Shafer Theory of Evidence [21]. A *coarse* frame of discernment Υ comprises propositions which are supersets of propositions in Θ (Θ is subsequently called a *refined* frame). For each member $v \in \Upsilon$ the *refining* operator $\omega_\Upsilon : 2^\Upsilon \to 2^\Theta$ gives the subset $\omega_\Upsilon(v)$ of Θ consisting of those possibilities into which v can be split.

Each of the $N-1$ coarse frame mass assignments $m_\Upsilon(Q_\Upsilon|z)$ obtained from an observation using Equation (6) or (7) can be seen as a source of information pertaining to the true state of the system and the true state must have common consensus for all sources. Therefore, the likelihood of x belonging to a state in Θ is represented by the conjunction of the $N-1$ coarse frame mass distributions. To illustrate, the three coarse frames in Figure 5 denoted A, B, and C are coarsenings of the underlying frame of discernment $\Theta = \{Q_0, Q_1, Q_2, Q_3\}$ such that:

$$\begin{aligned} \omega_A(\{A\}) &= \{Q_0\} & \omega_A(\{\neg A\}) &= \{Q_1, Q_2, Q_3\} \\ \omega_B(\{B\}) &= \{Q_0, Q_1\} & \omega_B(\{\neg B\}) &= \{Q_2, Q_3\} \\ \omega_C(\{C\}) &= \{Q_0, Q_1, Q_2\} & \omega_C(\{\neg C\}) &= \{Q_3\}. \end{aligned}$$

Since, for any individual binary coarse frame $\upsilon \subseteq \Upsilon$ [21]:

$$m_\Theta(\omega_\Upsilon(\upsilon)) = m_\Upsilon(\upsilon) \tag{8}$$

we have the following refined frame assignments from the mass assigned to the coarse frames by consensus:

$$\begin{aligned} m_\Theta(\{Q_0\}) &= k m_A(\{A\}) m_B(\{B\}) m_C(\{C\}) \\ m_\Theta(\{Q_1\}) &= k m_A(\{\neg A\}) m_B(\{B\}) m_C(\{C\}) \\ m_\Theta(\{Q_2\}) &= k m_A(\{\neg A\}) m_B(\{\neg B\}) m_C(\{C\}) \\ m_\Theta(\{Q_3\}) &= k m_A(\{\neg A\}) m_B(\{\neg B\}) m_C(\{\neg C\}) \end{aligned}$$

where k is a normalisation factor. Normalising by the mass assigned to the central state $m_\Theta(\{Q_1\})$, say, we obtain:

$$\frac{m_\Theta(\{Q_0\})}{m_\Theta(\{Q_1\})} = \frac{m_A(\{A\})}{m_A(\{\neg A\})}, \quad \frac{m_\Theta(\{Q_2\})}{m_\Theta(\{Q_1\})} = \frac{m_B(\{\neg B\})}{m_B(\{B\})}$$

and:

$$\frac{m_\Theta(\{Q_3\})}{m_\Theta(\{Q_1\})} = \frac{m_B(\{\neg B\})}{m_B(\{B\})} \frac{m_C(\{\neg C\})}{m_C(\{C\})}.$$

In general, we have, for singleton states Q and Q' in Θ:

$$\log \frac{m_\Theta(Q|z)}{m_\Theta(Q'|z)} = \sum_\Upsilon \log \frac{m_\Upsilon(\omega_\Upsilon^{-1}(\{Q\})|z)}{m_\Upsilon(\omega_\Upsilon^{-1}(\{Q'\})|z)}. \tag{9}$$

We next combine the ∞-norm filtering formalism developed so far with the Dempster-Shafer Theory of Evidence to produce the Qualitative filter. A straight forward interface is not possible as we seek the maximum mass assignment according to the ∞-norm independent opinion pool formalism developed in the previous section. To address this problem, we develop the ∞-norm Dempster-Shafer Theory of Evidential Reasoning.

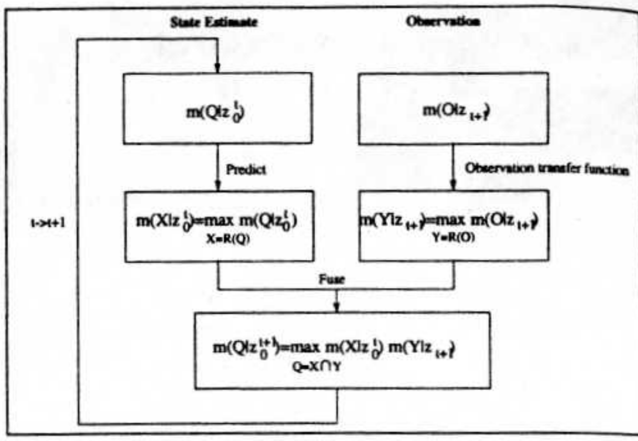


Figure 6: The Qualitative Filter Cycle. The state mass estimate is propagated to successor states X. Mass $m(O|z_{t+1})$ assigned by a new observation in quantity space O is transferred to the state quantity space and fused with $m(X|z_0^t)$.

3.3 The ∞-norm Filter Cycle

In accordance with most quantitative filters, the Qualitative filter consists of a *prediction stage* and an *update stage*: states are inferred and assigned mass during the prediction stage. Mass estimates from new observations are transferred using the observation transfer function (i.e. FSM) and fused with the predicted state mass values in the *update stage* using the ∞-norm Dempster rule of combination. Conceptually, the p-norm generalisation of the Dempster-Shafer theory is no different to the original Dempster-Shafer theory. The p-norm formalism generates an infinite class of theories of which the Dempster-Shafer theory is a special case. For any mass assignment function m on the frame of discernment Θ, we have [18]:

$$\begin{aligned} \forall A \subseteq \Theta \quad m_p(A) &\geq 0 \\ m_p(\emptyset) &= 0 \\ \left[\sum_{A \subseteq \Theta} m_p(A)^p\right]^{\frac{1}{p}} &= 1. \end{aligned}$$

with the following *belief* and *plausibility* functions:

$$Bel_p(A) = \left[\sum_{B \subseteq A} m_p(B)^p\right]^{\frac{1}{p}} \qquad Pl_p(A) = \left[\sum_{B \cap A \neq \emptyset} m_p(B)^p\right]^{\frac{1}{p}}$$

and the following Dempster Rule of Combination:

$$m_p(A|z_x, z_y) = k \left[\sum_{A = X \cap Y} mx_p(X|z_x)^p my_p(Y|z_y)^p\right]^{\frac{1}{p}}.$$

As can be seen, the original Dempster-Shafer theory corresponds to $p = 1$. In the limit $p \to \infty$ the Dempster combination rule is an ∞-norm information pool, a requirement from Section 3.2. Thus, the ∞-norm Dempster-Shafer theory is appropriate for our needs:

$$\forall A \subseteq \Theta \quad m_\infty(A) \geq 0 \qquad Bel_\infty(A) = \max_{B \subseteq A} m_\infty(B)$$
$$m_\infty(\emptyset) = 0 \qquad Pl_\infty(A) = \max_{B \cap A \neq \emptyset} m_\infty(B)$$
$$\max_{A \subseteq \Theta} m_\infty(A) = 1.$$

with the following rule of combination:

$$m_\infty(A|z_x, z_y) = k \max_{A = X \cap Y} mx_\infty(X|z_x) my_\infty(Y|z_y).$$

In the 1-norm formalism all evidence pointing towards a proposition contributes to the belief in that proposition whereas, in the ∞-norm formalism only the least corrigible evidence contributes.

The update stage of the filtering cycle is captured by the ∞-norm Dempster rule of combination. For the prediction stage of the cycle the observation and process model transfer functions can be clearly seen to obey, for some possible future state X:

$$m(X|z_0^t) = \max_{X = R(Q)} m(Q|z_0^t)$$

where $R(Q)$ are the successor states of Q. Figure 6 depicts the filter cycle diagrammatically.

In the next section we describe how, given a finite set of observations, we can identify the quantity-space region which contains the true state.

3.4 Meta-Reasoning and the Decision Rule

Confidence measures in the log likelihood ratio are required to indicate when a rational decision can be made. As described in Section 3.2, we can decide $x \in Q$ for a finite set of observations whenever:

$$\exists Q : x \in Q, p \,.\, \forall Q', p' \,.\, E\left(\log \frac{m_p(Q|z_0^t)}{m_{p'}(Q'|z_0^t)}\right) \geq 0. \quad (10)$$

In quantitative data fusion systems, covariance is a natural measure of confidence and, analogously, we can estimate the expected value in Equation (10) from the empirically determined $\log \frac{m_p(Q|z_0^t)}{m_{p'}(Q'|z_0^t)}$ and the variance of this value. If we choose to make the decision $x \in Q$ when [18]:

$$\forall Q' \neq Q, p' \,.\, \log \frac{m_p(Q|z_0^t)}{m_{p'}(Q'|z_0^t)} > G \sqrt{Var\left(\log \frac{m_p(Q|z_0^t)}{m_{p'}(Q'|z_0^t)}\right)} \quad (11)$$

then, we can show using Chebyshov's inequality that:

$$Pr\left(E\left(\log \frac{m_p(Q|z_0^t)}{m_{p'}(Q'|z_0^t)}\right) \geq 0\right) \geq 1 - \frac{1}{G^2} \quad (12)$$

We note that the variance the log likelihood ratio in Equation (11) is Lagrangian (as opposed to Eulerian) in the sense that it measures the variance of mass assigned to a state by a particular path and not the variance of the actual mass assigned to the state which is the maximum mass of all paths leading to the state. A minimum value for $Var\left(\log \frac{m_p(Q|z_0^t)}{m_{p'}(Q'|z_0^t)}\right)$ can be obtained by frugally combining the mass assignment functions as we will now see.

It turns out that the covariance of the log likelihood ratio for two observations z_1 and z_2 for binary coarse frames is directly related to the covariance of the observations themselves. For a mean estimator we have:

$$Cov\left(\log \frac{m_{mean}(Q|z_1)}{m_{mean}(\neg Q|z_1)}, \log \frac{m_{mean}(Q|z_2)}{m_{mean}(\neg Q|z_2)}\right) = Cov(z_1, z_2).$$

and for our median estimator we have:

$$Cov\left(\log \frac{m_{med}(Q|z_1)}{m_{med}(\neg Q|z_1)}, \log \frac{m_{med}(Q|z_2)}{m_{med}(\neg Q|z_2)}\right) \leq (\log \alpha)^2. \quad (13)$$

The variance of the log likelihood ratio in the refined frame Θ can be obtained using Equation (9). For two regions Q_i and Q_{i+n} in the refined frame Θ (see Figure 5) we have:

$$Var\left(\log \frac{m_\Theta(Q_i|z)}{m_\Theta(Q_{i+n}|z)}\right) = nVar\left(\log \frac{m_\Upsilon(Q|z)}{m_\Upsilon(\neg Q|z)}\right). \quad (14)$$

Before continuing to derive a recursive expression for $Var\left(\log \frac{m_p(Q|z_0^t)}{m_{p'}(Q'|z_0^t)}\right)$ we introduce some notation:

$$M_0^t = \log \frac{m_p(Q|z_0^t)}{m_{p'}(Q'|z_0^t)} \qquad V_0^t = Var\left(\log \frac{m_p(Q|z_0^t)}{m_{p'}(Q'|z_0^t)}\right)$$
$$M_0^{t-1} = \log \frac{m_p(Q|z_0^{t-1})}{m_{p'}(Q'|z_0^{t-1})} \qquad V_0^{t-1} = Var\left(\log \frac{m_p(Q|z_0^{t-1})}{m_{p'}(Q'|z_0^{t-1})}\right)$$
$$M_t = \log \frac{m_p(Q|z_t)}{m_{p'}(Q'|z_t)} \qquad V_t = Var\left(\log \frac{m_p(Q|z_t)}{m_{p'}(Q'|z_t)}\right)$$

and:

$$C = Cov\left(\log \frac{m_p(Q|z_0^{t-1})}{m_{p'}(Q'|z_0^{t-1})}, \log \frac{m_p(Q|z_t)}{m_{p'}(Q'|z_t)}\right).$$

By Equation (11), we make a decision whenever our decision metric $D = \frac{V_0^t}{(M_0^t)^2}$ satisfies $D \leq \frac{1}{G^2}$ for some gate G. As we noted in Section 3.2, the rate of convergence of our estimates decreases the closer the true state is to the landmark and so it is important to maintain minimal uncertainty in our estimate if an early decision is to be made. We can use the flexibility of our observation weights K_1 and K_2 in Equation (2) for two observation sets to tune the decision rate of our filter. We assign, without loss of generality, $K_1 = W - K$ and $K_2 = K$ with $W \geq K \geq 0$ so that $M_0^t = (W - K)M_0^{t-1} + KM_t$. Unfortunately, the value of K corresponding to the critical (minimum) value of our decision metric D is stochastic which violates the conditions for Equation (5). Alternatively, we can maximise $\frac{(E(M_0^t))^2}{V_0^t}$ and obtain $K = W \frac{SV_0^{t-1} - C}{SV_0^{t-1} + V_t - C[1+S]}$ where the *zero-noise decision strength* S between observation z_t and observations z_0^{t-1} is defined to be $S = \frac{E(M_t)}{E(M_0^{t-1})}$. Thus, we have [18]:

$$(V_0^t)_{mmse} = W^2 \frac{(V_t V_0^{t-1} - C^2)(V_t + S^2 V_0^{t-1} - 2CS)}{(SV_0^{t-1} + V_t - C[1+S])^2} \quad (15)$$

and:

$$(M_0^t)_{\text{mmse}} = W\frac{SM_tV_0^{t-1} + M_0^{t-1}V_t - C[M_t + SM_0^{t-1}]}{SV_0^{t-1} + V_t - C[1+S]}. \quad (16)$$

Thus, analogous to the Kalman filter, evidence is weighted inversely to its variance (i.e. consider $S = 1$ and $C = 0$). The zero-noise decision strength S reflects the fact that observations of a true state lying further from a landmark produce a faster converging estimate and, therefore, should have more weight. The filter is optimal when $S = \frac{E(M_t)}{E(M_0^{t-1})}$ but since the expected values of the masses are not known apriori, obtaining optimal values for S is problematic. This is an area of ongoing research.

We shall now see how the MMSE Qualitative filter can be extended to stochastic process models.

3.5 Estimators for Stochastic Processes

For many physical systems, it is desirable to associate stochastic components with the process and observation models themselves [2]. This can arise in situations when a complete qualitative deterministic model is unavailable. We demonstrate how the qualitative estimator can be generalised to accommodate stochastic models.

We may cater for stochastic mass propagation by introducing a random mass drift σ into the mass ratio update stage of the filter cycle. If $\frac{m^*(Q|z_0^t)}{m^*(Q'|z_0^t)}$ represents the deterministic mass assignment for a given observation sequence, and we assume that the drift of the true state x is uncorrelated over time, then we can write, for the actual mass assignment $\frac{m(Q|z_0^t)}{m(Q'|z_0^t)}$:

$$\frac{m(Q|z_0^t)}{m(Q'|z_0^t)} = \frac{m^*(Q|z_0^t)}{m^*(Q'|z_0^t)}\sigma(Q,Q') \quad (17)$$

where the logarithmic drift $\log\sigma$ is assumed to have zero mean. $\sigma > 1$ represents a drift towards or deeper into region Q and $\sigma < 1$ represents a drift towards or deeper into region Q'.

We will now see how stochastic mass assignment can model the underlying physical stochastic system. The log likelihood ratio $\log\frac{m(Q|x_0^t)}{m(Q'|x_0^t)}$ describes the mass assigned by $t+1$ observations under conditions of zero observation noise and indicates the progress of the stochastically driven true state x. It is this value which we aim to estimate. We have:

$$\log\frac{m(Q|x_0^t)}{m(Q'|x_0^t)} = (1-K)\left(\log\frac{m(Q|x_0^{t-1})}{m(Q'|x_0^{t-1})} + \log\sigma\right) + K\left(\log\frac{m(Q|x_t)}{m(Q'|x_t)}\right). \quad (18)$$

Maintaining a value for $\log\frac{m(Q|x_0^t)}{m(Q'|x_0^t)}$ is problematic since σ is not known. However, we can show, by induction, that $M_0^t = \log\frac{m(Q|z_0^t)}{m(Q'|z_0^t)}$ is an unbiased estimate for $\log\frac{m(Q|x_0^t)}{m(Q'|x_0^t)}$. In the following we adopt the notation of Section 3.4, and introduce:

$$\Sigma = Var(\log\sigma). \quad (19)$$

We assume that the estimate is unbiased at time $t-1$ so that:

$$E\left(M_0^{t-1} - \log\frac{m(Q|x_0^{t-1})}{m(Q'|x_0^{t-1})}\right) = 0.$$

If we choose $m(Q|z)$ to be the mean estimator m_{mean} then, since $E(\log\sigma) = 0$ and $E\left(\log\frac{m(Q|z_j)}{m(\neg Q|z_j)} - \log\frac{m(Q|x_j)}{m(\neg Q|x_j)}\right) = 0$ then, by Equation (18):

$$\begin{aligned} E&\left(M_0^t - \log\frac{m(Q|x_0^t)}{m(Q'|x_0^t)}\right) \\ &= (1-K)E\left(M_0^{t-1} - \log\frac{m(Q|x_0^{t-1})}{m(Q'|x_0^{t-1})} - \log\sigma\right) \\ &= 0. \end{aligned}$$

Hence, M_0^t is an unbiased estimate for $\log\frac{m(Q|x_0^t)}{m(Q'|x_0^t)}$. If we assume further that σ is uncorrelated with all other noise terms then the variance of the estimated value about the true log likelihood ratio is:

$$\begin{aligned} E&\left(\left(M_0^t - \log\frac{m(Q|x_0^t)}{m(Q'|x_0^t)}\right)^2\right) \\ &= E\left(\left[(1-K)\left(M_0^{t-1} - \log\frac{m(Q|x_0^{t-1})}{m(Q'|x_0^{t-1})} - \log\sigma\right)\right.\right. \\ &\quad \left.\left. + K\left(\log\frac{m(Q|z_t)}{m(Q'|z_t)} - \log\frac{m(Q|x_t)}{m(Q'|x_t)}\right)\right]^2\right) \\ &\quad + 2K(1-K)C \\ &= (1-K)^2 Var\left(M_0^{t-1} - \log\frac{m(Q|x_0^{t-1})}{m(Q'|x_0^{t-1})}\right) \\ &\quad + (1-K)^2\Sigma + K^2V + 2K(1-K)C. \end{aligned}$$

Thus, we can obtain an estimator for stochastic processes by replacing the update variance V_0^{t-1} with $V_0^{t-1} + \Sigma$ in Equations (15) and (16).

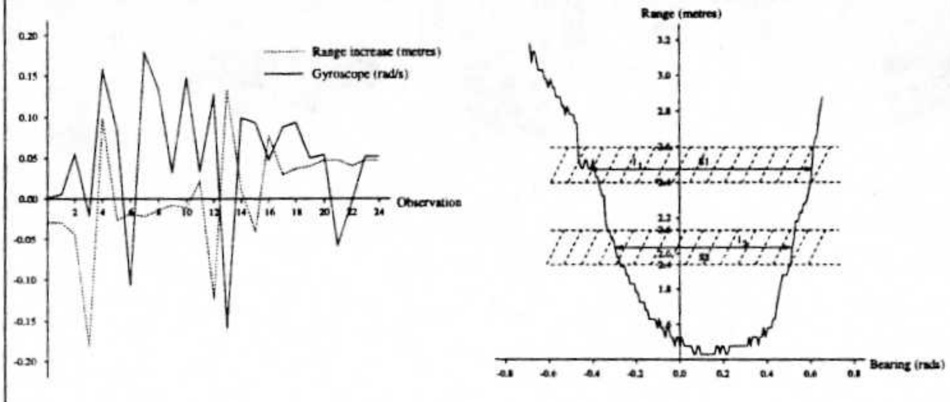
To determine a numeric value for Σ, we use Equation (6) to obtain:

$$\begin{aligned} \Sigma &= E\left(\left[\log\frac{m(Q|x)}{m(\neg Q|x)} - \log\frac{m^*(Q|x)}{m^*(\neg Q|x)}\right]^2\right) \\ &= E\left([x - l - (x^* - l)]^2\right) \\ &= E\left([x - x^*]^2\right) \end{aligned}$$

where x^* represents the noise-free value of the true-state and x is the true-state subject to a random variable conditioned on x^*. Thus, Σ is the drift variance of the true-state x which can be determined empirically.

4 The Robot Sensing Domain Revisited

Returning to the sensing domain introduced in Section 2, we implement a MMSE mean estimator as a



(a) Sonar and Gyroscope

(b) Laser

Figure 7: Laser range data and sonar range data and gyroscope data for a *convex* surface.

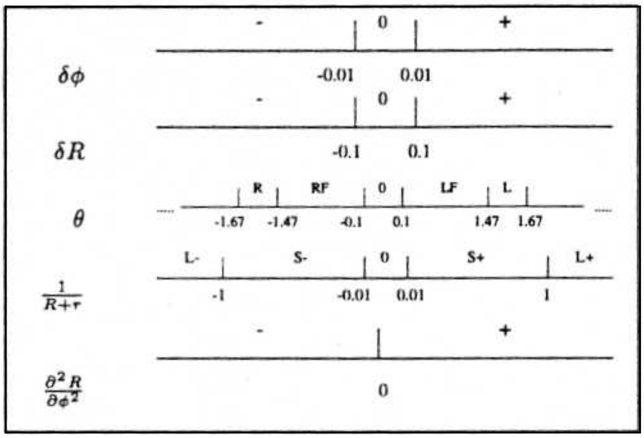


Figure 8: Quantity-spaces for robot navigation domain. ***L*+ and *S*+ denote *large positive* and *small positive* values respectively.**

Dempster-Shafer network and demonstrate this with an application in the domain of mobile robot feature recognition which was described in Section 2.

A network methodology provides a way of breaking the evidence that bears on a large problem into smaller items of evidence that bear on smaller parts of the problem so that these smaller problems can be dealt with one at a time. As such, this has applications in multi-sensor fusion when the problem state is too large to maintain, or when information is not available to maintain a complete state model or when decentralised (i.e. sensor parallel) computations are more appropriate in tasks which allow localisation of effort. The basic idea of local computation for propagating probabilities is due to Judea Pearl. Shenoy [22] extends this idea to belief functions in *Belief Networks*. In this section, we show that the Qualitative filter can be implemented as a belief network, thus demonstrating its applicability to large systems.

In the following experiment data from sparsely calibrated sensors is used to determine the curvature of some feature in the environment using the qualitative models described in Section 2. The calibrated readings (i.e. the landmarks) and the quantity-spaces for this domain are shown in Figure 8 [5] and the belief network is shown in Figure 9 (information flow is in the directions indicated). Estimates for $Qdir(\phi)$ and $Qmag(\dot{R}\tan\theta)$ are maintained using the MMSE mean estimator (Equations (15) and (16)).

Initially, the sonar mass estimates for $Qmag(\dot{R}\tan\theta)$ are propagated to the $\frac{1}{R+r}$ node and combined with the estimate for Qdir(ϕ) from the gyroscope thus:

$$m_{\frac{1}{R+r}}(A) = \max_{Qmult(A,B)=Qminus(B)} \{m_{\dot{\phi}}(B) m_{\dot{R}\tan\theta}(C)\}.$$

The sonar+gyroscope estimate for Qmag($\frac{1}{r+R}$) and the laser curvature estimate Qmag($\frac{\partial^2 R}{\partial\phi^2}$) are then passed to the r node which fuses them according to Table 2 in Section 2.

The data obtained from the sensors, while observing a convex surface, is shown in Figure 7. The following observation errors (standard deviations) were assumed for the sonar sensor and gyroscope: 0.01 metres for R, 0.02 rads for θ and 0.1 rads for $\delta\phi$. Figure 10 shows the second standard deviation limits for the expected log likelihood for the various curvature types discerned by the sonar+gyroscope combination. The decision that the surface is *convex* or *far-concave* can be made with 75% confidence after 3 observations have been received from each sensor at which point the following

[5] In order to filter to landmark values, each landmark is "blurred" and represented as a narrow interval on the real line (e.g. $Qmag(X) = 0 \Rightarrow X \in (-0.01, 0.01)$).

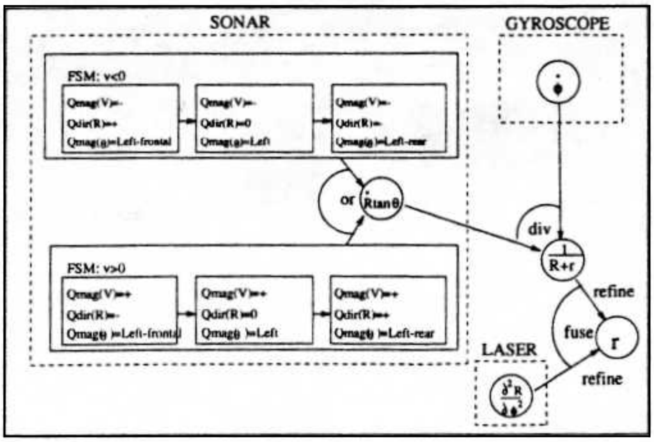


Figure 9: Robot domain belief network. SONAR node indicates direction of robot motion (velocity $v = -\frac{\dot{R}}{\cos\theta}$). The GYROSCOPE node output combined with the SONAR node output gives $m(Qmag(\frac{1}{r+R})|\cdot)$.

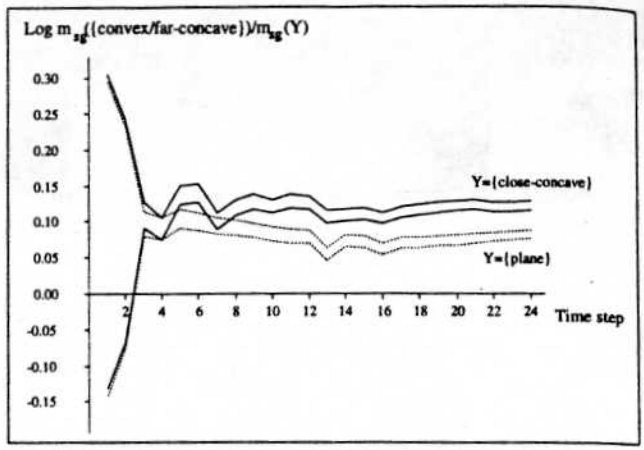


Figure 10: Mass assigned to curvature hypothesis by sonar sensor and gyroscope.

mass values were obtained for $Qmag(\frac{1}{r+R})$:

$$\log \frac{m_{sg}(\{convex, far\text{-}concave\})}{m_{sg}(\{planar\})} = 0.11 \pm 0.02$$

$$\log \frac{m_{sg}(\{convex, far\text{-}concave\})}{m_{sg}(\{close\text{-}concave\})} = 0.10 \pm 0.02.$$

To determine surface curvature from the laser data we identify two subsets of data points $S1$ and $S2$, (shown as shaded regions in Figure 7) such that the data in $S1$ have greater range values that those in $S2$. We then evaluate the mean bearing distance l between points within the same section and the variance of this distance. We can then determine the curvature from $E(l_1 - l_2)$:

$$Qmag(E(l_1 - l_2)) \begin{cases} > 0 & \iff Qmag(\frac{\partial^2 R}{\partial \phi^2}) = + \\ < 0 & \iff Qmag(\frac{\partial^2 R}{\partial \phi^2}) = -. \end{cases}$$

We assume that the laser sensor range information is accurate to $0.05m$ and the bearing inaccuracy is comparatively negligible [6]. Thus, from a single scan, the following estimates for $Qmag(\frac{\partial^2 R}{\partial \phi^2})$ were obtained from the laser sensor data:

$$\log \frac{m_l(\{close\text{-}concave, convex, planar\})}{m_l(\{far\text{-}concave\})} = \log \frac{m_{\frac{\partial^2 R}{\partial \phi^2}}(\{+\})}{m_{\frac{\partial^2 R}{\partial \phi^2}}(\{-\})} = 0.35 \pm 0.05.$$

Combining laser and sonar estimates using the Dempster rule we obtain the equations shown at the top of the next page from which the surface is correctly inferred to be *convex*.

5 CONCLUSIONS

In this paper, we have addressed the problem of data fusion and parameter estimation in systems which use qualitative models. We maintain that the more sensors available the better the estimate both for overcoming qualitative model ambiguity and for filtering noise in the data. We have developed a minimum-mean-square-error decision framework and we have explored four levels of (meta-) reasoning about uncertainty in this framework: the transformation of observation to mass via the mass assignment function, the uncertainty (variance) in this mass due to the random nature of the observations, the relationship of variance to mass for decision purposes and lower bounds on the reliability of the decision by Chebyshov's inequality.

We contrast our approach with the usual approach to noisy information in the QR literature [8]. In these methods the uncertainty in an observation is captured by an interval within the domain representation and correspondence (i.e. fusion) is the result of the intersection of intervals. Two problems exist with this approach: firstly, the extra imprecision can cause the interval constraining the true state to increase rapidly and information is lost; secondly, systems which assign large intervals to noisy data can be slow to converge (and may not converge at all) and those which assign too small an interval can rapidly become inconsistent when intervals do not overlap. In our approach, the intervals reflect imprecision in the models only and therefore, maintain a tight estimate of state. Further, our estimates converge even if initial large noise estimates are chosen.

Our methods avoid the often cited problem that the result of combining two dependent belief functions using Dempster's rule is not consistent with probability theory [15, 17]. Our approach does not rely on an interpretation of the mass values in terms of probability theory but it operates with state preference orderings and a variance driven decision method. Dependencies between mass functions, which indicate uncertainty in the preference ordering, are represented explicitly in our system by their covariances.

$$\log \frac{m(\{convex\})}{m(\{close\text{-}concave\})} = \log \frac{m_{sg}(\{convex, far\text{-}concave\})}{m_{sg}(\{close\text{-}concave\})} + \log \frac{m_l(\{close\text{-}concave, convex, planar\})}{m_l(\{close\text{-}concave, convex, planar\})} = 0.10 \pm 0.02$$

$$\log \frac{m(\{convex\})}{m(\{far\text{-}concave\})} = \log \frac{m_{sg}(\{convex, far\text{-}concave\})}{m_{sg}(\{convex, far\text{-}concave\})} + \log \frac{m_l(\{close\text{-}concave, convex, planar\})}{m_l(\{far\text{-}concave\})} = 0.35 \pm 0.05$$

$$\log \frac{m(\{convex\})}{m(\{planar\})} = \log \frac{m_{sg}(\{convex, far\text{-}concave\})}{m_{sg}(\{planar\})} + \log \frac{m_l(\{close\text{-}concave, convex, planar\})}{m_l(\{close\text{-}concave, convex, planar\})} = 0.11 \pm 0.02$$

Our framework supports mixed estimator type fusion and can be used in situations when the mean of one sensor output corresponds with the median of the output from a different sensor. Our framework is also amenable to network implementation and this has been demonstrated in the robot sensor fusion domain.

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