Qualitative Reasoning, Analogical Reasoning and Conceptualization in Physics : a psychological point of view

Sandra Bruno
% G. Vergnaud, Equipe Cognition et Activités Finalisées
CNRS - Université Paris VIII
2, rue de la Liberté 93 526 Saint Denis Cedex 2 - France
e-mail : brunosandra@yahoo.com

Patrice Mathieu
Laboratoire de Physique de la matière condensée de l'Ecole Normale Supérieure
F 75 231 Paris Cedex 05 – France

Abstract

Analogical Reasoning is a rich field to explore Qualitative Reasoning : on one hand, any physicist would state that the only evidence of an analogy is the mathematical identity of the equations ; on the other hand, analogical reasoning can take place outside mathematical formalisms and can provide an opportunity for conceptualization of facts (or construction of a model).
In our research, the subjects (University students in physics) were put in situations where the only way forward was to use Qualitative Reasoning (through either conceiving a simulation setting or through an interview with a psychologist who introduced himself as a novice in physics). In the study presented here, the physical domains surveyed were the fluid flow and the heat flow.
The aim of this paper is to illustrate, with a case study, the fundamental role of conceptualization for qualitative reasoning. Indeed, isn't it a logical point of view to consider that the validity of any reasoning (and of qualitative reasoning in particular) depends on the mental representations and understanding one has of the world?

Introduction

We hope that the presentation of our work can enrich as well as be enriched by the Qualitative Reasoning (Q.R.) approach, for 3 main reasons :
- our fundamental interest is Knowledge Acquisition by Analogical Reasoning (A.R.), and we assume that the use of an analogical system of simulation in physics constitutes a qualitative model
- the subjects of our experiments were observed in situations where it was not possible to reason from quantitative data or equations
- the physical domains surveyed (Hydraulic flux and Heat flux) have already been studied in a Q.R. approach (Forbus & Gentner (1986), Gentner (1989)).

Therefore, after having drawn up our frameworks, we would like to submit a problem that could lead to further questioning over situations which have already been modeled. The presentation of the problem will be done from an expert point of view, and we will analyse the way 2 students managed to resolve it.

Theoretical and Methodological Background

Theoretical background

When the research in psychology proceeded to study A.R., it mainly dwelt on defining its mechanism, principally through the issue of transfer : how and when transfer of knowledge appears and proceeds, and its different stages.
Recently, the question of conceptual changes which can occur during A.R. has been the main point of concern of some research.
Within this framework, research must deal with a fundamental matter, which goes beyond A.R. and is relevant to any study linked to knowledge, that is the relationship between the mechanism and thought content and, hence, the question of the various levels of generality in conceptions.
Some would say that :

> "The contemporary view tends to be that cognition is typically situated in a social and physical context and is rarely, if ever, decontextualised" (Butterworth, 1992).

According to some others, as to Lakoff & Johnson (1985), we project onto reality various ontological concepts, and so, we perceive the world essentially in a metaphorical manner.
For our part, our psychogenetical approach does not lead us to privilege one or the other positions, but rather leads us to look for descriptive tools of the evolution of the individuals' activity in the shorter or longer term. We think that in order to treat matters such as reasoning (through analogy or other) or conceptual changes, studies must take

place within an operative theory of cognition and more precisely within an operative theory of representation. According to Vergnaud (1987), we consider that representation is functional, that is to say, that it is rooted in action while sustaining this action. This is why the survey of mental representation must take into account situations where human activity unfolds, for it is in situ that we form our concepts and theories. This does not mean that concepts can not be independent from their context, as is the case in theoretical statements concerning a class of objects and situations, and not specific ones.

In Vergnaud's opinion, our activity, be it motor or cognitive, is mainly supervised by the schemas we build, we decompose and combine in our lifetime. As Piaget defined it, a schema is an invariant organisation of activity for a class of given situations. Vergnaud (1987) defines the different constructive elements of schemas, the core of which is made of "**operative invariants**" ("invariants" for their generic characteristic, "operative" because they enable action upon reality). He distinguishes as operative invariants the "**concepts-in-action**" and the "**theorems-in-action**" :

> "I mean by concept-in-action a concept (objects and predicates) implicitly deemed pertinent, and by theoremes-in-action, a proposal deemed true. There is no theorem in action without a concept in action and vice versa, as there is no theorem without a concept. But we are led to distinguish them because of their cognitive function which is distinct per se. Truth is different from pertinence and the same system of implicit concepts can enable different activities depending onh the theorems deemed true".

For exemple, if we consider the case of counting, the concepts in action associated with this competence are those of "object", "collection", "cardinal" etc...and an exemple of a theorem-in-action linked to this competence is that the cardinal of a collection is independent from the order in which the objects have been counted.

During an action, the choice of the operative invariants depends on the objectives one has, and the questions called to mind by the situation.

Nevertheless, and we do insist about this, even if these operative invariants are not necessarly explicited by the subject, still, they are present during his action and it matters to find the means to reveal them.

Thus, we will be able to point out the cognitive filiations and ruptures leading to the formation of concepts.

Methodological background

Among the methodologies developed to study conceptions (questionnaires, verbal associations, conceptual networks, ...) those dealing with the way in which individuals solve problems seem to be the most prevalent. This method enables us to define cognitive invariants, of which subjects may or may not be aware. The subject's statements, i.e. what he or she is capable of explaining, are indeed insufficient for understanding their knowledge.

For this purpose, an analogical construction task seems to be a condition that generates significant traces of operative invariants : first with the elements of the initial situation kept as pertinent enough to be simulated, also the constructed objects to make correspondences with these elements, then with the anticipations of the result of the simulation and finally with the questionings.

More generally, we have taken advantage both of the methods used in didactics (epistemological approach of the domain knowledge, ecological experimentations in classrooms) and of the methods used in psychology (definition of the cognitive task, observations, interviews).

With regard to this approach, Vergnaud (1982) stated :

> "One way to construct such a synthesis is to consider that knowledge is organised in "**conceptual fields**", the mastery of which develops over a long period of time through experience, maturation, and learning. By conceptual field, I mean an informal and heterogeneous set of problems, situations, concepts, relationships, structures, contents, and operations of thought, connected to one another and likely to be interwoven during the process of acquisition [...] The psychogenetic study of the acquisition of [such a] field requires the analysis of the different relationships involved, and the hierarchical study of the different classes of problems that may be offered to students. It requires also the study of the different procedures and the different symbolic representations that students may use"

To take these elements into account, one has to develop an expert frame of reference that includes aspects of the problems situations that are not usually taken into account by experts or by text authors. This is the core of the method! Vergnaud (1982) gives the exemple of the conceptual field of addition, for which

> "mathematicians are not interested in the concepts of time and dimension. But children take time aspects and dimensional aspects into consideration. One can hardly understand what they are doing if one keeps these aspects out of the frame of reference."

With the ambition to

> "capture both the common-sense knowledge of the person on the street and the tacit knowledge underlying the quantitative knowledge used by engineers and scientists" (Forbus, 1988),

we think that Q.R. would gain in searching for "conceptual fields" as defined by Vergnaud. We think that studying isolated ecological situations creates a rupture with more sophisticated scientific situations, whereas both remain relevant towards every day necessities (housing insulation for exemple).

A problem in physics : the heat flux, and its simulation

The expert scenario evolves through 5 phases, from the identification of the real-world questioning, to a possible resolution of the problem.

Phase 1 : real world situationsidentified as source of questioning

Firstly, we will define a range of situations whose factors we want to control and whose evolution we want to anticipate.

Let us consider apparently conflicting everyday facts : 3 different objects, a wooden spoon, a silver spoon and a piece of aluminium foil submitted to the same high temperature (after having been in the oven for a long time). When touching the objects out of the oven, the feeling of heat will differ from one object to the others. If we compare the two spoons (wood and silver), we can explain the heat difference felt when touching these objects by the different conductive quality of both materials. What about the piece of aluminium foil? Everybody knows that aluminium is a good conductor, but not everyone knows that they won't get burned when touching it. This last situation illustrates the fact that another property has to be taken into account. This is the "quantity of heat" accumulated in the piece of aluminium foil to reach the same high temperature as the other objects. Its thermal capacity is low because the piece of aluminium foil is very thin, therefore a very little quantity of heat is transmitted to the hand, even if its temperature is very high.

Consequently, two important material properties must be considered in order to model the transition period during which the heat spreads from one point to another. These are the conductivity and the thermal capacity.

Phase 2 : a complex experimental situation representing real world situations

At this stage, the situation remains complex in the sense that the measurements and the construction of an analog are difficult. Nethertheless, the disruptive effects are withdrawn from the real-world situations (by isolating the studied system), and the minimal constraints are preserved (a material in contact with a source of heat).

Figure 1 represents the setting for this thermal phenomenon [Th2], also showing the heat flow direction, which we know is directed towards the colder part of the material.

[Th2]

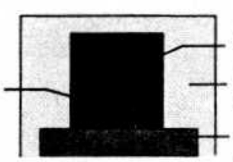


Fig. 1 : a complex thermal setting - any material (piece of wood for exemple)

The objective is to study the spread of heat in the material, using timed measurements of the temperature (T), at different spots (ideally an infinite number) in between the source of heat (kept at 80°C) and the top of the material. Thus, the function of the temperature has one variable : time (t), and one parameter : the distance from the source (d). At every spot, the magnitude (T) increases towards a balance state between the source (S) of heat and the receptor (R).

Phase 3 : modeling and reduction of the complex situation to a simple situation

As often done in physics, before studying a system in its entirety, which we said is complex, we will consider the "smallest" possible part of it : a "slice", which will constitute the simple system. A simple thermal setting ([Th1], figure 2) can be made to experiment on how the heat behaves in a slice of material, by modeling the decomposition of its two appropriate material properties towards the heat flux :

- conductivity (K, how much heat is passing through the material per unit of time) will be represented by a piece of paper (I) acting as an intermediary of low conductivity, and whose capacity is negligeable
- thermal capacity (C, number of heat units required to raise the temperature of the material by one degree) will be represented by a piece of copper (R), acting as a receptor, which can be considered isothermal because of its very high conductivity.

[Th1]

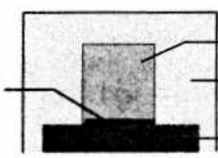


Fig2 : a simple thermal setting ("slice")

A thermometer is placed on top of the piece of copper, in order to measure the evolution of the temperature according to the timing which remains the variable. Because of the copper isothermal property, the distance parameter is no longer significant, and is therefore not taken into account.

The amount of heat transfered per unit of time through the intermediary (the piece of paper) is proportional to the difference of temperature between its two surfaces.

At the starting point (when copper and paper are placed over the heat source) the difference of temperature between the copper and the heat source is significant and the temperature of the piece of copper will increase rapidly. Then, as the temperature of the piece of copper increases, the amount of heat transfered decreases because the difference in temperature between the source and the copper decreases. Thus, the temperature of the piece of copper increases quickly at first, then increases slower and slower when it gets closer to balance with the temperature of the source.

Phase 4 : bringing on the analogies

With the simple thermal setting, we can now easily construct a hydraulic analog.
In a hydraulic setting ([Hy1], figure 3), the liquid flow can simulate the heat flow in a slice (the evolutions will be the same). The source (a large beaker containing a liquid) is connected to a receptor (a thin beaker) via a pipe of very small diameter. The conductivity of such setting is proportional to the diameter and length of the pipe, and to the viscosity of the liquid. The capacity of the receptor is proportional to the diameter of the small beaker.
Note that this analogy can easily be verified with the equations of conservation and transport, which are formally similar, but it is not our purpose here. The resolution of these equations leads to the equation of the evolution of the measured magnitude (temperature or height of liquid) in accordance to time.
Also note that another analogical setting can be made up with electrical material.

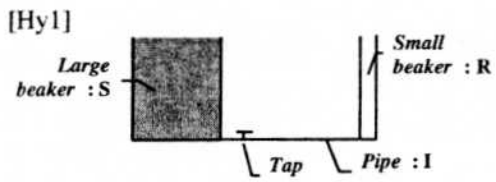
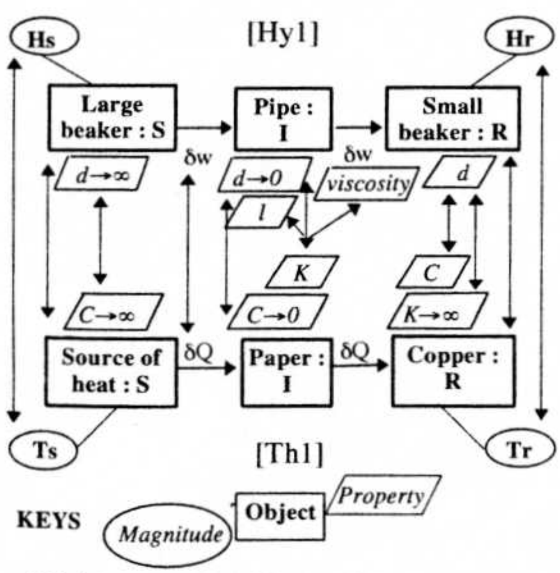


Fig. 3 : a simple hydraulic setting (simulating the heat slice)

Figure 4 sums up the simple analogy between the simple hydraulic and thermal systems.



H:Height ; w:weight (of liquid) ; d:diameter ; l:length
T:Temperature ; Q:heat ; C:Capacity ; K:conductivity

Fig. 4 : the simple analogy (correspondences between objects, properties, magnitudes and flux)

Phase 5 : using the analogies to simulate real world situations

The last stage of our scenario is aimed at simulating the complex thermal setting (see phase 2), in order to be closer to real-world situations.
Thus, it consists in applying to the hydraulic setting the opposite operation to the one applied in phase 3, because the complex settings (any material in the thermal case) are the concatenation of simple settings ("slices", made of paper-copper, in the thermal case).
The reader is invited to imagine the complex hydraulic setting he would propose. This was the task required from the subjects in our research.

Experiment

The aim of the experiment was to observe how the subjects constructed a complex hydraulic analog, after having been given knowledge of the simple analogy and the complex thermal system to be simulated.

Subjects and method

The 8 subjects taking part in the experiment were first year physics University students. They attended a 4 hours practical class, initially devised independently of our study and which consisted mainly in making experiments, taking down data and tracing graphs (this was followed, one week later, by a theoretical class, for the interpretation of the results).
In order to introduce a problem situation, the psychologist and the teacher adapted the pedagogical scenario to the following :
1- the teacher introduced the first, second and third phases of the expert scenario exposed above (from the real world situations to the simple thermal setting)
2- the teacher introduced "another phenomenon which can be characterized by an evolution towards a balance state between a source and a receptor : a simple hydraulic system"
3- the students (in pairs) made the simple experiments [Hy1] and [Th1]
4- the students were asked to list all the analogies between [Hy1] and [Th1] they had noticed during the experiments
5- the teacher exposed the simple relevant analogies (phase 4 of the expert scenario)
6- the teacher re-introduced the problem of heat propagation in any material, giving the example of a piece of wood (phase 2 of the expert scenario), and asked the students "to find a hydraulic analogy that simulates the wood setting and the heat evolution"
7- the students constructed the complex analogy.

Figure 5 summarizes the problem to be solved.

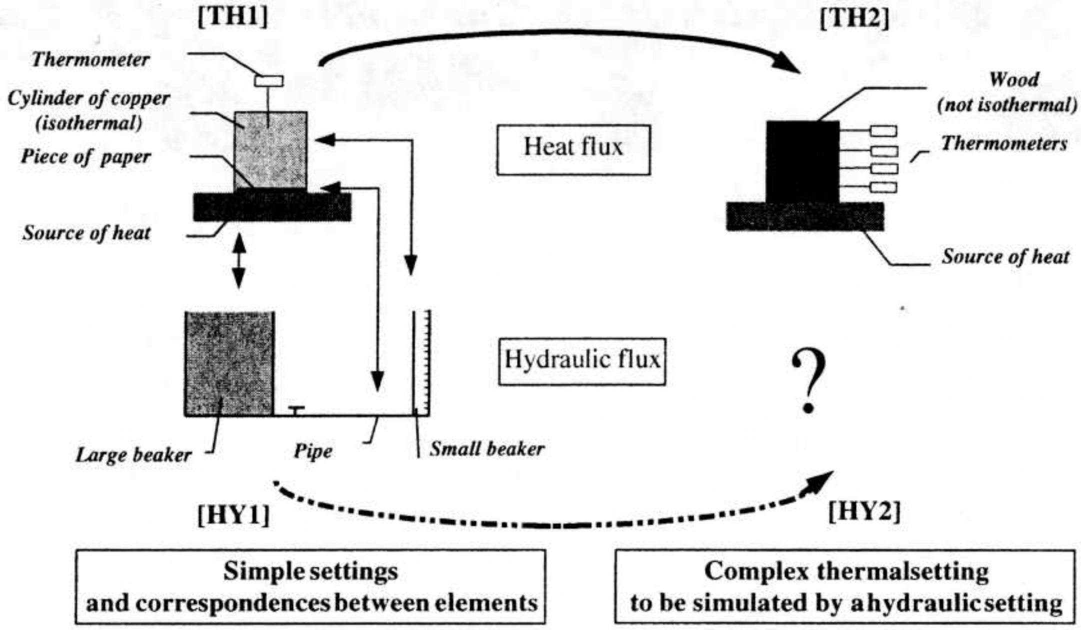


Fig. 5 : the problem to be solved

Data collecting and processing

An observer was present during the experiment. The pairs of students' conversations were tape recorded, and their drawings were collected.
The transcription of the audiotapes was processed in three stages, leading to :
- sequences : units of discourse (columns of the tables of presentation)
- micro-units : relevant extracts in sequences
- reading table : summary of the micro-units (ex : R[Hy1] indicates the receptor of the simple hydraulic setting), made of 3 types of information :
    - the systems considered by the students in the sequence
    - the conceptual register activated by the students about these systems
    - the type of cognitive process used by the students to progress in the reasoning

Results

We are concerned here with the last phase of the pedagogical scenario, when the subjects were required to simulate the complex thermal setting with a hydraulic setting. We are presenting and commenting a protocol of two subjects (table 1 and 2), organized around the three main stages of the resolution by the subjects : analysis, conception, improvement.
Note that the whole protocol lasted about half an hour.

STAGE	Analysis			
EXTRACTS FROM THE PROTOCOL	Lea : *"because here [TH2], the temperature depends on the height and on the time, and with the copper, as it goes very quickly, it was only the temperature. We neglected X [the distance from the source]"*	Lea : *"X would become H ... but ..."* *"the temperature corresponds to the height, well we keep the time, but X, I don't know what would be its correspondence"*	Obs : *"It [TH2] warms up little by little"* Gae : *"but this [HY1] fills up little by little also"* *" this R[HY1], we consider that it is the entire piece of wood"*	Gae : *"In fact, we need to divide the big volume by the amount of what comes in each time"*
SYSTEM CONSIDERED	[TH1] → [TH2]	[TH2] Corresp. ⇕ [HY2]	[TH2] ⇗⇙ [HY1]	[1] → [2] $\sum$
CONCEPTUAL REGISTER	Functions between : - Magnitudes : T - Variables : x, t TH2 : T = f(x,t) TH1 : T = f(t) Property : conductivity	Magnitudes : T, H Variables : x, t	Phenomenological : "wams up", "fills up" Objects	Concepts : "big volume", "what comes in" Relation
COGNITIVE PROCESS	Comparisons : differences, transformation	Correspondences : T⇔H, X⇔? Invariant : t	Comparisons Assimilation of objects	Operation : "divide" Abstraction

Table 1 : protocol of the subjects Lea & Gae, first stage : Analysis (4 sequences)

In their analysis, the subjects acknowledged that the problem was to find a hydraulic correspondence to the new variable in the complex thermal system : the distance from the source.

A kind of primitive function of spatial apprehension led the subjects in a first attempt, to think of making a correspondence between the distance (in the heat receptor) and the height of the water (in the hydraulic beaker), probably because both were vertical from the bottom to the top. For some time, this primitive function was predominant in Gae's thinking (" *this* [the receptor beaker], *we consider that it is the entire piece of wood*"). This spatial focusing, which could be considered as an obstacle to the correspondences HY-TH, played an important role, as Gae relied on it to elaborate the relationship between the simple and the complex systems : the wood is the result of the concatenation of the first simple concatenation "copper-paper" ; but he wanted to divide the volume of liquid in the same way as can be done with the wood. Lea tried to go beyond this primitive conception after taking the contradiction into account : the height of the water can not be both a parameter of a function and the measured value. Thanks to this approach, she was able to materially interpret the relationship established by Gae, as shown in the next stage.

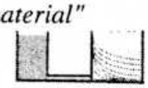
STAGE	Conception		Improvement	
EXTRACTS FROM THE PROTOCOL	Lea : *"you mean that we should put lots of little reservoirs?"*	Lea : *"we need this reservoir to be filled in order for it to give some to the other"*  *"the heat arrives here, it fills up here first, then it goes to the other one"* *"therefor, weve got at last the magnitude X "*	Gae : *"no, it [TH2] gives a little bit before"* *"the water comes in a first small beaker, and Hop ! it gives to the other one"* Lea : *"its filled in through minutes holes"* *"the more it fills up, the more it gives"* *"the front line is preceded by a little added heat"*	Lea : *"the holes should be more or less proportional to the wood conductivity "* Gae : *"you bring a K factor"* Lea : *"a porous material"*  Gae : *"the more [holes] there is, the more will drop"*
SYSTEM CONSIDERED	Σ [HY1] → [HY2]	[TH2] ↑ [HY2]	[TH2] ↓↑ [HY2]	[TH2] ↓ [HY2]
CONCEPTUAL REGISTER	Objects	Phenomenological	Phenomenological Principle	Objects Properties Principle
COGNITIVE PROCESS	Transfer, application of the operation to the setting	Transfer of the phenomenological evolution	Bidirectional transfer Abstraction	Correspondences : wood conductivity ⇔ holes diameter Abstraction

Table 2 : protocol of the subjects Lea & Gae, 2d stage : Conception (2 sequences), and 3d stage : Improvement (2 sequences)

The subject Lea applied the concatenation in order to use it as a transformation of the simple hydraulic setting towards the complex hydraulic setting (*"we should put lots of little reservoirs"*). It is important to note here that at this moment, the subject didn't mention how the reservoirs should be linked together. Then, in order to verify the physical coherence of the proposed setting, and to make its configuration precise, the subjects used the supposed evolution of the complex phenomenon in the heat and hydraulic systems (*"the heat arrives here, it fills up here first..."*).

A first outline of a complex analogy had been established, made with a certain number of basic elements (in particular with the concatenation relation), and others could then be removed or modified.

Indeed, without any resistance, the subjects dismissed the physics principle that justified their first setting and modified it, taking into account principles that had not been previously considered (*"we don't need to wait until it fills up completely for the water to go into the other one "* ; *"the holes should be proportional to the wood conductivity"*). In these extracts, it appears that the material analogy of the settings provided some kind of assistance and some thoughts materialization to verify the relevance of the proposed principle (a kind of meta-cognitive formula could be : "in order to validate the hypothesis of this principle, it must be applied to the thermal as well as to the hydraulic settings").

In parallel to this materialization, the subjects, using their spontaneous means of expression, referred to two fundamental laws which encompass the overall phenomenon : the outflow (of heat, water) is proportional to the difference of the considered magnitude (height, temperature) measured at the source and at the receptor : *"the more it fills up, the more it gives"*, and the flux is proportional to a conductivity property which depends on the objects, and which can be measured (the number of holes, and their diameter ; the conductivity of the material).

Interpretation

The overall cognitive progress aroused by the analogical simulation

Pertaining to the systems considered, we can see a progress from what is known to what is "unknown", that is, from what lies around the goal of the reasoning towards this goal which becomes more and more sophisticated.

Pertaining to the conceptual registers, we can see a progress steming from a focus on concepts (magnitudes, properties) to relations as expressed in principles

(theorems), through phenomenological aspects.
As to the cognitive processes, we note that the theorems mentioned above are built on a progressive abstraction process.

The interactionbetween analogical reasoning and conceptualizations

The students' approach can be analysed with regard to the expert scenario. In the third phase of this scenario, two aspects are fundamental for the analogical simulation process.

- TH1 is "a slice" of TH2
The opposite operation of "slicing" is an operation of concatenation. Then, HY2 being the analog of TH2, it must satisfy the concatenation operation, which must be applied on HY1. In other words, the TH concatenation must be transfered in the HY systems, as shown in the next figure.

$$\begin{tikzcd} TH1 \arrow[rr, "\text{concatenation}"] & & TH2 \\ & \Downarrow & \\ HY1 & & HY2 \end{tikzcd}$$

The students understood this process well and materialised it with the idea of "putting lots of little reservoirs"

- TH1 materialises the breaking up of two properties
The two elements chosen for TH1 (the paper and the copper) stand on the materialisation of two main properties (capacity and conductivity), which are represented in HY1 by the pipe and the small beaker. The difficult problem is to maintain this correspondence in the complex systems, even if in TH2, these two properties form part of one unique material (the wood). This fact won't be the case for HY2, for which the decomposition must be kept.

$$\begin{tikzcd} TH1 \arrow[dd, leftrightarrow] & & TH2 \\ & \longrightarrow & \\ HY1 & & HY2 \end{tikzcd}$$

During this stage of the analogical process, some correspondences TH1/HH1 are no more considered as constraints by the students for the complex systems correspondences (TH2/HY2). In particular, the "capacity" property will be abandonned to the benefit of a focus on the "conductivity" property. We think that this is the reason why the subjects did not construct the expert solution.
More precisely, it seems that the concept of **pressure** is not integrated with all the phenomenological consequences it implies, and, in particular, with regard to its causal effect towards the evolution of the phenomenon, and its analogy with the temperature. If it had been the case, the subjects would not have thought of the necessity of adding pipes (or holes), in order to increase the outflow proportionaly to the height of liquid (*"the more there is, the more it gives"*). Actually, the increase of pressure (due to the increase of height) is enough to increase the outflow (in this case, the pipes are located at the bottom of each small beaker).

Finally, even if not all the constraints were taken into account, the analogical simulation resulted in the construction by the subjects of two theorems-in-action, materialized by the solution of "porous partitions" for the hydraulic system.
- the outlflow is proportional to the height of water. The higher the water is, the more water will outflow because it will encounter more holes.
- the outflow is proportional to the physical characteristics of the intermediary objects located between the source and the target. The more holes are made in the partitions, the more significant the outflow is.

Conclusion and perspectives

The A.R. can generate new knowledge.

A paradoxical aspect of AR, in which one needs to anticipate "what is going on" in order to build a physical setting and, at the same time, needs to build a setting in order to understand "what is going on", has compelled the subjects to build a model of the different situations.
With this model, the subjects tested the concrete validity of the concepts and theorems recalled or made up.

Some limits

Despite the fact that the form of the subjects' A.R. can be considered as right, (with mapping, identification of the unknown element in the target, tranfer, verification, improvement), the proposed solution was not the expert analogy.
With regard to this, we consider that if the subjects did not deal with some correspondences during their reasoning, it is not for lack of their pertinence but because they opposed conceptual difficulties.
While knowing the existence of the "pressure" concept, the subjects lacked situations of reference to exploit this concept (which was not yet really a concept for them). Nevertheless, from the point of view of a learning theory, we consider that the experience of the subjects led, through this exercise, to put them in a position where they could in future consider this concept better.

The modeling of reasoning

Cognitive modeling cannot only rely on a formal network of a domain knowledge. According to Weil-Barais and Vergnaud (1990), an operative theory of representation should :
- consider formal aspects of the representation through the way they are actually and really helpfull to the reasoning
- take into account two additional aspects : operative invariants and situations of reference.

References

Butterworth, G. (1992) Context and cogniton in models of cognitive growth. In P. Light & G. Butterworth, (Eds.), *Context and Cognition*, Harvester Wheatsheaf.

Forbus, K. & Gentner, D. (1986) *Learning physical domains : toward a theoretical framework.*

Forbus, K. (1988) Qualitative Physics : Past, Present and Future. In *Exploring Artificial Intelligence.*

Gentner, D. (1989) *The mecanisms of analogical learning.* In S. Vosniadou & A. Orthony (Eds.) : Similarity and Analogical Reasoning. New York, NY : Cambridge University Press.

Lakoff, G. & Johnson, M. (1985) *Les métaphores dans la vie quotidienne*. Les Editions de Minuit.

Vergnaud, G. (1982) A classification of cognitive tasks and operation of thought involved in addition and substraction problems. In T.P. Carpenter, J.M. Moser & T.A. Romberg : *Addition and substraction : a cognitive perspective*. Hillsdale (New Jersey) Erlbaum.

Vergnaud, G. (1987) Les fonctions de l'action et de la symbolisation dans la formation des connaissances chez l'enfant. In J. Piaget, P. Mounoud & J.P. Bronckart, (Eds.), *Psychologie, Encyclopédie la Pléiade,* XLVI (pp. 821-844). Paris: Gallimard.

Weil-Barais, A. & Vergnaud, G. (1990) Students' conceptions in physics and mathematics : biases and helps. In J.P. Caverni, J.M. Fabre & M. Gonzalez (Eds.) : *Cognitive Biases*. Elsevier Science Publishers B.V. (North-Holland).

We would like to thank E. Marechal and S. Mordacq, for the translation