Improving Semiquantitative Simulation by Using Lyapunov Analysis*

Michael Hofbaur, Nicolaos Dourdoumas
Department of Automatic Control
Technical University Graz, Austria
{hofbaur, dourdoumas}@irt.tu-graz.ac.at

Abstract

Semiquantitative simulation is a powerful method to analyze uncertain dynamic systems using reasoning techniques. However, simulating systems which exhibit oscillatory behavior, impose big difficulties on the current semiquantitative simulation methods. Reasoning about "energy" can be very helpful for analyzing such systems using semiquantitative simulation. The simulation program QSIM provides such a mechanism, called kinetic-energy filter, which can be used for second order systems of a specific type. This paper proposes a more general approach on the basis of Lyapunov functions. By exploiting the similarities of the descriptions of the system used in semiquantitative simulation and nonlinear control theory it is possible to apply powerful methods for the latter to deduce a Lyapunov function for the system under investigation. Simple yet powerful filtering methods based on Lyapunov functions are presented and demonstrated by example.

Introduction

Semiquantitative simulation is a method to analyze uncertain dynamic systems. The basis for the simulation is a system description in the form of a semiquantitative differential equation (SQDE) which is an abstraction of an uncertain ordinary differential equation (ODE) model of the dynamic system under investigation. The simulation is performed using reasoning methods and predicts the possible behaviors of the system starting from an uncertain initial state. QSIM (Kuipers 1986) and its semiquantitative extensions (Berleant 1991)(Kay 1996) provide such a simulation environment for analyzing uncertain dynamic systems. Due to the uncertainty of the system we get a set of possible behaviors as simulation result. The predicted set of behaviors contains every real behavior of the system. However, it is very often the case that the reasoning mechanism also predicts behaviors which are not possible for the system. This is due to the fact that current reasoning mechanisms are not powerful enough to detect and filter every inconsistent behavior. These so-called *spurious* behaviors can spoil the simulation result and make simulation of systems of the order $n \geq 2$ very difficult if not impossible. Especially systems which show oscillatory behavior impose big difficulties on current reasoning/simulation techniques. Besides many advanced reasoning methods (Williams 1991)(Bousson and Trave-Massuyes 1992)(Clancy and Kuipers 1997) it was shown that reasoning about "energy" in the system under investigation is very helpful for analyzing such systems (Fouche and Kuipers 1992). QSIM provides such a mechanism, called kinetic-energy filter, which can be used for second order systems of very specific type. This paper proposes a more general approach which uses *Lyapunov functions* for improving the reasoning capabilities. In the first step, we shall demonstrate the similarities of the representation of the system used in semiquantitative simulation and the representation used in nonlinear control theory. By exploiting these similarities we can reformulate our analysis problem in a form such that powerful methods from nonlinear control theory (Boyd and Yang 1989) can be applied to construct a Lyapunov function for the system under investigation. In a second step, we shall show how reasoning based on Lyapunov functions can detect many spurious behaviors so that semiquantitative simulation can be enhanced. The application of the methods described is demonstrated on the basis of two examples.

Semiquantitative Modeling and Simulation

Using semiquantitative simulation it is our goal to analyze an uncertain nonlinear initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \;\; \mathbf{x}(t_0) =: \mathbf{x}_0, \qquad (1)$$

where $\mathbf{x}$ represents the state vector $\mathbf{x} := [x_1, ..., x_n]^T$. The initial value problem consists of an ODE with an *uncertain* nonlinear function $\mathbf{f}(\mathbf{x}) := [f_1, ..., f_n]^T$ and an uncertain initial state $\mathbf{x}_0$, which lies in a box $\mathcal{D}_0$ defined by the vector pair $\mathbf{x}_0^- := [x_{0,1}^-, ..., x_{0,n}^-]^T$, $\mathbf{x}_0^+ := [x_{0,1}^+, ..., x_{0,n}^+]^T$ so that $\mathbf{x}_0^- \preccurlyeq \mathbf{x}_0 \preccurlyeq \mathbf{x}_0^+$ (the sign $\preccurlyeq$ denotes the componentwise inequality). The nonlinear functions f_i are composed using arithmetic

*Paper also submitted to European Control Conference ECC99 operations $(+,-,*,/)$ and nonlinear functions of a scalar variable which are of a specific class, called M^+ in literature (Kuipers 1986). Such an M^+ function describes an uncertain, nonlinear, continuously differentiable, time-invariant, strictly monotonically increasing functional relationship $u = f(y)$. The uncertainty of an M^+ function $f(y)$ is specified by envelope functions $f^-(y)$ and $f^+(y)$ and numerical bounds for the slope ζ and η so that $f^-(y) \leq f(y) \leq f^+(y)$ and $\zeta \leq \frac{df(y)}{dy} \leq \eta$. We shall use the notation f_j^M to represent the set of M^+ functions which satisfy the inequalities with envelope functions and slope $\{f_j^-(y), f_j^+(y), \zeta_j, \eta_j\}$. The set of vector functions with M^+ elements will be denoted by $\mathbf{f}_\mathbf{j}^\mathbf{M} := [f_{j,1}^M, ..., f_{j,p}^M]^T$.

Semiquantitative simulation uses an abstraction of the uncertain ODE in the form of an SQDE which describes the variables of the system, the constraints among them, and the uncertain numerical information in the form of interval values and envelope functions. Based on such a description it is possible to reason about the behavior of the system using a constraint-satisfaction mechanism. As a result of the simulation we get a set of possible behaviors which describes the possible trajectories $\mathbf{x}(t)$ of the initial value problem (1) in a semiquantitative form.

Lyapunov Analysis

The aim of this paper is to show how methods from nonlinear control theory can be used to improve semiquantitative simulation. For this purpose we shall demonstrate the similarities between the standard nonlinear feedback system shown in figure 1 and the representation of the system used in semiquantitative simulation.

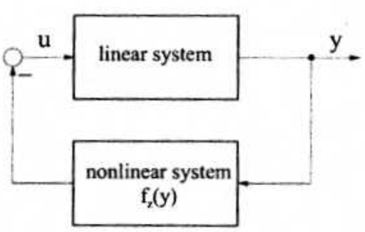


Figure 1: Feedback connection of a linear system and a nonlinear element

The feedback system consists of a linear plant and a nonlinear feedback connection and can be described by

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{Az} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cz} \\ \mathbf{u} &= -\mathbf{f_z}(\mathbf{y}).\end{aligned} \tag{2}$$

The nonlinear function $\mathbf{f_z} := [f_{z,1}, \ldots, f_{z,p}]^T$ should be decoupled in the sense that the j^{th} component $f_{z,j}$ depends only on the j^{th} component y_j of the vector $\mathbf{y}$. Its components are so-called *sector-nonlinearities*, meaning that they satisfy the inequality

$$\alpha_j y_j^2 \leq y_j f_{z,j}(y_j) \leq \beta_j y_j^2, \; \forall j = 1, \ldots, p \tag{3}$$

with constants α_j and β_j (the nonlinearities $f_{z,j}$ are also said to lie in the sector $[\alpha_j \;\; \beta_j]$). In order to use the theory developed for systems of the form (2) we shall restrict the semiquantitative analysis to systems which can be described by

$$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{Bf_x}(\mathbf{Cx}), \tag{4}$$

where $\mathbf{f_x}$ is an M^+ vector function described by $\mathbf{f_x^M}$. The main consequence of the use of M^+ vector functions instead of sector-nonlinearities is that system (4) does not necessarily have its equilibrium point at the origin $\mathbf{x} = \mathbf{0}$. In fact, we have to distinguish between two different types of equilibrium points: We shall say that an isolated point $\mathbf{x}_e$ is an *exact equilibrium point* of (4) if for *all* M^+ functions $\mathbf{f_x}$ which are members of $\mathbf{f_x^M}$ it is true that $\mathbf{Ax}_e - \mathbf{Bf_x}(\mathbf{Cx}_e) = \mathbf{0}$. In contrast to the exact equilibrium point we shall say that a point $\mathbf{x}_e$ is an *uncertain equilibrium point* if there exists *at least one* M^+ function $\mathbf{f_x}$ which is a member of $\mathbf{f_x^M}$ so that $\mathbf{Ax}_e - \mathbf{Bf_x}(\mathbf{Cx}_e) = \mathbf{0}$. An uncertain equilibrium point will lie in a box $\mathcal{D}_e$ defined by the vector pair $\mathbf{x}_e^-$, $\mathbf{x}_e^+$ so that $\mathbf{x}_e^- \preccurlyeq \mathbf{x}_e \preccurlyeq \mathbf{x}_e^+$.

Regardless of the specific value and type of an equilibrium point, we can introduce a new state vector $\mathbf{z} := \mathbf{x} - \mathbf{x}_e$ so that equation (4) becomes

$$\dot{\mathbf{z}} = \mathbf{Az} - \mathbf{Bf_z}(\mathbf{Cz}) \tag{5}$$

which is the same form as the standard control loop given in (2) except that we represent $\mathbf{f_z}$ by an M^+ vector function with the property $\mathbf{f_z}(\mathbf{0}) = \mathbf{0}$. Due to the similarities of M^+ functions and sector-nonlinearities we can always represent $\mathbf{f_z}$ by sectors so that we can use the comprehensive theory developed for the standard feedback loop (2). The values of α_j and β_j which define the sector for the components $f_{z,j}$ of the vector function $\mathbf{f_z}$ can be evaluated using the envelope functions and slope bounds. If system (4) has an exact equilibrium point we have to find the smallest sector $[\alpha_j \;\; \beta_j]$ so that $\alpha_j y_j^2 \leq y_j f_{z,j}^-(y_j) \leq \beta_j y_j^2$ and $\alpha_j y_j^2 \leq y_j f_{z,j}^+(y_j) \leq \beta_j y_j^2$ holds, where $f_{z,j}^-$ and $f_{z,j}^+$ represent the shifted envelope functions. Otherwise, if (4) has an uncertain equilibrium point, the sector is defined by the slope so that $\alpha_j = \zeta_j$ and $\beta_j = \eta_j$.

To reason about the behavior of the system it is important to know about the stability of the equilibrium point(s). We shall say that the system (2) is *absolutely stable* if the origin $\mathbf{z} = \mathbf{0}$ is globally asymptotically stable. This implies for the original uncertain ODE system (4) that for every initial state $\mathbf{x}_0 \in \mathcal{D}_d$, where $\mathcal{D}_d \subseteq \Re^n$ denotes the domain of $\mathbf{f_x}$, it is true that $\lim_{(t-t_0)\to\infty} \mathbf{x}(t) = \mathbf{x}_e$, $\mathbf{x}_e \in \mathcal{D}_e$. One method to test whether (2) is absolutely stable is to find a quadratic function

$$V(\mathbf{z}) = \mathbf{z}^T \mathbf{Pz}, \tag{6}$$

with a positive definite and symmetric matrix $\mathbf{P}$, so that $V(\mathbf{z})$ is a Lyapunov function for the system, i.e. $V(\mathbf{z}) > 0,\ \dot{V}(\mathbf{z}) < 0,\ \forall \mathbf{z} \neq \mathbf{0}$.

The problem of finding a quadratic Lyapunov function can be formulated as a linear matrix inequality (LMI) problem for which powerful solvers are available (a summary of the underlying theory is given in the Appendix).

Our current implementation is able to perform the previously described analysis automatically. If the simulation program is provided with the SQDE of the system under investigation, it checks whether the system can be represented in the form of equation (4), calculates the equilibrium point(s), performs the state variables change, and constructs a quadratic Lyapunov function for the system if it exists. The SQDE together with the Lyapunov function is then used by the simulation engine.

Lyapunov-Filtering

A quadratic Lyapunov function for the uncertain ODE system allows several filtering methods to be applied in order to improve semiquantitative simulation. The most important and also most effective method is based on the fact that by knowing a Lyapunov function for the ODE of an uncertain initial value problem it is possible to calculate bounds for the state variables so that all real behaviors of the system stay within these bounds. Behaviors predicted by semiquantitative simulation which reach the bounds can be identified as spurious behaviors and are filtered from the simulation result.

Let us outline this method for a system with an exact equilibrium point first. The Lyapunov function $V(\mathbf{x}) = (\mathbf{x}-\mathbf{x}_e)^T\mathbf{P}(\mathbf{x}-\mathbf{x}_e)$ describes a hyperellipse centered at the equilibrium point $\mathbf{x}_e$. All we have to do is to find the smallest hyperellipse defined by $V_{\max}$ such that $V_{\max} \geq (\mathbf{x_0}-\mathbf{x}_e)^T\mathbf{P}(\mathbf{x_0}-\mathbf{x}_e)$ for every initial state $\mathbf{x_0} \in \mathcal{D}_0$. This can be done by checking the Lyapunov function at the 2^n vertices $\mathbf{x}_0^{(1)}, \ldots, \mathbf{x}_0^{(2^n)}$ of $\mathcal{D}_0$ and taking the maximum value: $V_{\max} = \max_{j=1,\ldots,2^n} V(\mathbf{x}_0^{(j)})$. The hyperellipse found in this way not only contains all possible initial states of the uncertain initial value problem, it also contains all possible trajectories $\mathbf{x}(t)$. However, the mathematical representation of a region in the form of a hyperellipse is not very helpful for our purpose as QSIM uses boxes for the description of uncertainty. Therefore we approximate the hyperellipse with a box $\mathcal{D}_b$. Figure 2 shows the construction of such a *bounding-box* $\mathcal{D}_b$ for a second order system.

A slightly modified approach must be taken for systems with an uncertain equilibrium point. In such a case we have to find the smallest hyperellipse which, centered at *any* point of $\mathcal{D}_e$, contains the box $\mathcal{D}_0$. It is possible to show that we can restrict our search to hyperellipses which are centered at the 2^n vertices

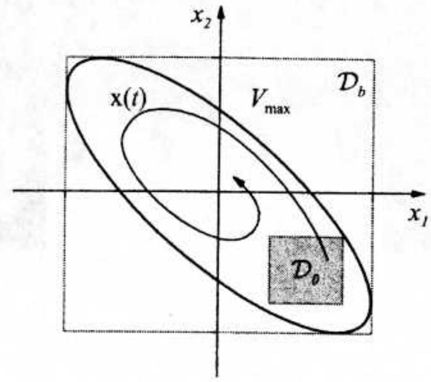


Figure 2: Bounding-box construction

$\mathbf{x}_e^{(1)}, \ldots, \mathbf{x}_e^{(2^n)}$ of the box $\mathcal{D}_e$. The value of $V_{\max}$ can be evaluated by taking the maximum value of the 2^{2n} Lyapunov functions $V(\mathbf{x}_0^{(i)}, \mathbf{x}_e^{(j)})$. All we have to do then is to find a bounding-box $\mathcal{D}_b$ which contains the hyperellipse defined by $V_{\max}$, no matter at which point $\mathbf{x}_e \in \mathcal{D}_e$ it is centered.

The bounding-box can be recalculated for every predicted semiquantitative state. In this way it is possible to shrink the box during simulation. The numerical information represented by the bounding-box not only allows very efficient filtering of all behaviors which reach the box, it also provides additional numerical information which improves the reasoning about the numerical ranges of the system variables.

It is also possible to identify certain spurious behaviors which stay within the box as the one shown in figure 3. This can be done by checking behaviors with semiquantitative states where all state variables except one are at their equilibrium value.

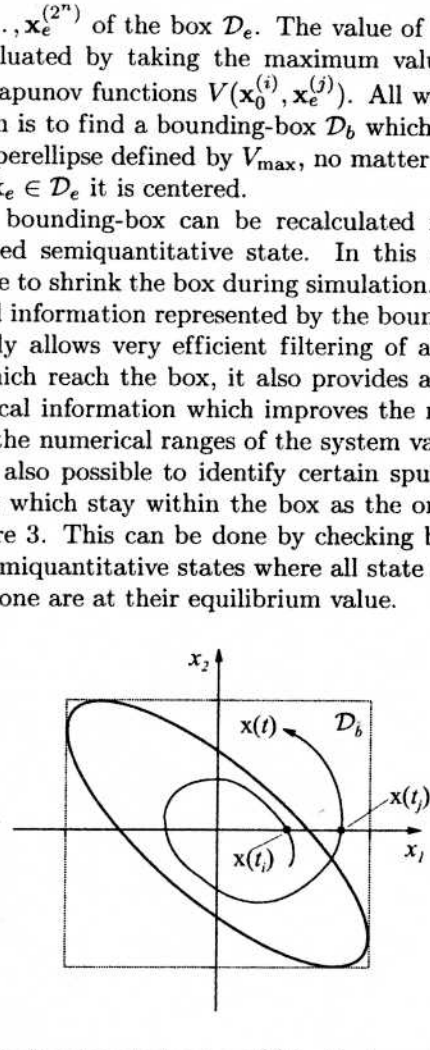


Figure 3: Spurious behavior $\mathbf{x}(t)$ in the bounding-box

Consider the two time-points $t_i,\ t_j,\ (t_i < t_j)$ where all state variables except x_k should be at their equilibrium value, i.e. $x_l(t_i) = x_l(t_j) = x_{e,l},\ l \neq k$. The expression for the Lyapunov function at these time-points is reduced to

$$V = (\mathbf{x}-\mathbf{x}_e)^T\mathbf{P}(\mathbf{x}-\mathbf{x}_e) = p_{kk}(x_k - x_{e,k})^2. \quad (7)$$

For an absolutely stable system it must be true that

$V(t_j) < V(t_i)$. As the coefficient p_{kk} of the positive definite matrix $\mathbf{P}$ is always positive we can write

$$(x_k(t_j) - x_{e,k})^2 < (x_k(t_i) - x_{e,k})^2. \tag{8}$$

The value of a variable at time-points is represented symbolically in the form of a so called landmark with an associated numerical range. It is possible to identify some spurious behaviors which violate (8) by checking the ordering of the landmarks for $x_k(t_j)$ and $x_k(t_i)$ with respect to the landmark for $x_{e,k}$. Behaviors which satisfy one of the following two conditions

$$\begin{array}{ccccc} x_{e,k} & < & x_k(t_i) & \leq & x_k(t_j) \\ x_k(t_j) & \leq & x_k(t_i) & < & x_{e,k} \end{array} \tag{9}$$

violate (8) and can be filtered from the simulation result[1].

A third filtering method can be based on the fact that an absolutely stable system cannot exhibit cyclic behaviors with constant amplitude. These behaviors can be identified with the cycle detection mechanism of QSIM and filtered from the simulation result.

It should also be noted that the existence of a Lyapunov function for the system under investigation increases the expressiveness of an attainable envisionment simulation. This simulation type determines all possible semiquantitative states of the system which can be reached from an initial state and links these states in a transition graph. The advantage of such an approach is that the possible behaviors can be represented in a finite graph. Its disadvantage is that normally it is not possible to decide whether cyclic behaviors, which are represented as cycles in the graph, describe increasing, steady, or decreasing oscillations. By knowing a Lyapunov function for the system under investigation it is possible to overcome this ambiguity since the system can only exhibit oscillatory behaviors with decreasing amplitude and therefore all cycles in the graph can be classified correctly.

Examples

Spring-mass system

We want to demonstrate the presented Lyapunov methods with a damped spring-mass system first[2]. This allows us to compare the described filtering methods with the kinetic-energy filter (Fouche and Kuipers 1992) which can handle second order systems of this specific type. The SQDE for the damped spring-mass system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -f^M_{x,1}(x_1) - f^M_{x,2}(x_2), \end{aligned} \tag{10}$$

where $f^M_{x,1}$ describes the spring characteristic and $f^M_{x,2}$ specifies the damping term. The envelopes for these M^+ functions define the sector [30.0 32.0] for $f^M_{x,1}$ and [0.18 0.24] for $f^M_{x,2}$. Providing our extended QSIM simulation environment with this SQDE, the system predicts an exact equilibrium point at the origin ($x_{e,1} = x_{e,2} = 0$) which is classified as locally asymptotically stable using a stability test based on Lyapunov's indirect method (Hofbaur 1997). Our system then deduces the standard nonlinear feedback system representation of (10)

$$\begin{aligned} \dot{\mathbf{z}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u} \\ \mathbf{y} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{z} \\ \mathbf{u} &= -\left[\ f^M_{z,1}(y_1), \ f^M_{z,2}(y_2) \ \right]^T \end{aligned} \tag{11}$$

with the state vector $\mathbf{z} := [x_1, \ x_2]^T$ and the sectors [30.0 32.0] for $f^M_{z,1}(y_1)$, and [0.18 0.24] for $f^M_{z,2}(y_2)$ and calculates the quadratic Lyapunov function

$$V(\mathbf{z}) = \mathbf{z}^T \begin{bmatrix} 0.6642 & 0.0019 \\ 0.0019 & 0.0214 \end{bmatrix} \mathbf{z}. \tag{12}$$

A semiquantitative simulation using the previously described Lyapunov filtering methods together with the non-intersection filter (Lee and Kuipers 1988) predicts, when starting at the uncertain initial state

$$x_{0,1} \in [1.0 \ 2.0], \quad x_{0,2} = 0, \tag{13}$$

a set of behaviors which represents an oscillatory behavior with decreasing amplitude (see figure 4) which can become overdamped after a finite number of half-cycles.

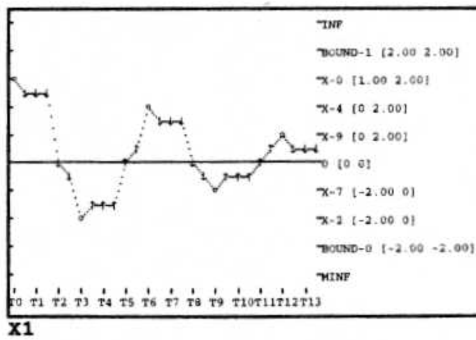


Figure 4: Semiquantitative time-plot of $x_1(t)$

The possible ranges of the state variables for $t > t_0$

$$x_1(t) \in (-2.0 \ 2.0), \quad x_2(t) \in (-11.1 \ 11.1) \tag{14}$$

[1] The reason for testing these conditions symbolically and not numerically is that it is possible that two landmarks `lm-1` and `lm-2` with the symbolic ordering `lm-1` < `lm-2` have overlapping numerical ranges! (E.g. see figure 4, where the landmarks `X-4` and `X-9` with the symbolic ordering `X-9` < `X-4` have the same numerical range (0.0 2.0).)

[2] The damped spring-mass system is one of the simplest mechanical systems that exhibits oscillations and it should be noted that it is difficult to simulate semiquantitatively without additional energy based reasoning!

are defined by the bounding-box $\mathcal{D}_b$ which is calculated using the Lyapunov function (12), the initial state (13), and the equilibrium point of the system. A semiquantitative simulation using the kinetic-energy filter, on the other hand, provides the same set of possible behaviors but with the weaker ranges

$$x_1(t) \in (-\infty\ 2.0), \quad x_2(t) \in [-356\ \infty). \qquad (15)$$

Controlled tank system

The second example should demonstrate the application of our analysis and filtering methods with a second order system modeling a PI-controlled tank. Although this system seems to be rather simple, it is important to note that current semiquantitative simulation using the kinetic-energy filter can handle such a system only in a simplified and revised form (e.g. see (Clancy, Branjnik and Kay 1997)). Such an additional effort is not necessary with our approach since the Lyapunov analysis is performed automatically by our extended QSIM simulation environment.

The mathematical model of a fluid tank with a PI-controller for a fixed set-point x_s is given by

$$\begin{aligned} \dot{x}_1 &= -f(x_1) + V_g\ u \\ \dot{x}_2 &= x_s - x_1 \\ u &= K\ (x_s - x_1) + \tfrac{K}{T_i}\ x_2, \end{aligned} \qquad (16)$$

where x_1 represents the fluid level, x_2 the integral part of the controller, $V_g = 0.7771$ the input gain of the tank system, and $K = 0.1498$, $T_i = 18.1857$ are the parameters of the PI-controller which was designed for a fixed set-point $x_s = 20.0$. The uncertainty of the system should lie in the inexact knowledge of $f(x_1)$, which is of type M^+ and represents the outflow characteristic of the tank and the maximum fluid level of the tank $x_{1,\max} \in [43.0\ 45.0]$. This model can be described by the SQDE

$$\begin{aligned} \dot{x}_1 &= -f^M_{x,1}(x_1) - f^M_{x,2}(x_1) + f^M_{x,3}(x_2) \\ \dot{x}_2 &= -f^M_{x,4}(x_1), \end{aligned} \qquad (17)$$

where the linear and exact relationships are also modeled as M^+ functions. The uncertain function $f^M_{x,1}$ is defined for $x_1 \geq 0$ and characterized by the envelopes $f^-_{x,1} = 0.385\left(\sqrt{x_1 + 0.25} - 0.5\right)$, $f^+_{x,1} = 0.385\left(\sqrt{x_1 + 0.25} + 0.5\right)$ and the slope $\zeta_1 = 0.0285$, $\eta_1 = 0.39$. The functions $f^M_{x,2}$, $f^M_{x,3}$, and $f^M_{x,4}$ are given by $f^-_{x,2} = f^+_{x,2} = 0.1164\,(x_1 - 20)$, $\zeta_2 = \eta_2 = 0.1164$, $f^-_{x,3} = f^+_{x,3} = 0.0064\,x_2$, $\zeta_3 = \eta_3 = 0.0064$, and $f^-_{x,4} = f^+_{x,4} = x_1 - 20$, $\zeta_4 = \eta_4 = 1$.

Providing our extended QSIM simulation environment with this SQDE, the system predicts an uncertain equilibrium point

$$x_{e,1} = 20.0, \ x_{e,2} \in [240.6\ 300.8]. \qquad (18)$$

Our system then deduces the feedback loop representation of (17)

$$\begin{aligned} \dot{\mathbf{z}} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{u} \\ \mathbf{y} &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^T \mathbf{z} \\ \mathbf{u} &= -\left[\ f^M_{z,1}(y_1),\ f^M_{z,2}(y_2),\ f^M_{z,3}(y_3),\ f^M_{z,4}(y_4)\ \right]^T \end{aligned} \qquad (19)$$

with the sectors $[0.0285\ 0.39]$ for $f^M_{z,1}$, $[0.1164\ 0.1164]$ for $f^M_{z,2}$, $[0.0064\ 0.0064]$ for $f^M_{z,3}$, and $[1\ 1]$ for $f^M_{z,4}$ and calculates the quadratic Lyapunov function

$$V(\mathbf{z}) = \mathbf{z}^T \begin{bmatrix} 0.6649 & -0.0247 \\ -0.0247 & 0.0083 \end{bmatrix} \mathbf{z}. \qquad (20)$$

The goal of our analysis is to evaluate whether the system can overflow when filled from empty. Performing an attainable envisionment simulation without the Lyapunov filtering methods cannot prove this as behaviors which cause an overflow are deduced. The application of the described filtering methods, however, provides a set of behaviors where the fluid level does not reach the maximal value.

The numerical bounds for the state variables for $t > t_0$ which can be drawn from the bounding-box $\mathcal{D}_b$

$$x_1(t) \in [0.0\ 54.8), \ x_2(t) \in (-71.6\ 613.0). \qquad (21)$$

allow QSIM to exclude the possibility of an overflow. This is due to the fact that a semiquantitative state with a fluid level $x_1 \in [43.0\ 45.0]$ and an integral part within the valid range $x_2 \in (-71.6\ 613.0)$ cannot be reached in the course of a semiquantitative simulation as the numerical ranges do not agree with the information given by the SQDE (17).

The obtained set of behaviors is described by an attainable envisionment graph with 3 branches shown in figure 5.

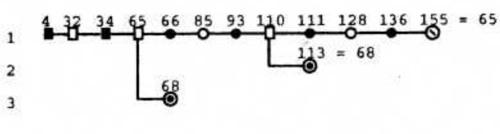
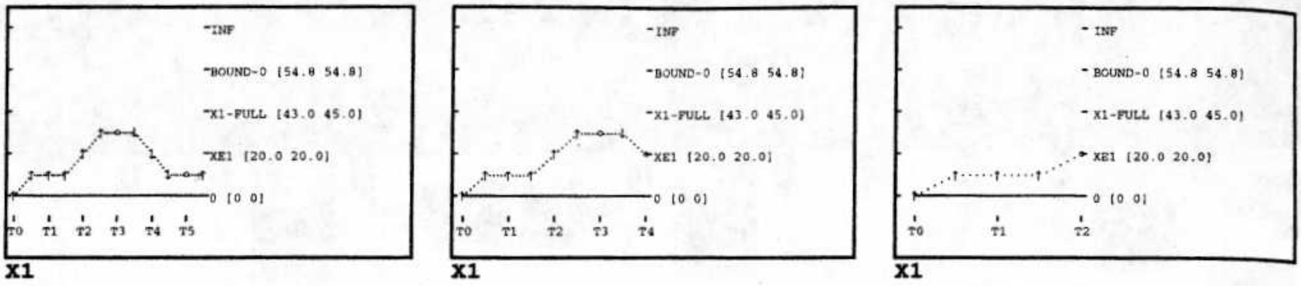


Figure 5: Envisionment graph for the controlled tank

The graph describes a set of possible behaviors where the fluid level reaches the set-point $x_s = x_{e,1}$ without an overshoot (behavior 3), with one overshoot (behavior 2), or the fluid level exhibits a decreasing oscillation around the set-point (behavior 1) and can reach the set-point after any finite number of half-cycles. The corresponding semiquantitative time-plots are shown in figure 6.

Furthermore, our method allows the application of semiquantitative simulation for the analysis of oscillatory 3^{rd} order systems for the first time. We analyzed



(a) behavior 1 (b) behavior 2 (c) behavior 3

Figure 6: Semiquantitative time-plots of the fluid level $x_1(t)$

several 3^{rd} order systems and experienced an immense increase in complexity of the simulation when moving from the analysis of 2^{nd} order systems to 3^{rd} order systems resulting in a high demand on system memory[3]. The simulation predicts a large set of possible behaviors for the systems (e.g. for a spring-mass system controlled via integrating state-feedback we obtained an attainable envisionment graph with 183 branches). The interpretation of such a large set of behaviors is difficult and it is not yet clear whether it is possible to gain additional insight from it as many behaviors are surely spurious.

Conclusion

This paper demonstrated how methods from modern nonlinear control theory can contribute to ongoing research in semiquantitative simulation. The similarities between the descriptions of the system used in semiquantitative simulation and nonlinear control theory allows the application of powerful methods from the latter. In this way it is possible to formulate a method which can calculate Lyapunov functions for many systems of interest. This additional information can be used with simple but powerful filtering methods, which improve the reasoning capabilities of the simulation engine. Compared to kinetic-energy filtering, the advantage of our method is that we are not limited to second order systems of a very specific type and that the semiquantitative reasoning capabilities can be improved. The application of the methods described, which are implemented as an extension to the QSIM simulation platform, is demonstrated by example. Simulation studies with various systems showed that filtering based on Lyapunov functions performs very well with 2^{nd} order systems and also allows the simulation of higher order systems. However, simulating oscillatory 3^{rd} order systems showed that the application of semiquantitative simulation using the current QSIM implementation is limited by the immense demand on system resources and additional research is required to overcome this difficulty so that systems of the order $n \geq 3$ can be simulated successfully.

[3]Especially advanced methods such as Q3 and chatter abstraction cause a high demand on system memory which exceeds our current computing resources (we use QSIM running under Allegro Common Lisp on a DEC Alpha workstation 255 with 320 MB Ram and 400MB swap-space).

Appendix

We shall give a brief summary of the underlying theory (Boyd and Yang 1989) which can be used to find a positive definite symmetric matrix $\mathbf{P}$ so that the quadratic function

$$V(\mathbf{z}) = \mathbf{z}^T \mathbf{P} \mathbf{z} \tag{22}$$

is a Lyapunov function for (2)-(3), i.e. its derivative $\dot{V}(\mathbf{z})$ along the trajectories $\mathbf{z}(t)$ of (2) which is given by

$$\dot{V}(\mathbf{z}) = \mathbf{z}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{z} - 2\mathbf{z}^T\mathbf{P}\mathbf{B}\mathbf{f}_\mathbf{z}(\mathbf{y}) \tag{23}$$

is negative definite for all nonlinearities $\mathbf{f}_\mathbf{z}(\mathbf{y})$ which satisfy the sector-condition (3).

Let us define the *time-varying gains* $k_j(t)$ for a given trajectory $\mathbf{z}(t)$ of the system (2)-(3) by

$$k_j(t) := \begin{cases} \frac{f_{z,j}(y_j(t))}{y_j(t)} & y_j(t) \neq 0 \\ \alpha_j & y_j(t) = 0 \end{cases}, \quad j = 1, \dots, p. \tag{24}$$

It is clear that the gains $k_j(t)$ are differently defined for each trajectory, however, irrespective of the particular trajectory traced it is always true that

$$\alpha_j \leq k_j(t) \leq \beta_j. \tag{25}$$

By using the time-varying gains we can rewrite (23) by

$$\begin{aligned} \dot{V}(\mathbf{z}) &= \mathbf{z}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{z} - 2\mathbf{z}^T\mathbf{P}\mathbf{B}\,\mathrm{diag}(\mathbf{k}(t))\,\mathbf{y} \\ &= \mathbf{z}^T[(\mathbf{A} - \mathbf{B}\,\mathrm{diag}(\mathbf{k}(t))\,\mathbf{C})^T\mathbf{P} + \\ &\quad + \mathbf{P}(\mathbf{A} - \mathbf{B}\,\mathrm{diag}(\mathbf{k}(t))\,\mathbf{C})]\,\mathbf{z} \end{aligned} \tag{26}$$

where the vector $\mathbf{k}(t) := [k_1(t), \dots, k_p(t)]^T$ can take on values in the box $\mathcal{D}_k$ which is defined by the vector pair $\alpha := [\alpha_1, \dots, \alpha_p]^T, \beta := [\beta_1, \dots, \beta_p]^T$ so that $\alpha \preccurlyeq \mathbf{k}(t) \preccurlyeq \beta$. If the right-hand side of (26) is negative definite for all time-varying gains satisfying (25) than (23) is negative definite for all sector-nonlinearities satisfying (3). In this way it is possible to reformulate the problem of finding a quadratic Lyapunov function for the nonlinear system (2) to the problem of finding one for the linear time-varying system

$$\dot{\mathbf{z}} = (\mathbf{A} - \mathbf{B}\,\mathrm{diag}(\mathbf{k}(t))\,\mathbf{C})\,\mathbf{z}. \tag{27}$$

Let $\mathbf{k}^{(1)}, \ldots, \mathbf{k}^{(2^p)}$ denote the 2^p vertices of $\mathcal{D}_k$ which can be used to define the vertex-matrices $\mathcal{A}^{(i)}$ of the time-varying matrix $\mathcal{A}(t) := (\mathbf{A} - \mathbf{B}\,\mathrm{diag}(\mathbf{k}(t))\,\mathbf{C})$ by

$$\mathcal{A}^{(i)} := \mathbf{A} - \mathbf{B}\,\mathrm{diag}(\mathbf{k}^{(i)})\,\mathbf{C}. \tag{28}$$

Than it is possible to show that (26) is negative definite if and only if

$$(\mathcal{A}^{(i)})^T\mathbf{P} + \mathbf{P}\mathcal{A}^{(i)} < 0,\ \forall i = 1, \ldots, 2^p. \tag{29}$$

This matrix inequality provides a necessary and sufficient condition for the existence of a quadratic Lyapunov function for (2)-(3) and can be used to define a *linear matrix inequality* (LMI) problem

$$\begin{array}{ll} \text{find} & \mathbf{P} \\ \text{subject to} & \mathbf{P} > 0, \\ & (\mathcal{A}^{(i)})^T\mathbf{P} + \mathbf{P}\mathcal{A}^{(i)} < 0,\ \forall i = 1, \ldots, 2^p \end{array}$$

which can be solved using semidefinite programming methods (Vanderberghe and Boyd 1996). The advantage of the LMI problem formulation is that it provides a necessary and sufficient condition so that solvers for semidefinite programs can either find the quadratic Lyapunov function or provide evidence that there does not exist *one* quadratic Lyapunov function *for all* ODE systems which are defined by (2)-(3).

References

D. Berleant. *The Use of Partial Quantitative Information with Qualitative Reasoning.* PhD thesis, Artificial Intelligence Laboratory, The University of Texas at Austin, 1991.

K. Bousson and L. Trave-Massuyes. Modeling perspective for qualitative simulation. In *IFAC Symposia on Intelligent Components and Instruments for Control Applications*, pages 217–222, Malaga, Spain, 1992.

S. Boyd and Q. Yang. Structured and simultaneous lyapunov functions for system stability problems. *International Journal of Control*, 49:2215–2240, 1989.

D. Clancy, G. Brajnik, and H. Kay. Model revision: Techniques and tools for analyzing simulation results and revising qualiative models. In *Proceedings of the 11th International Workshop on Qualitative Reasoning (QR97)*, pages 53–65, 1997.

D. Clancy and B. Kuipers. Static and dynamic abstraction solves the problem of chatter in qualitative simulation. In *Proceedings of the 14th National Conference on Artificial Intelligence (AAAI-97)*, 1997.

P. Fouche and B. Kuipers. Reasoning about energy in qualitative simulation. *IEEE Transactions on Systems, Man, and Cybernetics*, 22(1):47–63, 1992.

M. Hofbaur. Analysis of control systems using qualitative and quantitative simulation. In *Proceedings of the 11th International Workshop on Qualitative Reasoning (QR97)*, pages 115–120, 1997.

H. Kay. SQSIM: A simulator for imprecise ode models. Technical Report TR AI96-247, Artificial Intelligence Laboratory, The University of Texas at Austin., 1996.

B. Kuipers. Qualitative simulation. *Artificial Intelligence*, 29:289–338, 1986.

W. Lee and B. Kuipers. Non-intersection of trajectories in qualitative pahse space: a global constraint for qualitative simulation. In *Proceedings of AAAI-88*, pages 286–290, 1988.

L. Vanderberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38:49–95, March 1996.

B. Williams. A theory of interactions: unifying qualitative and quantitative algebraic reasoning. *Artificial Intelligence*, 51:39–94, 1991.