# Using Qualitative Reasoning to Solve Dynamic Problems

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**Abstract**

A significant amount of research has been done in solving static engineering problems. Dynamic problems, where at least one of the parameters of the system is changing, pose a larger challenge that has only been addressed in part by previous work. We describe two new qualitative reasoning techniques that we use to solve additional problems. In the first technique, we use qualitative reasoning to determine when parameters stay constant between states, which allows us to eliminate variables from equations. This is useful in solving problems that have partial information and require information to be shared between different states in order to simplify equations. In the second technique, we use a guided attainable envisionment to determine which equations are applicable in a particular state and to guide calculations using these equations. This technique helps to solve problems that require the use of different equations in different states. In some situations, the envisionment also allows us to conclude some simplifying modeling assumptions. We also describe how to use quantitative information from the problem solver to focus this envisionment and prevent state explosion.

## Introduction

Most earlier problem solvers had some forms of dynamic problem solving mechanisms. De Kleer (de Kleer 1975) was the first to argue that qualitative reasoning is needed for problem solving. His system, NEWTON, solved roller coaster problems by using attainable envisionment. Since the envisionment constraints were hard coded, the system could only solve more problems of the same type. Another solver, MECHO (Bundy 1978; Bundy 1983), used schemata and a strategy called "hypothesize and test" involving envisionment and numerical testing. MECHO solved some static problems, such as pulley systems, and some roller coaster problems. CASCADE (VanLehn, Jones, & Chi 1991) reasoned about some dynamic physics problems by using similar solved examples as a starting point. The lack of a general mechanism for directly reasoning about changes limited CASCADE to problems that had analogues in its database.

Engineers reason about how different devices interact with each other by selecting device models appropriate for a particular task. These models provide the framework for deciding which equations are applicable. Instead of focusing on model selection though, engineering textbooks focus on teaching quantitative reasoning with emphasis on solving problems (Bhaskar & Simon 1977). Students develop expertise in selecting models through solving problems. Qualitative reasoning provides a framework for organizing the knowledge necessary for model selection. Because model selection is critical for many tasks, a large amount of research in qualitative reasoning has concentrated on how to select models (Falkenhainer & Forbus 1991; Iwasaki & Levy 1994).

Recently, qualitative reasoning has been used successfully for analyzing complex systems (Sacks & Joskowicz 1993; Yip 1991), designing a chemical separation system (Sgouros 1993) and designing controllers (Kuipers & Shults 1994). In SCHISM (Skorstad & Forbus 1990), qualitative reasoning was used to formulate the equations describing steady-state systems, such as a closed refrigeration cycle, and then to calculate the desired parameter. Although these systems focused on modelling or controlling the behavior of a system rather than problem solving, these successful applications demonstrate the superior technologies that are available for performing qualitative envisionment now.

We take advantage of the advances in qualitative reasoning theory and technology to solve additional problems in two subclasses of dynamic problems. The ideas described here have been implemented as a part of a thermodynamics problem solver (TPS) system (Pisan 1996). First, we argue why we want to solve dynamic problems, and why qualitative reasoning is needed to solve them. Then, we describe the use of qualitative reasoning in determining which parameters remain constant between states and how this knowledge is applied to simplify equations. We describe the situations when hidden states arise and present a technique for discovering them. Next, we give two examples of how these ideas are used when solving problems. Finally, we discuss our conclusions.

## Qualitative Reasoning and Engineering Problems

In most engineering domains, education is based on problem solving. Discussions with a domain expert (Dr.

Peter B. Whalley, University of Oxford, personal communication) suggest that textbook problems are reasonable approximations for engineering analyses performed in industry, because they incorporate practical applications with basic engineering knowledge to derive quantitative solutions.

Dynamic problems, where at least one of the parameters of the system is changing, comprise a large number of engineering problems. In physics, textbooks cover kinematics, dynamics, projectile motion, gravity, waves, and a number of other dynamic phenomena. Thermodynamics textbooks use static problems as a part of the introduction to the domain and then build up to dynamic problems involving multiple states as students develop expertise in the domain.

Limiting a problem solver to static problems sacrifices the ability of the problem solver to examine interesting behavior. In a sense, the problem solver is reduced to examining photographs of our usual everyday experience. It is not a surprise that previous work has examined static problems. This limitation often makes for much simpler solutions. For example, in thermodynamics, a typical static problem, such as determining the value of a parameter of a gas given other parameters of the gas, will often require only one equation. Still, the great majority of problems are not static problems. These include more interesting problems where the system is changing, such as real world examples.

To solve these more interesting and complex problems, we must incorporate a mechanism for reasoning about change into our problem solver. Qualitative reasoning provides this mechanism.

We use the qualitative reasoning technique of modeling change by using histories which are extended through time and spatially bounded (Hayes 1979). This approach allows us to create world models that do not require complete knowledge and also greatly reduces the state explosion problem by specifying that only objects in the same location can interact with each other. Qualitative Process Theory (Forbus 1984) introduces processes as the agents of change and describes a framework for reasoning with multiple models. Heat transfer between objects, acceleration due to forces (such as gravity) and the effects of friction forces are examples of processes that have been used in qualitative physics. One common and significant problem with qualitative analysis is that ambiguity can cause too many states to be generated. In our case, however, numbers and ordinal relations given in the problem enable us to constrain the qualitative analysis so that we can avoid this problem.

Qualitative reasoning provides the framework we need for reasoning about change in the problem solving context. In the next section, we describe how the information from qualitative reasoning about a parameter being constant is exploited to simplify problem solving.

## Finding and Using Constant Parameters

When solving problems, students struggle with how to combine equations to eliminate unknown parameters. In domains like thermodynamics where there can be over one hundred applicable equations, this presents a formidable challenge for students. Problems with multiple states further complicate matters by introducing the concern of how equations that are applicable to different states should be combined. To reduce these equations, it is necessary to know what parameters stay constant between states. Knowing the parameters that are constant lets us simplify and combine equations and solve problems despite incomplete state information. Using Qualitative Process Theory we can determine what parameters are constant by verifying that there are no active processes influencing those parameters. Knowing that these parameters are constant allows us to share information between the states. This can help us to simplify and combine equations.

For example, a typical problem in thermodynamics is to consider the consequences of heating a body of a given substance. In the simplest case, where an ideal gas is heated, the ideal gas law (Equation 1) applies to both the initial and final states. Because there are no active processes influencing the mass of the gas (m), a problem solver can conclude that m remains constant when the gas is heated. Domain knowledge asserts that the gas constant (R) is constant. This knowledge allows combining the initial state ideal gas law with the final state ideal gas law to create a simpler equation. In cases where the mass of the gas or the gas constant is not known, this simplification is necessary for the problem solver to be able to solve this problem. If the initial and final state of the substance being heated is known to be saturated, the problem solver can derive that the process is isobaric and isothermal. These findings serve as modeling assumptions to introduce more equations into the analysis and simplify existing equations.

Performing this kind of analysis is also necessary for simplifying equations that involve parameters from two states, such as the first law of thermodynamics (Equation 2). When applying the first law of thermodynamics, modeling assumptions and knowledge of what parameters remain constant between the states must be used to simplify the equation. For example, when applying the steady-state modeling assumption to a device such as a turbine, the

**Ideal Gas Law applied to initial state:**

$P_{Initial}V_{Initial} = m_{Initial}R_{Initial}T_{Initial}$

**Ideal Gas Law applied to final state:**

$P_{Final}V_{Final} = m_{Final}R_{Final}T_{Final}$

**Results of qualitative analysis:**

R and m are constant.

($R_{Initial} = R_{Final}$ and $m_{Initial} = m_{Final}$)

**Ideal Gas Law combined:**

$P_{Initial}V_{Initial}/T_{Initial} = P_{Final}V_{Final}/T_{Final}$

**Equation 1: Combining Ideal Gas Law Applications**

modeling assumption implies that the parameters of the substance inside the turbine are constant. This implies that all variables for substance in the initial state (subscript 1) and the final state (subscript 2) are equal, and cancel each other in the equation. We use modelling assumptions for the turbine that eliminate kinetic energy and potential energy, since their effects are often negligible. Alternatively, these modelling assumptions could be made automatically by using order of magnitude reasoning techniques, such as decompositional modelling (Williams & Raiman 1994). Eliminating all of these variables reduces Equation 2 to a version of first law that is typically used for turbines (Equation 3).

In both of these cases, qualitative reasoning determined which parameters were constant and allowed the problem solver to combine equations and eliminate variables. It also allowed the problem solver to make a domain modeling assumption that allowed it to further conclude the future behavior of a system. Next, we will look at using a form of envisionment to help describe the behavior of systems in problems.

## Discovering Hidden States

We have discussed problems that have an initial state and a final state, but the behavior of a system for a complex problem can have any number of states. Some of these states may be mentioned in the problem. We refer to the other states that are not mentioned in the problem as "hidden states". To generate the states for the behavior of a system defined by a problem, we perform *problem guided envisionment*. This requires a qualitative domain theory for the area covered by the problem.

A well-defined domain theory identifies important ordinal relations in the domain. These ordinal relations may involve a constant and one parameter of one object, such as determining that a liquid is totally saturated, or many parameters from multiple objects, such as a spatial configuration change. The changing of these ordinal relations is used to define state transitions. For example, we call a spring compressed, relaxed, or stretched depending on the relationship between length of spring and the rest length of spring. In thermodynamics, the comparison between temperature and saturation temperature is one of the ordinal relations used in identifying the state of a substance.

In addition to identifying important ordinal relations in a domain, a well-defined domain theory also defines what equations are applicable when certain ordinal relations are true. For example, when a liquid is heated it will go through liquid, saturated, and gas phases. Each of these phases has a different qualitative state with corresponding modeling assumptions and applicable equations. Since these states require different equations, it is important to differentiate between them. Identifying the relevant assumptions and equations requires envisionment.

The most basic of envisionments is a total envisionment. A total envisionment involves generating all possible behaviors for a system using all possible initial conditions. Even for small examples, a total envisionment can produce a large number of states, making it infeasible for most applications. Instead of generating all behaviors from the set of all possible starting states, an attainable qualitative envisionment examines all possible following behaviors from a set of states satisfying a set of given initial conditions. Although the number of states produced in an attainable envisionment is significantly less than total envisionment, an attainable envisionment still does more work than what a problem solver needs.

Hence, we use problem guided envisionment. *Problem guided envisionment* uses the approach we lay out in Figure 1 to limit state generation. We begin performing a typical attainable envisionment. However, we stop exploring a path when a state is found to be contradictory with the problem statement. Also, we stop the envisionment when a state is found that satisfies the final conditions given in the

$$Q_{c.v.} + \Sigma m_i(h_i + V_i^2/2 + gZ_i) = \Sigma m_e(h_e + V_e^2/2 + gZ_e) + [m_2 (h_2 + V_2^2/2 + gZ_2) - m_1(h_1 + V_1^2/2 + gZ_1)] + W_{c.v}$$

**Equation 2: The First Law of Thermodynamics**

$$Q_{c.v.} + m_i h_i = m_e h_e + W_{c.v}$$

**Equation 3: A simplified version of the first law that is typically used for turbines**

| Legend for Variables | | Subscripts Used | |
|---|---|---|---|
| m | mass | c.v. | control volume |
| h | enthalpy | i | parameter at inlet |
| V | velocity | e | parameter at exit |
| g | gravity | 1 | parameter for the initial state |
| Z | height | | |
| Q | heat | 2 | parameter for the final state |
| W | work | | |

**Table 1: Explanation for Equation 2 and Equation 3**

**Step 1:** Using the initial conditions in the problem, generate a set of qualitative states.
**Step 2:** On each state, perform limit analysis to find the set of states that could follow that state.
**Step 3:** Examine the new states for pruning. If any of the states have ordinal relations that are contradictory to the values in the problem statement, discard those states.
**Step 4:** Examine the remaining states to determine if they match the final state as given in the problem statement. If there is no match go to **Step 2** with the remaining states.

**Figure 1: Problem Guided Attainable Envisionment**

problem statement.

In all cases, the problem guided envisionment produces a subset of the attainable envisionment. In the worst case, the set of states generated is equal to those in an attainable envisionment. If no satisfactory final state was present, then the problem was malformed, since it requires that an impossible state be reached. Usually, the problem guided envisionment produced is a strict subset of the attainable envisionment, since problems rarely concern themselves with all possible outcomes from a given state.

It is possible that the problem guided envisionment could produce multiple qualitative states that could qualify as final states. In these cases, the envisionment finds only the first matching qualitative state. Since problem solving is a relatively constrained activity, finding the first qualitative matching state is sufficient for analysis. As support for this statement, consider a problem that generates multiple final states, in which calculations will yield different answers for the problem posed. In order for this to occur, the initial problem state would have to be underspecified. We currently do not handle underspecified problems.

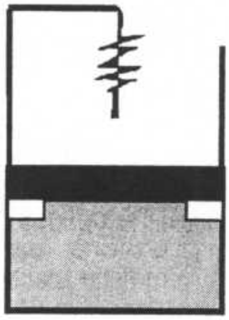
Once we have found a hidden state, we can ensure that the proper equation will be used for it. For example, when analyzing an oscillating spring where the initial state specifies the compressed length and the final state specifies the partially stretched length of the spring, the envisionment will identify when the spring is relaxed. This state is important, because it represents the division between the two behaviors of compression and stretching, for which the direction of the force is opposite. Similarly, the envisionment for a thermodynamics problem in which liquid water is heated in a closed container needs to identify the saturated phase where the temperature and pressure of the substance will be constant. Then the system can use the appropriate equations for calculating the relationships between pressure, temperature, and volume for each state (liquid, saturated, gas).

The problem guided envisionment enables the problem solver to find hidden states in the piston example, which allowed the problem solver to use the proper equations for work.

## Example One

In Figure 2, we show a typical problem involving a piston and a spring taken from an introductory thermodynamics textbook (Wylen & Sonntag 1985).

When the problem solver performs the problem guided envisionment, it determines that two different models of work are required. The first model is applicable to the piston rising before hitting the spring, and the second model is applicable to the piston when it is pushing against the spring. This problem needs to be broken down to a minimum of three states: piston resting, piston rising, and piston compressing the spring. Doing the envisionment, we find there are five different qualitative states. The relevant qualitative relations and the corresponding work equation are given in Equation 4.



A frictionless piston having cross-sectional area of $0.06m^2$ rests on stops on the cylinder walls such that the contained volume is 30L. The piston mass is such that 300kPa pressure is required to raise the piston against ambient pressure outside. When the piston has moved to the point where the contained volume is 75L, the piston encounters a linear spring that requires 360kN to deflect it 1m. Initially the cylinder contains 4kg of saturated (liquid + vapor) water at 35C. If this system is heated until the final pressure inside is 7MPa, determine the work done during this process.

**Figure 2: Problem 4.21**

**Adiabatic Expansion:**

$W = \text{Pressure}\ (\text{Volume}_{\text{Final}} - \text{Volume}_{\text{Initial}})$

**Work against Spring:**

$W = \text{Pressure}_{\text{Initial}} + \text{Pressure}_{\text{Final}}/2$
$\times\ (\text{Volume}_{\text{Final}} - \text{Volume}_{\text{Initial}})$

**Equation 4: Different Applicable Work Equations**

| Description | Qualitative State Information |
|---|---|
| 1. The piston is resting on the stops. | pressure(atmosphere) > pressure(gas)<br>pressure(gas) is constant.<br>volume(gas) is constant.<br>W=0, no change in volume. |
| 2. The piston is just touching the stops. | pressure(atmosphere) = pressure(gas)<br>pressure(gas) is constant.<br>volume(gas) is constant.<br>W=0, no change in volume. |
| 3. The piston is rising. | pressure(atmosphere) = pressure(gas)<br>pressure(gas) is constant.<br>volume(gas) is increasing.<br>$W = \text{pressure}(\text{volume}_{\text{Final}} - \text{volume}_{\text{Initial}})$ |
| 4. The piston is just touching the spring. | pressure(atmosphere) < pressure(gas)<br>pressure(gas) is constant.<br>volume(gas) is increasing.<br>$W = \text{pressure}(\text{volume}_{\text{Final}} - \text{volume}_{\text{Initial}})$ |
| 5. The piston is pushing the spring. | pressure(atmosphere) < pressure(gas)<br>pressure(gas) is increasing.<br>volume(gas) is increasing.<br>$W = \text{pressure}_{\text{Initial}} + \text{pressure}_{\text{Final}}/2 \times (\text{volume}_{\text{Final}} - \text{volume}_{\text{Initial}})$ |

**Table 2: States generated by constrained envisionment for Problem 4.21**

In this example, the initial state is well constrained and there is only one possible initial qualitative state. The final state is given only in terms of pressure. Examining how the value of pressure changes using Table 2, we see that pressure only increases when piston is pushing against the spring. Since the initial pressure is less than 7MPa, we know that the piston pushing against the spring is the first qualitative state that can satisfy our criteria for being the final state of the problem.

After deriving the equations for work, the problem solver can then find the answer to the question. By being able to model the different states of the piston, the problem solver was able to use all relevant applicable equations to solve the problem.

## Example Two

In Figure 3, we show a typical problem involving a pedestrian running after a bus taken from an introductory physics textbook (Haber-Schaim, Dodge, & Walter 1986).

The problem solver performs a problem guided envisionment of the problem to determine what can happen. There is only one initial state (state 0). The problem mentions two different possible outcomes. Likewise, our intuitions on this problem are that the pedestrian can either catch the bus, or miss it. However, the envisionment shows that there are actually three different paths through the state space for this problem, two of which correspond to the pedestrian catching the bus. The difference between these states is that in one, the man overtakes the bus (state 2), and in the other, he just barely catches up to it before it pulls away from him (state 4). The two final states mentioned in the problem correspond to states 2 and 4 (when he catches the bus), and state 3 (when the bus escapes).

In Figure 4, we show the attainable qualitative envisionment for this problem. Although there are ten states in the attainable envisionment for this problem, only half of these need to be examined in this problem (states 0-4). These are generated in only two iterations of the algorithm in Figure 1.

In this example, the problem guided envisionment enables the problem solver to take a systematic approach to evaluating the possible behaviors of the system in the problem. The problem includes constraints on what the relationships between the parameters have to be. Evaluating these constraints (step 3 in Figure 1) allows the problem solver to find which states actually occur. After finding the actual final state, the problem solver can use the equations from the state path to find the solution.

Qualitative reasoning enables the problem solver to consider the possible outcomes of the situation, and to find the one that actually occurred. Then it helps by stipulating which equations can be used in those situations.

A pedestrian is running at his maximum speed of 6.0 m/s to catch a bus stopped by a traffic light. When he is 25 meters from the bus the light changes and the bus accelerates uniformly at 1.0 m/s2. Find either (a) how far he has to run to catch the bus or (b) his frustration distance (closest approach). Do the problem either by use of a graph or by solving the appropriate equations.

**Figure 3: Chapter 1, Problem 24**

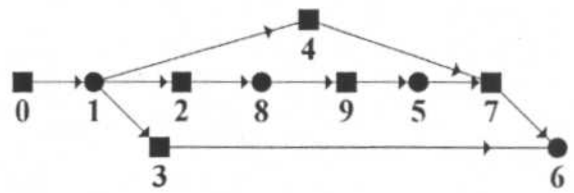



**Figure 4: Attainable envisionment for Problem 24 [Squares are instants. Circles are intervals.]**

| State Description | Qualitative State Information |
|---|---|
| 0. The bus is stationary. | position(person) < position(bus)<br>velocity(person) > 0<br>velocity(bus) = 0 |
| 1. The bus begins to move. | position(person) < position(bus)<br>velocity(bus) > 0<br>velocity(bus) < velocity(person) |
| 3. The bus begins to escape from the person. | position(person) < position(bus)<br>velocity(bus) = velocity(person) |
| 4. The person just catches the bus. | position(person) = position(bus)<br>velocity(bus) = velocity(person) |
| 2. The person is overtaking the bus. | position(person) = position(bus)<br>velocity(bus) < velocity(person) |
| 8. The person is past the bus. | position(person) > position(bus)<br>velocity(bus) < velocity(person) |
| 9. The bus begins to catch up to the person. | position(person) > position(bus)<br>velocity(bus) = velocity(person) |
| 5. The bus is catching up to the person. | position(person) > position(bus)<br>velocity(bus) > velocity(person) |
| 7. The bus is overtaking the person. | position(person) = position(bus)<br>velocity(bus) > velocity(person) |
| 6. The bus has escaped from the person. | position(person) < position(bus)<br>velocity(bus) > velocity(person) |

**Table 3: Explanation of Figure 4**

## Conclusion

In the past, the difficulties in constructing qualitative models and the lack of tools for applying qualitative analysis made these techniques generally unavailable for use in problem solving. The recent advances in qualitative reasoning have shown that building large qualitative domain models is possible and necessary for analyzing complicated systems.

In this paper, we argued that qualitative reasoning is a necessary technique for dealing with multi-state problems. We showed that qualitative reasoning is essential for at least two classes of multi-state problems that are commonly encountered in engineering problem solving.

For the first class of problems, we use qualitative reasoning to find constant parameters that can be used to simplify and combine equations. For the second class of problems, we use problem guided envisionment to discover important hidden states. By identifying hidden states we can break down the problem and limit the application of each equation appropriately.

Problem guided envisionment also reduces the size of the envisionment and the number of states that must be explored. The space can further be reduced in problems that specify events that occur within the behaviors of the system for a specific problem, by using temporal constraints such as those in TeQSIM. (Brajnik & Clancy 1996) Beginning with SUDEs instead of attainable envisionments could also assist the problem solver in focusing on the important behavioral distinctions for each problem. (DeCoste 1994) Part of our future work is to extend the problem solver to make use of more restrictions on envisionment, such as these.

Currently, we have successfully implemented constant parameter detection and our algorithm for discovering hidden states as a part of a thermodynamics problem solver (Pisan 1996). Although our examples were taken from physics and thermodynamics, the same principles are applicable to other domains where qualitative models can be built. For example, in history-oriented envisioning, the goal is to estimate and plan system behaviors. Our technique of problem guided envisionment could be adapted to assist in history-oriented envisioning, by reducing the number of contingencies that need to be considered. (Washio & Motoda 1996)

## References


Bhaskar, R., & Simon, H. A. (1977). Problem solving in Semantically Rich Domains: An Example of Engineering Thermodynamics. *Cognitive Science, 1*, 193-215.

Brajnik, G., & Clancy, D. J. (1996). Trajectory Constraints in Qualitative Simulation. In *Proceedings of AAAI*

Bundy, A. (1978). Will it Reach the Top? Prediction in the Mechanics World. *Artificial Intelligence, 10*(2), 129-146.

Bundy, A. (1983). *The Computer Modelling of Mathematical Reasoning* (United States ed.). New York, New York: Academic Press Inc.

de Kleer, J. (1975). *Qualitative and quantitative knowledge in classical mechanics* (Technical Report 352.). Cambridge, MA.: MIT AI Lab.

DeCoste, D. M. (1994). *Goal-Directed Qualitative Reasoning with Partial States* (57): Institute for Learning Sciences.

Falkenhainer, B., & Forbus, K. (1991). Compositional modeling: Finding the right model for the job. *Artificial Intelligence, 51*, 95-143.

Forbus, K. (1984). Qualitative Process Theory. *Artificial Intelligence, 24*(3), 85-168.

Haber-Schaim, U., Dodge, J. H., & Walter, J. A. (1986). *PSSC Physics* (sixth ed.). Lexington, Massachusetts: D.C. Heath and Company.

Hayes, P. J. (1979). Naive Physics Manifesto .

Iwasaki, Y., & Levy, A. Y. (1994). Automated Model Selection for Simulation. In *Proceedings of AAAI*, Seattle, Washington, 1183-1190.

Kuipers, B. J., & Shults, B. (1994). Reasoning in logic about continuous systems. In *Proceedings of The 8th International Workshop on Qualitative Reasoning about Physical Systems*, Nara, Japan., 164-175.

Pisan, Y. (1996). Using Qualitative Representations In Controlling Engineering Problem Solving. In *Proceedings of 10th International Workshop on Qualitative Reasoning about Physical Systems*, 190-197.

Sacks, E., & Joskowicz, L. (1993). Automated modeling and kinematic simulation of mechanisms. *Computer Aided Design, 25*(2), 106-118.

Sgouros, N. M. (1993). *Representing physical and design knowledge in innovative engineering design*. Ph.D. Thesis, Department of Computer Science, Northwestern University, Evanston, IL.

Skorstad, G., & Forbus, K. (1990). Qualitative and quantitative reasoning about thermodynamics. In *Proceedings of The 10th Annual Conference of the Cognitive Science Society*

VanLehn, K., Jones, R. M., & Chi, M. T. H. (1991). Modeling the self-explanation effect with Cascade 3. In *Proceedings of The Thirteenth Annual Conference of the Cognitive Science Society*, Hillsdale, NJ, 137-142.

Washio, T., & Motoda, H. (1996). A History-oriented Envisioning Method. In *Proceedings of Fourth Pacific Rim International Conference on Artificial Intelligence*, Cairns, Australia

Williams, B. C., & Raiman, O. (1994). Decompositional Modeling through Caricatural Reasoning. In *Proceedings of AAAI*, Seattle, Washington

Wylen, G. J. V., & Sonntag, R. E. (1985). *Fundamentals of Classical Thermodynamics* (3rd ed.): John Wiley & Sonc Inc.

Yip, K. (1991). *KAM: A system for Intelligently Guiding Numerical Experimentation by Computer*. Cambridge, MA: MIT Press.